Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY
PROCEEDINGS
Volume 26, 2001–2002
Pages 721–739

SPECTRAL DECOMPOSITION OF ULTRAMETRIC SPACES AND TOPOS THEORY

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Abstract. The paper proves that any complete ultrametric space is isometric to a limit of inverse sequence of uniformly discrete ultrametric spaces and that any non-expanding map between complete ultrametric spaces is a limit of non-expanding maps between the terms of the corresponding spectra. Thus a category of complete ultrametric spaces is a limit of its subcategories of a simpler structure. Restricted to the spaces of finite diameter, each of these subcategories is a pseudo-topos. Thus the whole category of complete ultrametric spaces of a given diameter is a projective pseudo-topos.

It was about one hundred years ago when several different examples of metric spaces with the same property appeared in different areas of mathematics. These were: the rings $\mathbf{Z}_p$ and the fields $\mathbf{Q}_p$ of p-adic numbers in number theory (Kurt Hensel, 1904), the Baire space $B_{\aleph_0}$ in real analysis (René Baire, 1909), and so called "nichtarchimedische Metrik" in topology (Felix Hausdorff, 1914). All these satisfy the strengthened triangle inequality

$$d(x, z) \leq \max[d(x, y), d(y, z)].$$

Such spaces were called "non-Archimedean" in German literature [23], "ultrametric" – in French studies [28], and "isosceles"

2000 *Mathematics Subject Classification.* Primary 54E35, 18B25; Secondary 11E95, 11F85, 12J25, 18B30, 26E30, 54B30, 54B35, 54C10, 54E40, 54F50.

Key words and phrases. Metric space, ultrametric space, inverse spectrum, p-adic numbers, category, sums and products in a category, equalizers and co-equalizers, pullbacks and pushouts, exponentiation, subobject classifier, topos, pseudotopos.

– in Russian [34]. Marc Krasner was the first who defined them axiomaticaly [28], described their basic properties, and found fruitful applications to number theory, algebra, and algebraic geometry [29]. Later on the theory of ultrametric spaces was closely related to Euclidean geometry [5, 37, 42], geometry of Hilbert spaces [25, 37, 42], Lipschitz geometry [51-54], Lebesgue measure and integral theory [42], p-adic analysis and functional analysis [3, 17, 26, 57, 61, 65], theory of Boolean algebras [34, 44], lattice theory [43], graph theory [11, 30, 41], category theory [7, 10, 35, 36, 39, 40, 43-45, 49], topology [8-10, 34, 35, 44, 45, 48, 68, 69], set theory and foundations [45, 68], and many other realms in mathematics [10, 17, 21, 22, 26, 55, 58-60, 65], physics [2, 11, 26, 27, 63], biology [26], statistics [14, 69], and computer science [1, 6, 12, 13, 15, 16, 19, 47, 49, 62, 64, 66, 67].

One of the directions in studying ultrametric spaces is to present any space by means of simpler constructed spaces. E.g., Theorem 2.5 [45] states that any ultrametric space is isometric to a subset of a metric product of two-point spaces (an analogue to the Alexandroff Embedding Theorem [18, Theorem 2.3.26]). Another example: for any cardinal τ there are constructed the ultrametric spaces LV_τ and LW_τ, which contain isometrically all ultrametric spaces of weight at most τ [45, Theorem 1.6 and 48, Main Theorem].

In the present paper, we prove that any complete ultrametric space is isometric to a limit of countable inverse spectrum of uniformly discrete ultrametric spaces (and vice versa) and any non-expanding map between complete ultrametric spaces is a limit of maps between the terms of the corresponding spectra. Thus a category of complete ultrametric spaces and non-expanding maps is a limit of its subcategories of a simpler structure. Further we show that, restricted to spaces of finite diameter, each of these subcategories is a pseudo-topos (it satisfies three of four requirements in the definition of topos and a weakened fourth requirement). Thus the whole category of complete ultrametric spaces of a given diameter is a projective pseudo-topos.

Theorem 1 [38, Theorem 3.4]. *Any complete ultrametric space* (X, d) *is isometric to a limit of a countable inverse sequence of uniformly discrete ultrametric spaces and non-expanding projections,* $(X, d) = \lim\{(X_n, d_n), p_n, \mathbb{N}\}$*; and vice versa.*

Proof. Let (X, d) be an ultrametric space. Denote by $B(x, r) = \{y | d(x, y) < r\}$ an open balls of radius r with a center located at x. The following properties of balls lying in ultrametric spaces are well known.

(i) *Any point of a ball is its center, i.e.,* $B(x, r) = B(y, r)$ *for any* y *in* $B(x, r)$.
(ii) *Any two balls are either disjoint or one is the subset of the other.*
(iii) *If the balls* B *and* B' *are disjoint then* $d(x, x') = d(y, y')$ *for any* x *and* y *in* B *and any* x' *and* y' *in* B'.

(see [43, Lemmas 1, 3, 4] or [45, Theorem 2.5], or any standard text-book on p-adic analysis). These imply that for any x and y in X and any $r > 0$, a relation $d(x, y) < r$ is an equivalence relation on $X \cdot X$. Denote it by $\sim_r$. Equivalence classes of the relation $\sim_r$ are none other than open balls of radius r. Denote by $[x]_r$ and X_r the equivalence class of the point x and a quotient space $X_r = X/\sim_r$. It follows from (iii) that a function $d_r([x]_r, [y]_r) = d_r(B(x, r), B(y, r)) = d(x', y')$ for any x' in $B(x, r)$ and y' in $B(y, r)$ is well-defined for any $[x]_r \neq [y]_r$. Moreover, it is an ultrametric on X (certainly we put $d_r([x]_r, [x]_r) = 0$). A quotient map $f_r : (X, d) \to (X_r, d_r)$ has got the following additional property

(iv) *If* $d(x, y) \geq r$ *then* $d_r(f_r(x), f_r(y)) = d_r([x]_r, [y]_r) = d(x, y)$; *otherwise* $d_r(f_r(x), f_r(y)) = 0$.

Thus the ultrametric $d_r([x]_r, [y]_r)$ is, in a certain sense, inherited from (X, d). This implies that for any $s > r > 0$, $(X_s, d_s) = X/\sim_s = (X/\sim_r)/\sim_s = (X_r, d_r)/\sim_s$.

By the definition [18], a metric space X is *uniformly discrete* if there exists $c > 0$ such that $d(x, y) > c$ for any $x \neq y$. Such spaces are necessarily complete.

Let (X, d) be a complete ultrametric space. For any natural n, consider a partition of the space X into a family of disjoint open balls of radius $1/n$. Denote by (X_n, d_n) the corresponding quotient space and let $p_n : (X_{n+1}, d_{n+1}) \to (X_n, d_n)$ be a natural projection. Consider an inverse spectrum $S = \{(X_n, d_n), p_n, \mathbb{N}\}$. First we prove that it has a limit. If X is of finite diameter we use the following lemmas, where I denotes an arbitrary index set and **METR**(c), (**ULTRAMETR**(c)) denotes the category of all (ultra) metric spaces of diameter at most c and non-expanding maps.

Lemma 1 ([35], °5.0)**.** *For any family* $\{(X_\alpha, d_\alpha)|\alpha \in I\}$ *of metric spaces* $(X_\alpha, d_\alpha) \in$**METR**(c), *there exists a product of these spaces in the category* **METR**(c) *(called a metric product),* $m\Pi\{(X_\alpha, d_\alpha)|\alpha \in I\} = (X_\Pi, d_\Pi)$. *Here* $X_\Pi = \Pi X_\alpha$ *is a product of the sets* X_α *in the category* **SET** *(Cartesian product), and* $d_\Pi(\{x_\alpha\}, \{y_\alpha\}) = \sup\{d_\alpha(x_\alpha, y_\alpha)|\alpha \in I\}$.

Lemma 2 ([35], °6.0)**.** *For any inverse spectrum* $S = \{(X_\alpha, d_\alpha), p_\alpha^\beta, I\}$ *there exists a limit of the spectrum* S, $\lim \underleftarrow{S} = (\underleftarrow{X}, d_\Pi)$, *where* $\underleftarrow{X}$ *is a set of threads in a product of the spaces* X_α, $X_\Pi = m\Pi\{(X_\alpha, d_\alpha)|\alpha \in I\}$, *and a metric* d_Π *is inherited from* $m\Pi\{(X_\alpha, d_\alpha)\}$.

By Lemma 5.1 [35], a product of ultrametric spaces is ultrametric and a product of complete spaces is complete.

Let a diameter of X be infinite. Denote by **METR** (**ULTRA-METR**) the category of all (ultra) metric spaces and non-expanding maps.

Lemma 3 ([49], Proposition 3)**.** *A metric product of an infinite family* $\{(X_\alpha, d_\alpha)|\alpha \in I\}$ *of metric spaces exists in* **METR** *if and only if there is* $c > 0$ *such that for almost all of* (X_α, d_α), $\operatorname{diam}(X_\alpha, d_\alpha) < c$. *If so, the product metric* d_Π *on* X_Π *is equal to* $d_\Pi(\{x_\alpha\}, \{y_\alpha\}) = \sup\{d_\alpha(x_\alpha, y_\alpha)|\alpha \in I\}$.

Lemma 4. *Given an inverse spectrum* $S = \{(X_\alpha, d_\alpha), p_\alpha^\beta, I\}$ *in the category* **METR**, *a limit of the spectrum* S *exists if and only if for any two threads* $\{x_\alpha\}$ *and* $\{y_\alpha\}$ *in* $\Pi\{X_\alpha\}$, *the distance* $d_\Pi(\{x_\alpha\}, \{y_\alpha\})$ *is finite.*

The proof of Lemma 4 is similar to that for Lemmas 1, 2, and 3, [35, 38]. To complete the proof of Theorem 1 note that a thread $\{[x]_n\}$ in the spectrum $S = \{(X_n, d_n), p_n, \mathbb{N}\}$ is a sequence $\{B_n\}$ of enclosed balls of radius $1/n$, $B(x_1, 1) \supset (B(x_2, 1/2) \supset \ldots \supset B(x_n, 1/n) \supset \ldots$. Since X is complete an intersection $\cap\{B(x_n, 1/n)|\mathbb{N}\} = \{x\}$ is not empty. On the other hand, for any x in X, a sequence of balls $\{B(x, 1/n)\}$ is obviously a thread of S.

This states a bijection between X and $\lim \underleftarrow{S}$. For any two different points x and y in X, the distance $d_n([x]_n, [y]_n) = 0$ for any n such that $d(x,y) < 1/n$ and $d_n([x]_n,[y]_n) = d(x,y)$ otherwise. Therefore, $d_\Pi(\{[x]_n\},\{[y]_n\}) = \sup\{d_n([x]_n,[y]_n)|\mathbb{N}\} = d(x,y)$. Hence X and $\lim\limits_{\longleftarrow} S$ are isometric. □

Corollary 1 [40]**.** *Every compact ultrametric space is isometric to a limit of inverse sequence of skeletons of finite dimensional isosceles simplexes lying in Euclidean spaces.*

Proof. For any totally bounded ultrametric space (X,d) the quotient space (X_r,d_r) is finite. By [37, Theorem 1], *every ultrametric space consisting of* $n+1$ *points can be isometrically embedded in the* n*-dimensional Euclidean space* E^n*. No ultrametric space consisting of* $n+1$ *points can be isometrically embedded in the* k*-dimensional Euclidean space* E^k *for* $k < n$ (see also [42 and 48]). Thus an isometric image $i(X_r)$ of a finite ultrametric space in E^n, is a set of points in general position, i.e. it is a skeleton of a simplex. It is well-known that the ultrametric Axiom is equivalent to the property that any three points x, y, and z form an isosceles triangle with the base being no greater that the sides. Thus any two-dimensional face of the simplex $i(X_r)$ is an isosceles triangle with the same property. We call such simplexes *isosceles.* □

Side by side with Theorem 1, another version of the Spectral decomposition theorem is proved in [38] (see also [35] for spaces of diameter at most 1). Let the index set $I = \mathbb{Q}\cap(0, \text{diam } X]$ be a subset of positive rationals. For any r in I, consider the equivalence relation $\sim_r$ and the quotient space $X_r = X/\sim_r$ defined above. Let us introduce another metric on X_r, namely, $m_r([x]_r,[y]_r) = r$ for any $[x]_r \neq [y]_r$. This makes the spaces (X_r,m_r) *metrically discrete* (i.e., $d(x,y) = const$). The projections $p^r_s : (X_r,m_r) \to (X_s,m_s)$ are not non-expanding in this case but they are uniformly continuous (even Lipschitz). This implies the following.

Theorem 1′ ([38, Theorem 3.1], [35, Theorem 6.4] for spaces of diameter ≤ 1)**.** *Every complete ultrametric space is isometric to the limit of a countable inverse spectrum of metrically discrete spaces and uniformly continuous maps.*

Substituting the index set I by $\{1/n|n \in \mathbb{N}\}$ we get a rougher version of the theorem.

Theorem 1″([38, Theorem 3.2]. *Every complete ultrametric space is uniformly homeomorphic to the limit of inverse sequence of metrically discrete spaces and uniformly continuous projections.*

This strengthens W. Kulpa's theorem [31], that characterizes complete ultrametric spaces up to homeomorphism as limits of sequences of topologically discrete spaces and continuous projections (note that, for the spaces (X_r, m_r), a topology of metric product coincides with that of topological product). Theorem 1″ immediately implies a strengthening of another Kulpa's theorem [32].

Theorem ([38, Theorem 3.3]). *Every complete ultrametric space admits a uniformly continuous one-to-one map onto compactum.*

On the other hand, J. Luukkainen and H. Movahedi-Lancarani [52] proved a thinner form of Theorem 1″ and characterized complete ultrametric spaces up to Lipschitz equivalence. If a space X is not complete, one can take its completion and prove that X is Lipschitz equivalent [52, Section 2] (uniformly equivalent, isometric) to a dense subset of the limit of corresponding spectrum. In [52, Section 3], [53], [54], and [51], a nice criterion is proved for the embedding of ultrametric spaces in Euclidean spaces up to Lipschitz equivalence.

For spaces of diameter at most 1, Spectral Decomposition Theorem was first proved in [35, Theorem 6.4]. Since a diameter of the ring of p-adic integers $\mathbf{Z}_p$ is 1 for any p, one can apply the Theorem to $\mathbf{Z}_p$ (but not to $\mathbf{Q}_p$) to prove the following Corollary ([37, 42, and 48, Corollary 2]).

Corollary 2. *There exists an isometric (and closed) imbedding of the ring* $\mathbf{Z}_p$ *in Hilbert space* H, $i : \mathbf{Z}_p \to H$, *under which the image* $i(\mathbf{Z}_p)$ *is located on the sphere* $S_{r(p)}$ *of radius* $r(p) = p/\sqrt{2(p^2+p+1)}$ *and the images* $i(\mathbb{N})$ *of the positive integers form a basis in* H.

Proof. For any natural n, consider a partition of the set $\mathbf{Z}_p$ into a family of disjoint closed balls of radius $1/p^n$ (instead of $1/n$). The corresponding quotient space Z_n is none other than a quotient ring $\mathbf{Z}_p/p^n \cdot \mathbf{Z}_p$, and the inverse limit described in Theorem 1 is $Z_0 \leftarrow Z_1 \leftarrow Z_2 \leftarrow \ldots Z_n \leftarrow Z_{n+1} \leftarrow \ldots$. Here Z_0 is a singleton.

Z_1 consists of p points at a distance $1/p$ from one another. Z_2 consists of p^2 points joined in p groups; each pair of points in the same group are at a distance $1/p^2$ from one another, the other distances are $1/p$, etc. Each Z_n can be imbedded isometrically in (p^n-1)-dimensional Euclidean space as a vertexes of simplex. The straightforward calculation shows that the radius r_n of sphere circumscribed around this simplex satisfies the recursive relation $r_{n-1}^2 + p^2(p-1)/2 = p^3 r_n^2$. A limit of the sequence r_n^2 equals $p^2/2(p^2+p+1)$. It remains to justify the passing to the limit, use Theorem 2 below, and show that the cluster map $f : \mathbf{Z}_p \to H$ would be not only non-expanding but also an isometric imbedding. The latter is based on the property (iv), see [46] for the rest of the proof and details. □

A few other applications of Theorem 1 can be found in [50].

Theorem 2. *Any non-expanding map between complete ultrametric spaces $f : X \to Z$ is isomorphic to a limit of a countable inverse spectrum of non-expanding maps between uniformly discrete ultrametric spaces, $S = \{f_n : X_n \to Z_n, p_n, \mathbb{N}\}$; and vice versa.*

Proof. Let (X, d) and (Z, d) be complete ultrametric spaces and let

$$(X_1, d_1) \overset{p_1}{\longleftarrow} (X_2, d_2) \overset{p_2}{\longleftarrow} ...(X_n, d_n) \overset{p_n}{\longleftarrow} (X_{n+1}, d_{n+1}) \longleftarrow ...(X, d)$$

and

$$(Z_1, d_1) \overset{p_1}{\longleftarrow} (Z_2, d_2) \overset{p_2}{\longleftarrow} ...(Z_n, d_n) \overset{p_n}{\longleftarrow} (Z_{n+1}, d_{n+1}) \longleftarrow ...(Z, d)$$

be their decompositions in countable inverse spectra stated in Theorem 1. Since $f : X \to Z$ is non-expanding, $d(f(x), f(y)) \le d(x, y)$ for any x and y in X. Hence for any $r > 0$, $f(B(x, r)) \subset B(f(x), r)$. Therefore, a map $f_r([x]_r) = [f(x)]_r$ is well-defined and it maps X_r to Z_r. Moreover, for any natural n, the maps $f_n : X_n \to Z_n$, $f_{n+1} : X_{n+1} \to Z_{n+1}$, $p_n : (X_{n+1}, d_{n+1}) \to (X_n, d_n)$, and $p_n : (Z_{n+1}, d_{n+1}) \to (Z_n, d_n)$ form a commutative square. This implies, first, that the family $S = \{f_n : X_n \to Z_n, p_n, \boldsymbol{N}\}$ is an inverse spectrum in the category of morphisms of the category **ULTRAMETR** (see [24]) and, second, that the map $f : X \to Z$ is isomorphic to a limit of this inverse spectrum. □

Decomposition theorems for uniformly continuous (Lipschitz) maps lie beyond the scope of the present paper. Our main goal is to relate the theory of ultrametric spaces to topos theory. The notion of topos was introduced by Alexander Grothendieck [4]. F. W. Lawver and M. Tierney used the following equivalent definition [33], [20, Chapter 4.3]: a category $\mathcal{C}$ is said to be a topos provided it satisfies the following conditions

(1) $\mathcal{C}$ *is finitely complete*
(2) $\mathcal{C}$ *is finitely co-complete*
(3) $\mathcal{C}$ *admits an exponentiation*
(4) $\mathcal{C}$ *has a subobject classifier.*

Although it is proved [56] that (2) follows from the other conditions we'll study separately all the conditions (1) - (4) in turn since our categories are not exactly the topoi. For another definition of topos see [70] and [20, Chapter 4.7].

(1) Finite completeness.

A category $\mathcal{C}$ is called *finitely complete* provided there exists a limit of any finite diagram in $\mathcal{C}$ [20, Chapter 3.15], [24].

Theorem [24, Theorem 27.3]**.** *For any category $\mathcal{C}$, the following are equivalent*
(1) $\mathcal{C}$ *is finitely complete*
(2) $\mathcal{C}$ *has pullbacks and a terminal object*
(3) $\mathcal{C}$ *has finite products and pullbacks*
...
(6) $\mathcal{C}$ *has finite products and equalizers*
...

Let $\mathcal{M}$ denote any of the categories **ULTRAMETR**, **ULTRAMETR**(c), **METR**, or **METR**(c). It is obvious that a singleton 1 $= \{\emptyset\}$ is a terminal object in $\mathcal{M}$. By [49, Proposition 2], $\mathcal{M}$ has finite products (compare with Lemma 3 above). A map $f : X \to Y$ is a monomorphism in $\mathcal{M}$ if and only if it is a non-expanding injection (i.e., is one-to-one onto its range $f(X)$). For any two maps f and $g : X \to Y$, an isometric imbedding of a set $E = \{x | f(x) = g(x)\}$ in X, $e : E \hookrightarrow X$, is an equalizer of f and g. Thus $e : E \to X$ is an equalizer (or *regular monomorphism*, [24, definition 16.3]) if and only if it is a closed isometric imbedding. Hence we have

Lemma 5. *A category $\mathcal{M}$ is finitely complete.*

The Theorem above implies that there exists a pullack in $\mathcal{M}$. It can be described explicitly. Suppose $f : X \to Z$ and $g : Y \to Z$ are non-expanding maps with a common codomain. Take a product $X \cdot Y$ and consider a subset $D = \{(x, y)|f(x) = g(y)\}$ in $X \cdot Y$. Restrictions of the natural projections of the product onto the factors, $p_X|_D = g' : D \to X$ and $p_Y|_D = f' : D \to Y$ complete the diagram below and form a Cartesian square. Thus it is a pullback.

$$\begin{array}{ccc} D & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

(2) Finite cocompleteness.

Let $\mathcal{M}$ be the same as above. The empty set is obviously the initial object in $\mathcal{M}$.

Lemma 6 ([35, °4.1])**.** *For any family $\{(X_\alpha, d_\alpha)|\alpha \in I\}$ of metric spaces (X_α, d_α)* in **METR**(c), *there exists a sum of these spaces in the category* **METR**(c) *(called a metric sum), $(X_\Sigma, d_\Sigma) = m\Sigma\{(X_\alpha, d_\alpha)|\alpha \in I\}$, where X_Σ is a sum of the sets X_α in the category* **SET** *and $d_\Sigma(x_\alpha, y_\alpha) = d_\alpha(x_\alpha, y_\alpha)$, $d_\Sigma(x_\alpha, y_\beta) = c$ for $\alpha \neq \beta$.*

By [35, °4.2], a sum of ultrametric spaces is ultrametric and a sum of complete spaces is complete .

Proposition 1 [49]**.** *A sum of objects does not exist in* **METR** *even for two singletons.*

The following proposition is well known. Let $M(x, y) : X \cdot X \to \boldsymbol{R}_+$ be an arbitrary non-negative function and let $\{d_\alpha(x_\alpha, y_\alpha) : X \cdot X \to \boldsymbol{R}_+|\alpha \in I\}$ be a family of functions such that $d_\alpha(x_\alpha, y_\alpha) \leq M(x, y)$ for any x, y in X and any α in I. If each of $d_\alpha(x_\alpha, y_\alpha)$ is a metric (an ultrametric, a pseudo-metric, a pseudo-ultrametric) then so is $d(x, y) = \sup\{d_\alpha(x_\alpha, y_\alpha)|\alpha \in I\ \}$.

Corollary 3. *For any function $M(x,y) : X \cdot X \to \boldsymbol{R}_+$, there exist a greatest pseudo-metric $d(x,y)$ smaller than $M(x,y)$ and a greatest pseudo-ultrametric $u(x,y)$ smaller than $M(x,y)$; clearly $u(x,y) \leq d(x,y)$.*

If $M(x,y)$ is a metric and $u(x,y)$ satisfies the identity axiom ("$u(x,y) = 0$ implies $x = y$") then it is called *a subdominant ultrametric*, [8, 63]. The problem of existence of the subdominant ultrametric (the Hausdorff-Bayod Problem) plays a significant role in Physics [63]; it is completely solved in [44]. In general, we get a notion of ultrametrization of arbitrary metric space and an epi-reflective functor from **METR** to **ULTRAMETR**, [44].

Lemma 7. *The category $\mathcal{M}$ has coequalizers.*

Proof. For any pair of non-expanding maps f and $g : X \to Y$, consider a set $S = \{(f(x), g(x)) | x \in X\}$ in $Y \cdot Y$. Let $R \subset Y \cdot Y$ be a least equivalence relation which contains S and let $q : Y \to Y/R$ be a quotient map. For any $[x]$ and $[y]$ in Y/R, put $M([x],[y]) = \inf\{d(x,y) | x \in [x], y \in [y]\}$. Clearly, $M([x],[y]) \geq 0$, $M([x],[y]) = M([y],[x])$, and $M([x],[y]) \leq d(x,y)$ for any x and y. Let $d_{/R}$ ($u_{/R}$) denote a greatest pseudo-metric (pseudo-ultrametric, respectively) smaller than M. If $d_{/R}$ ($u_{/R}$) turns out to be a metric (an ultrametric, respectively) then a quotient map from (Y, d) to $(Y/R, d_{/R})$ (to $(Y/R, u_{/R})$, respectively) is a coequalizer of the maps f and g in the category **METR** (**ULTRAMETR**). Otherwise, take a quotient space of the set Y/R with respect to the pseudometric $d_{/R}$ (to the pseudoultrametric $u_{/R}$, respectively) and a composition of the latter two quotient maps. □

Combining Lemmas 6 and 7 we get the following.

Corollary 4. *The categories* **METR**(c) *and* **ULTRAMETR**(c) *are finitely cocomplete.*

This implies that both the categories have got pushouts. For **METR** and **ULTRAMETR**, existence of pushouts can be deduced from the following Theorem (compare with the Hausdorff Theorem [23] and [18, Theorem 4.5.20]).

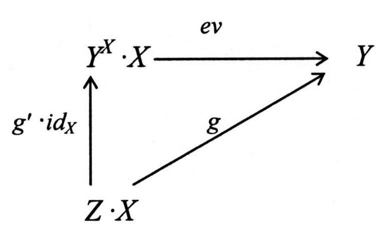
Theorem 3. *Let $(A, D|_A)$ be a subset of a metric space (X, D) and $d : A \cdot A \to \boldsymbol{R}_+$ be a metric on A smaller than $D|_A$. Then there exists a metric $d' : X \cdot X \to \boldsymbol{R}_+$, which is smaller than D on X, coincides with d on A, and is a greatest of such metrics.*

Conversely, if $i : (A, D|_A) \to (X, D)$ is an isometric imbedding and $f : (A, D|_A) \to (A, d)$ is a one-to-one contraction then a pushout of the left diagram below is a set X with the extended metric $d'(x, y)$. The map $i' : (A, d) \to (X, d')$ is an isometric imbedding and $f' : (X, D) \to (X, d')$ is a one-to-one contraction. Note that f' is a local isometry outside of the closure $[(A, d)]$.

$$\begin{array}{ccc} (A,D) & \xrightarrow{\ i\ } & (X,D) \\ {\scriptstyle f}\downarrow & & \\ (A,d) & & \end{array} \qquad\qquad \begin{array}{ccc} (A,D) & \xrightarrow{\ i\ } & (X,D) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (A,d) & \xrightarrow[\ i'\ ]{} & (X,d') \end{array}$$

(3) An exponential.

By the definition [20, Chapter 3.16], a category $\mathcal{C}$ admits an exponentiation provided, for any objects X and Y in $\mathcal{C}$, there is an object Y^X (called *an exponential*) and a morphism $ev : Y^X \cdot X \to Y$ (called *an evaluation*) with the following property. For any object Z in $\mathcal{C}$ and any morphism $g : Z \cdot X \to Y$, the is a unique morphism $g' : Z \to Y^X$ such that the product $g' \cdot id_X$ completes the triangle diagram below.



Let X and Y be arbitrary metric spaces and $f : X \to Y$ be a non-expanding map. Substituting Z by a singleton 1, we see that $g'(1) = f : X \to Y$. Thus an exponential Y^X contains any non-expanding map $f : X \to Y$ if exists. For any f and $g : X \to Y$, the distance $d^X(f,g)$ between them in Y^X satisfies the inequality $d_\Pi(ev(f,x), ev(g,y)) = \max[d^X(f,g), d(x,y)] \geq d(f(x), g(y))$ since ev is non-expanding. In particular, $\max[d^X(f,g), d(x,x)] = d^X(f,g) \geq d(f(x), g(x))$ for any x in X. Hence $d^X(f,g) \geq \sup\{d(f(x), g(y)) | x \in X\}$ and, since d^X is the smallest metric on the set of maps from X to Y (the map g' is non-expanding!), $d^X(f,g) = \sup\{d(f(x), g(y)) | x \in X\}$. Thus the exponential Y^X, if exists, should be equipped with the metric of uniform convergence.

Proposition 2. *If a space Y is ultrametric then the evaluation map is non-expanding for any metric space X. Otherwise, there exists a metric space (moreover, an ultrametric space) X such that ev is not non-expanding.*

Proof. Actually, if Y is ultrametric then $d(f(x), g(y)) \leq \max[d(f(x), f(y)),\ d(f(y), g(y))] \leq \max[d(x,y), d^X(f,g)] = d_\Pi(ev(f,x), ev(g,y))$. Otherwise there are points x, y, and z in Y such that $d(x,z) > d(x,y) \geq d(y,z)$. Let $X = \{a,b\}$ be a two-point space with the metric $d(a,b) = d(x,y)$. Define f and $g : X \to Y$ as follows: $f(a) = x, f(b) = y = g(a)$,and $g(b) = z$. Then $d(f(a), g(b)) = d(x,z) > d(x,y) = d_\Pi(ev(f,a), ev(g,b))$. □

A map $f : X \to Y$ is called *bounded* provided a diameter of $f(X)$ is finite. If at least one of the spaces X and Y is bounded then all non-expanding maps from X to Y are bounded and the distance $d^X(f,g)$ is finite for any f and g. On the other hand, if both X and Y are unbounded and if there is at least one unbounded map form X to Y then the exponential Y^X does not exist. Actually, if $\operatorname{diam}(f(X)) = \infty$ then there is an increasing sequence of natural numbers n_k and points y_k in $f(X)$ such that $n_k \leq d(y_0, y_k) < n_k + 1$. Take x_k in $f^{-1}(y_k)$ and let $g : X \to Y$ be a constant map, $g(X) = y_0$. Since the evaluation map is non-expanding we have $d_\Pi(ev(f,x_k), ev(g,x_k)) = \max[d^X(f,g), d(x_k,x_k)] = d^X(f,g) \geq d(f(x_k), g(x_k)) = d(y_0, y_k) \geq n_k$ for any k. Hence $d^X(f,g) = \infty$.

Proposition 3. *If an ultrametric space Y is unbounded then there exists an ultrametric space X and an unbounded map $f : X \to Y$. If, in addition, Y is spherically complete then for any unbounded ultrametric X, there is an unbounded map $f : X \to Y$.*

Proof. Take the sequences n_k in $\mathbb{N}$ and y_k in Y as above, and consider a subset $X = \{n_k | k \in \mathbb{N}\}$ in $\mathbb{N}$ endowed with the max-metric, $d(n_k, n_m) = \max(n_k, n_m)$. The map $f(n_{k+1}) = y_k$ is unbounded and non-expanding. For spherically complete Y, take n_k in $\mathbb{N}$, and y_k in Y, and x_k in X such that $n_k \leq d(y_0, y_k) \leq d(x_0, x_k) < n_{k+1}$, put $f(x_k) = y_k$ and use Theorem 1 [7]. □

Combining the presiding discourses we get the following.

Lemma 8. *Among the categories* **METR**, **METR**(c), **ULTRAMETR**, and **ULTRAMETR**(c), *the category* **ULTRAMETR**(c) *only admits an exponentiation.*

(4) A subobject classifier.

In general category theory, a morphism $f : X \to Y$ is called a *subobject* provided it is a monomorphism [24, definition 6.22]. This definition is absolutely adequate for algebraic categories such as **GROUP**, **RING**, **MONOID**, etc. However, for categories of topological type (such as **TOP**, **UNIF**, **METR**...) such notion seems to be too wide. First, it is well known that it is more adequate to consider only closed subsets of topological (uniform, metric) space X as elements of a hyperspace $\mathcal{P}(X)$. It is just the set of closed subsets where the Vietoris topology (the Weil uniformity, the Hausdorff metric, respectively) is well defined and satisfies the natural separation properties. Second, even if we restrict ourselves by closed monomorphisms $f : X \to Y$, the set X could have a stronger topology (a finer uniformity, a greater metric) than a subspace $f(X)$. For categories of topological type, it is more natural to consider only closed homeomorphisms (uniform homeomorphisms, isomotric imbeddings, respectively) as subobjects of Y. From the categoricalpoint of view, these are just equalizers (= regular monomorphisms = regular subobjects).

A category $\mathcal{C}$ with the terminal object 1 has a subobject classifier Ω provided there exists an object Ω and a morphism $1 \to \Omega$ (called *a true*) such that for any subobject $f : X \to Y$, there is a unique morphism $\chi_f \colon Y \to \Omega$ (called *a characteristic function*) that completes the diagram below and makes it a Cartesian square [20, chapter 5]. Here I_X denotes a unique morphism from X to 1.

It is well known that almost no topological category has a subobject classifier (the trivial exception is a category of discrete spaces, i.e., **SET**). We'll say that a category $\mathcal{C}$ has *a regular subobject classifier* Ω provided the definition above holds for regular subobjects (=equalizers); $\mathcal{C}$ is a pseudo-topos provided it satisfies requirements (1) - (3) above and has a regular subobject classifier.

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle I_X}\downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \xrightarrow[true]{} & \Omega \end{array}$$

In the beginning of the paper we considered the category of uniformly discrete spaces, i.e., the spaces where $d(x,y) \geq r$ for any $x \neq y$. Denote it by $\mathbf{METR}_r$.

Lemma 9. *For any positive r, the category* $\mathbf{METR}_r$ *has a regular subobject classifier.*

Proof. Put $\Omega_r = \{0, r\}$ and define $true : 1 \to \Omega_r$ and $\chi_f \colon Y \to \Omega_r$ as follows: $true(1) = r$ and $\chi_f|_{f(X)} = r$, and $\chi_f|_{Y \setminus f(X)} = 0$. In view of r-uniform discretness of Y, χ_f is non-expanding. For any non-expanding map $f : X \to Y$, the map χ_f (and only χ_f) makes the diagram above commutative. Direct categorical verification shows that the diagram is a Cartesian square if and only if f is an isometric imbedding (note that any subset in Y is closed). $\square$

Combining Lemmas 5-9 we get the main results of the paper.

Theorem 4([39]). *For any positive r and c, the category* **ULTRAMETR**$_r(c)$ *is a pseudo-topos.*

Combining this with Theorems 1 and 2 we have.

Theorem 5 ([40]). *The category co-***ULTRAMETR**(c) *of complete ultrametric spaces of diameter at most c and non-expanding maps is a projective pseudo-topos.*

Close relationships between topos theory and logic are well known [33, 20]. A logical interpretation could be partially given for pseudotopoi **ULTRAMETR**$_r(c)$. The Kripke scale in the corresponding

Kripke model turns out to be real-graduated (i.e., to be endowed with a monotonic real valued positive function). On the other hand, the Main Theorem [43] states the isomorphism between the category **ULTRAMETR** and the category **LAT*** of complete, atomic, tree-like, and real graduated lattices (i.e., the lattices equipped with a monotonic real valued positive function). This relation will be considered elsewhere.

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