# Topology Proceedings

**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
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**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

**TOPOLOGY PROCEEDINGS**
Volume 26, 2001–2002
Pages 741–750

# ON GENERALIZATIONS OF RADON-NIKODÝM COMPACT SPACES

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Abstract. There are two notions weaker than that of Radon-Nikodým compact spaces and stronger than that of fragmentable compact spaces. They are the notions of "strongly fragmentable" compact spaces due to Reznichenko and that of "quasi-Radon-Nikodým" compact spaces due to Arvanitakis. We show below that the last two notions are equivalent. We also give an exposition of Arvanitakis's short topological proof of the fact that each quasi-Radon-Nikodým Corson compact space is Eberlein compact using the concept of "almost neighborhoods of the diagonal".

## 1. Introduction

Let $(X, \tau)$ be a topological space and let $\varphi$ be a non-negative function on $X \times X$ such that $\varphi(x, x) = 0$ for each $x \in X$. The space $(X, \tau)$ is said to be *fragmented by* $\varphi$ if, whenever $\varepsilon > 0$ and $A$ is a non-empty subset of $X$, there is a $\tau$-open set $U$ in $X$ such that $U \cap A \neq \emptyset$ and $\varphi$−diam$(U \cap A) < \varepsilon$. Here, for a non-empty subset $S$ of $X$, $\varphi$−diam$(S) = \sup\{\varphi(x, y) : x, y \in S\}$.

2000 *Mathematics Subject Classification.* 54D30, 54E99.
*Key words and phrases.* Radon-Nikodým compact, quasi-Radon-Nikodým compact, strongly fragmentable, Eberlein compact, Corson compact and Almost neighborhood.

In [4] and [5], the author defined a compact Hausdorff space $(K, \tau)$ to be *Radon-Nikodým (RN) compact* if $(K, \tau)$ is homeomorphic to a weak* compact subset of the dual of an Asplund space. It is proved that the compact space $(K, \tau)$ is RN compact if and only if it is fragmented by a metric which is lower semicontinuous on $(K \times K, \tau \times \tau)$. Generalizing this, Ribarska [7] has defined a topological space $(X, \tau)$ to be *fragmentable* if $(X, \tau)$ is fragmented by some metric on $X$. There exist fragmentable compact Hausdorff spaces which are not RN compact (see [6]).

There are two notions that are, at least formally, between the two mentioned above. The first one, which is attributed to Reznichenko in [2], is the following: A topological space $(X, \tau)$ is *strongly fragmentable* if $(X, \tau)$ is fragmented by a metric $\rho$ satisfying the Condition (R):

(R) whenever $x, y \in X$ and $x \neq y$, there are open neighborhood $U$ and $V$ of $x$ and $y$ respectively such that
$$\inf\{\rho(u, v) : u \in U,\ v \in V\} > 0.$$

The second notion, due to Arvanitakis [3], is that of quasi-Radon-Nikodým compact spaces. A compact Haudorff space $(K, \tau)$ is said to be *quasi-Radon-Nikodým compact (quasi-RN compact)* if $(K, \tau)$ is fragmented by a lower $(\tau \times \tau)$-semicontinuous quasi-metric on $K$, where a *quasi-metric* on $K$ is a function $\varphi : K \times K \to [0, 1]$ such that $\varphi(x, y) = \varphi(y, x)$ for all $x, y \in K$ and $\varphi(x, y) = 0$ if and only if $x = y$.

One of the aims of the present note is to show that a compact Hausdorff space is strongly fragmentable if and only if it is quasi-RN. In so doing, we find the notion of "almost neighborhoods of the diagonal" introduced in [5] useful.

In response to Problem 4 in [5], Orihuela, Schachermayer and Valdivia [6] have proved that an RN compact space is Eberlein compact if and only if it is Corson compact (terms recalled in §3). Their proof relies on Banach space techniques and it is rather involved. Recently Arvanitakis [3] gave a relatively short topological proof of a stronger result: A quasi-RN compact space is Eberlein compact if and only if it is Corson compact. Our second aim is to give an exposition of Arvanitakis's proof stated in terms of almost neighborhoods of the diagonal.

## 2. The equivalences

Let $(X, \tau)$ be a topological space. Then we denote the *diagonal* $\{(x, x) : x \in X\}$ by $\Delta_X$. Let $C$ be a subset of $X \times X$ containing $\Delta_X$. Then we say that a subset $A$ of $X$ is *$C$-small* if $A \times A \subset C$. The set $C$ is called an *almost neighborhood of* $\Delta_X$ if, whenever $A$ is a non-empty subset of $X$, there is a non-empty relatively $\tau$-open subset of $A$ which is $C$-small. This notion was given in [5]. We note that if $C$ is an almost neighborhood of $\Delta_X$, then so is its inverse $C^{-1} = \{(y, x) : (x, y) \in C\}$. Also if $C_1, C_2, \ldots, C_k$ are almost neighborhoods of $\Delta_X$, then so is the intersection $\bigcap_{j=1}^k C_j$. The following is the main result of this section.

**Theorem 1.** *Let $(X, \tau)$ be a topological space. Then the following conditions are equivalent.*

- *(a) $(X, \tau)$ is fragmented by a lower semicontinuous quasi-metric on $X$.*
- *(b) $(X, \tau)$ is fragmented by a lower semicontinuous function $\varphi : X \times X \to [0, \infty)$ such that $\varphi(x, y) = 0$ if and only if $x = y$.*
- *(c) $(X, \tau)$ is strongly fragmentable.*
- *(d) There is a sequence $\{C_j : j \in \mathbb{N}\}$ of $(\tau \times \tau)$-closed almost neighborhoods of $\Delta_X$ such that $\bigcap\{C_j : j \in \mathbb{N}\} = \Delta_X$.*

Before we begin the proof, we restate Condition (R) in a convenient form. The proof is straightforward and is omitted.

**Lemma 2.** *A metric $\rho$ on the topological space $(X, \tau)$ satisfies Condition (R) if and only if*

$$\bigcap_{n=1}^{\infty} \overline{\{(x, y) : \rho(x, y) \leq 1/n\}}^{\tau \times \tau} = \Delta_X.$$

*Proof of Theorem 1.* (a)$\Rightarrow$(b) is trivial.
(b)$\Rightarrow$(d): Let $C_j = \{(x, y) : \varphi(x, y) \leq 1/j\}$. Then clearly each $C_j$ is a closed almost neighborhood of $\Delta_X$ and $\bigcap\{C_j : j \in \mathbb{N}\} = \Delta_X$.
(d)$\Rightarrow$(a): Let $\{C_j : j \in \mathbb{N}\}$ be as in (d). By replacing $C_j$ with $\bigcap\{C_n \cap C_n^{-1} : 1 \leq n \leq j\}$ we may assume that $C_j = C_j^{-1}$ and $C_{j+1} \subset C_j$ for each $j \in \mathbb{N}$.

For each $j$, let $\varphi_j : X \times X \to [0,1]$ be defined by

$$\varphi_j(x,y) = \begin{cases} 0, & \text{if } (x,y) \in C_j; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\varphi_j$ is lower semicontinuous function such that $\varphi_j(x,y) = \varphi_j(y,x)$ for $x, y \in X$. Let $\varphi = \sum\{2^{-j}\varphi_j : j \in \mathbb{N}\}$. Then $\varphi(x,y) = 0$ if and only if $\varphi_j(x,y) = 0$ for each $j \in \mathbb{N}$ and this is the case if and only if $(x,y) \in C_j$ for each $j$ or equivalently $(x,y) \in \bigcap\{C_j : j \in \mathbb{N}\} = \Delta_X$. Hence $\varphi$ is a lower semicontinuous quasi-metric as defined above. It remains to show that $(X,\tau)$ is fragmented by $\varphi$.

Let $A$ be a non-empty subset of $X$ and let $\varepsilon > 0$. Choose an $n \in \mathbb{N}$ so that $2^{-n} < \varepsilon$. Then there is a $\tau$-open subset $U$ of $X$ such that $U \cap A$ is non-empty and $C_n$-small. Since $U \cap A$ is also $C_j$ small for $j \leq n$, if $(x,y) \in U \cap A$, then

$$\varphi(x,y) = \sum_{j=n+1}^{\infty} 2^{-j}\varphi_j(x,y) \leq 2^{-n}.$$

Hence $\varphi-\operatorname{diam}(U \cap A) \leq 2^{-n} < \varepsilon$. This proves that $(X,\tau)$ is fragmented by $\varphi$.
(c)⇒(d): Let $\rho$ be a metric satisfying Condition (R) by which $(X,\tau)$ is fragmented. Let

$$C_j = \overline{\{(x,y) \in X \times X : \rho(x,y) \leq 1/j\}}^{\tau\times\tau}.$$

Since $(x,\tau)$ is fragmented by $\rho$, each $C_j$ is an almost neighborhood of $\Delta_X$. Clearly each $C_j$ is closed and $\bigcap\{C_j : j \in \mathbb{N}\} = \Delta_X$ by Lemma 2.
(d)⇒(c): It is shown in the proof of [5, Theorem 6.1] that, if $C$ is an almost neighborhood of $\Delta_X$, then there is a $\{0,1\}$-valued pseudo-metric $d$ on $X$ such that the set $\{(x,y) : d(x,y) = 0\}$ is an almost neighborhood of $\Delta_X$ contained in $C$. Such $d$ is said to be *associated with* $C$.

Now let $\{C_j : j \in \mathbb{N}\}$ be as in (d) and, for each $j$, let $d_j$ be a pseudo-metric associated with $C_j$. Let $\rho = \sum\{2^{-j}d_j : j \in \mathbb{N}\}$. Then as shown in the proof of [5, Theorem 6.1], $\rho$ is a metric and $(X,\tau)$ is fragmented by $\rho$. If $x, y \in X$ and $\rho(x,y) < 2^{-j}$, then $d_j(x,y) = 0$, *i.e.* $(x,y) \in C_j$. Since $C_j$ is $\tau \times \tau$-closed,

$$\overline{\{(x,y) : \rho(x,y) < 2^{-j}\}}^{\tau\times\tau} \subset C_j.$$

It follows that

$$\bigcap_{n=1}^{\infty} \overline{\{(x,y) : \rho(x,y) \leq 1/n\}}^{\tau\times\tau} \subset \bigcap\{C_j : j \in \mathbb{N}\} = \Delta_X.$$

Hence by Lemma 2, $\rho$ satisfies Condition (R).

**Remarks. (i)** It follows from Theorem 1 that a compact Hausdorff space is quasi-RN compact if and only if it is strongly fragmentable. Clearly each RN compact space is quasi-RN but the converse is unknown.

**Problem 1.** Is a quasi-RN compact Hausdorff space necessarily RN compact?

It is yet unknown if each Haudorff quotient of an arbitrary RN compact space is RN compact (Problem 1 in [5]). However, each Hausdorff quotient of each quasi-RN compact space is again quasi-RN compact ([3], or implicitly in the proof of [5, Theorem 6.8]). Hence, if Problem 1 is solved affirmatively, then the problem of quotients of RN compact spaces is also solved.

**(ii)** The relationship among the notions discussed thus far can be summarized in terms of almost neighborhoods of the diagonal as follows:

(a) A topological space $(X,\tau)$ is fragmentable if and only if there is a sequence $\{C_n : n \in \mathbb{N}\}$ of almost neighborhoods of $\Delta_X$ with $\bigcap\{C_n : n \in \mathbb{N}\} = \Delta_X$ (*cf.* [5, Theorem 6.1]).

(b) A compact Hausdorff space $(K,\tau)$ is quasi-RN compact if and only if there is a sequence $\{C_n : n \in \mathbb{N}\}$ of $(\tau\times\tau)$-closed almost neighborhoods of $\Delta_K$ with $\bigcap\{C_n : n \in \mathbb{N}\} = \Delta_K$ (*cf.* Theorem 1).

(c) A compact Haudorff space $(K,\tau)$ is RN compact if and only if (b) is satisfied with the additional condition that $C_{n+1} \circ C_{n+1} \subset C_n$ for each $n$ (*cf.* [5, Theorem 6.6]).

## 3. Arvanitakis's Theorem

Recall that a compact Haudorff space is said to be *Eberlein compact* if it is homeomorphic to a weakly compact subset of a Banach space. Recall also that a compact Hausdorff space is said to be *Corson compact* if it is homeomorphic to a compact subset $K$ of the product $[0,1]^\Gamma$ such that, for each $x \in K$, $\{\gamma \in \Gamma : x(\gamma) > 0\}$ is countable. Equivalently, a compact Hausdorff space $K$ is Corson compact if and only if there is a family $\{f_\gamma : \gamma \in \Gamma\}$ of continuous functions $f_\gamma : K \to [0,1]$ separating points of $K$ such that $\{\gamma \in \Gamma : f_\gamma(x) > 0\}$ is countable for each $x \in K$. In [1], Alster has proved that a Corson compact space is a "strong-Eberlein compact space" if and only if it is scattered. The crucial step of this proof consists of an interesting transfinite induction. Arvanitakis has adapted the pattern of Alster's proof in order to prove the result quoted in the introduction. We give in this section an exposition of ideas of Alster and Arvanitakis using the notion of almost neighborhoods of the diagonal.

Let $X$ be a Hausdorff topological space and let $C$ be a fixed almost neighborhood of $\Delta_X$. For each non-empty compact set $F \subset X$ and each ordinal $\alpha$, we define inductively a compact subset $F(\alpha)$ of $F$ as follows: Let $F(0) = F$, and suppose that $F(\beta)$ is defined for all $\beta < \alpha$. If $\alpha = \beta + 1$, then let

$$F(\alpha) = F(\beta+1) = \\ F(\beta) \setminus \bigcup\{U : U \text{ is a } C\text{-small relatively open subset of } F(\beta)\}.$$

If $\alpha$ is a limit ordinal, let $F(\alpha) = \bigcap_{\beta<\alpha} F(\beta)$. Clearly, $\alpha \le \alpha'$ implies $F(\alpha') \subset F(\alpha)$. Since $C$ is an almost-neighborhood of $\Delta_X$, $F(\alpha) = \emptyset$ eventually. Let $\alpha_0$ be the least ordinal with $F(\alpha_0) = \emptyset$. Then by compactness, $\alpha_0$ is not a limit ordinal. Let $\beta_0 = \alpha_0 - 1$. Then $F(\beta_0) \neq \emptyset$ and it can be covered by a family (hence, by a finite family) of $C$-small relatively open subsets of $F(\beta_0)$. Define $Z(F) = F(\beta_0)$ and $\alpha(F) = \beta_0$. Then $Z(F) = F(\alpha(F))$. Let $\mathcal{K}$ denote the family of all compact subsets of $X$.

**Lemma 3.** *Using the notation above,*

*(i) If $F \in \mathcal{K}$ and non-empty, then $Z(F)$ is a compact non-empty subset of $F$ and can be covered by finitely many $C$-small relatively open subsets of $Z(F)$.*
*(ii) If $F_1, F_2 \in \mathcal{K}$ and $F_1 \subset F_2$, then $F_1(\alpha) \subset F_2(\alpha)$ for all $\alpha$ and $\alpha(F_1) \leq \alpha(F_2)$.*
*(iii) If $\{F_n : n \in \mathbb{N}\}$ is a sequence in $\mathcal{K}$ with $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, then there is a $k \in \mathbb{N}$ such that*

$$Z(F_1 \cap \cdots \cap F_k) \cap F_m \neq \emptyset \quad \textit{for all} \quad m \in \mathbb{N}.$$

*Proof.* (i) is a summary of the facts stated above. The first statement of (ii) can be established by an obvious transfinite induction, and clearly the second statement follows from the first. To see (iii), let $\{F_n\}$ be as given, and let $G_k = \bigcap_{n=1}^{k} F_k$. Then by (ii), $\alpha(G_1) \geq \alpha(G_2) \geq \dots$. Hence, there is a $k \in \mathbb{N}$ with $\alpha(G_k) = \alpha(G_m)$ for all $m \geq k$. Then using (ii), for $m \geq k$, $Z(G_k) \cap F_m = G_k(\alpha(G_k)) \cap F_m \supset G_m(\alpha(G_k)) = G_m(\alpha(G_m)) = Z(G_m) \neq \emptyset$. For $m < k$, $Z(G_k) \cap F_m \supset Z(G_k) \cap G_k = Z(G_k) \neq \emptyset$. □

An indexed family $\{A_\gamma : \gamma \in \Gamma\}$ of subsets of $X$ is said to be *point-finite* (*point-countable*) if for each $x \in X$, $\{\gamma \in \Gamma : x \in A_\gamma\}$ is finite (resp. countable). It is said to be *$\sigma$-point-finite* if $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ and for each $n, \{A_\gamma : \gamma \in \Gamma_n\}$ is point-finite.

**Theorem 4.** *Let $X$ be a Hausdorff topological space and let $C$ be an almost neighborhood of $\Delta_X$. Let $\{A_\gamma : \gamma \in \Gamma\}$ and $\{B_\gamma : \gamma \in \Gamma\}$ be indexed families of non-empty subsets of $X$ such that, for each $\gamma \in \Gamma$, $A_\gamma$ is compact, $A_\gamma \subset B_\gamma$ and $(A_\gamma \times (X \setminus B_\gamma)) \cap C = \emptyset$. If $\{B_\gamma : \gamma \in \Gamma\}$ is point-countable, then $\{A_\gamma : \gamma \in \Gamma\}$ is $\sigma$-point-finite.*

*Proof.* We use the following notation: For each non-empty subset $\Psi$ of $\Gamma$, let

$$\mathcal{Z}(\Psi) = \{Z(A_{\gamma_1} \cap \cdots \cap A_{\gamma_k}) : \gamma_1, \dots \gamma_k \in \Psi,\ k \in \mathbb{N}\}.$$

Here, we let $Z(\emptyset) = \emptyset$. Clearly $|\mathcal{Z}(\Psi)| \leq |\Psi|^*$, where $|\Psi|$ is the cardinality of $\Psi$ and $|\Psi|^* = \max(\aleph_0, |\Psi|)$. We also let

$$H(\Psi) = \{\eta \in \Gamma : Z \cap A_\eta \neq \emptyset \ \text{ for some } \ Z \in \mathcal{Z}(\Psi)\}.$$

Clearly, $\Psi \subset H(\Psi)$ and we claim that

$$|H(\Psi)| \leq |\Psi|^*. \tag{1}$$

To see (1), it is sufficient to show that for each $Z \in \mathcal{Z}(\Psi)$, there are at most countably many $\eta$'s such that $Z \cap A_\eta \neq \emptyset$. Suppose that it is not the case, i.e. $Z \cap A_\eta \neq \emptyset$ for uncountably many $\eta$'s. By Lemma 3 (i), $Z = \bigcup_{n=1}^k U_n$ where each $U_n$ is $C$-small, i.e. $U_n \times U_n \subset C$. Hence for some $n \in \{1, 2, \ldots, k\}$, $U_n \cap A_\eta \neq \emptyset$ for uncountably many $\eta$'s. By hypothesis $(A_\eta \times (X \setminus B_\eta)) \cap (U_n \times U_n) = \emptyset$ for each $\eta$. It follows that $U_n \cap (X \setminus B_\eta) = \emptyset$ whenever $U_n \cap A_\eta \neq \emptyset$. Hence $\emptyset \neq U_n \subset B_\eta$ for uncountably many $\eta$'s, contradicting the assumption that $\{B_\gamma : \gamma \in \Gamma\}$ is point-countable.

The proof of Theorem 4 is by induction on $|\Gamma|$. If $|\Gamma| \leq \aleph_0$, there is nothing to prove. Now let $\aleph$ $(> \aleph_0)$ be a cardinal such that Theorem 4 is true whenever $|\Gamma| < \aleph$. We must prove the theorem when $|\Gamma| = \aleph$. We may assume that $\Gamma$ is the least ordinal with $|\Gamma| = \aleph$. (So, in particular, $\gamma \in \Gamma \iff \gamma < \Gamma$ and $\alpha = \{\gamma : \gamma < \alpha\}$ for each $\alpha \leq \Gamma$.)

We define inductively $\Psi_\alpha \subset \Gamma$ for $\alpha < \Gamma$ so that

$$|\Psi_\alpha| \leq |\alpha|^*. \tag{2}$$

Let $\Psi_0 = 0\,(= \emptyset)$. Suppose that $\Psi_\beta$ is defined for all $\beta < \alpha$. If $\alpha = \beta + 1$, then let $\Psi_\alpha = \Psi_{\beta+1} = \alpha \cup H(\Psi_\beta)$. If $\alpha$ is a limit ordinal, then let $\Psi_\alpha = \bigcup_{\beta<\alpha} \Psi_\beta$. Clearly $\{\Psi_\alpha : \alpha < \Gamma\}$ is increasing and $\bigcup\{\Psi_\alpha : \alpha < \Gamma\} = \Gamma$. (2) follows from (1) and the inductive hypothesis.

Now for each $\alpha < \Gamma$, let $\Gamma_\alpha = \Psi_{\alpha+1} \setminus \Psi_\alpha$. Then $\{\Gamma_\alpha : \alpha < \Gamma\}$ is a disjoint partition of $\Gamma$, and $|\Gamma_\alpha| \leq |\alpha|^* < \Gamma$ for each $\alpha < \Gamma$ by (2). Hence by the inductive hypothesis, $\Gamma_\alpha = \bigcup_{n=1}^\infty \Gamma_\alpha^n$ where $\{A_\gamma : \gamma \in \Gamma_\alpha^n\}$ is point-finite for each $n$. Let $\Gamma_n = \bigcup\{\Gamma_\alpha^n : \alpha < \Gamma\}$. Clearly $\Gamma = \bigcup\{\Gamma_n : n \in \mathbb{N}\}$ and the proof is complete once $\{A_\gamma : \gamma \in \Gamma_n\}$ is shown to be point finite. By construction, it suffices to show that if $\alpha_1 < \alpha_2 < \cdots < \Gamma$ and if $\gamma_j \in \Gamma_{\alpha_j}$ for $j \in \mathbb{N}$, then $\bigcap_{j=1}^\infty A_{\gamma_j} = \emptyset$. Suppose, on the contrary, $\bigcap_{j=1}^\infty A_{\gamma_j} \neq \emptyset$. Then by Lemma 3(iii) there is a $k \in \mathbb{N}$ such that

$$Z(A_{\gamma_1} \cap \cdots \cap A_{\gamma_k}) \cap A_{\gamma_m} \neq \emptyset \quad \text{for} \quad m \in \mathbb{N}.$$

Since $\gamma_1, \ldots, \gamma_k \in \Psi_{\alpha_k+1}$, $\gamma_m \in H(\Psi_{\alpha_k+1}) \subset \Psi_{\alpha_k+2}$ for all $m \in \mathbb{N}$. It follows that $\gamma_m \notin \Gamma_{\alpha_m} = \Psi_{\alpha_m+1} \setminus \Psi_{\alpha_m}$ whenever $\alpha_m \geq \alpha_k + 2$. Since $\alpha_j \uparrow$, the last condition holds for large enough $m$'s. This contradicts our choice of $\gamma_j$'s. This completes the proof. $\square$

**Corollary 5 (Arvanitakis [3]).** *Let $K$ be a quasi-RN compact Hausdorff space and let $\{A_\gamma : \gamma \in \Gamma\}$ and $\{V_\gamma : \gamma \in \Gamma\}$ be indexed families of non-empty subsets of $K$ such that, for each $\gamma \in \Gamma$, $A_\gamma$ is closed, $V_\gamma$ is open and $A_\gamma \subset V_\gamma$. If $\{V_\gamma : \gamma \in \Gamma\}$ is point-countable, then $\{A_\gamma : \gamma \in \Gamma\}$ is $\sigma$-point-finite.*

*Proof.* Let $\{C_n : n \in \mathbb{N}\}$ be a sequence of closed almost neighborhoods of $\Delta_K$ such that $\bigcap_{n=1}^{\infty} C_n = \Delta_K$. By the remark at the beginning of §2, we may assume that $C_n \supset C_{n+1}$ for each $n$. For each $\gamma$, $(A_\gamma \times (K \setminus V_\gamma)) \cap \Delta_K = \emptyset$. Hence by compactness, $(A_\gamma \times (K \setminus V_\gamma)) \cap C_n = \emptyset$ for some $n$. Therefore, it is possible to partition $\Gamma$ as $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, where $(A_\gamma \times (K \setminus V_\gamma)) \cap C_n = \emptyset$ for each $\gamma \in \Gamma_n$. By Theorem 4, $\{A_\gamma : \gamma \in \Gamma_n\}$ is $\sigma$-point-finite. Therefore $\{A_\gamma : \gamma \in \Gamma\}$ is $\sigma$-point-finite. $\square$

Recall that Rosenthal [8] proved that a compact Hausdorff is Eberlein compact if and only if it admits a $\sigma$-point-finite separating family of open $\mathcal{F}_\sigma$-subsets. Here a family $\mathcal{S}$ of subsets of a space $X$ is said to be *separating* if, whenever $x, y \in X$ and $x \neq y$, $\{x, y\} \cap S$ is a singleton for some $S \in \mathcal{S}$.

**Theorem 6 (Arvanitakis [3]).** *If a compact Hausdorff space $K$ is both quasi-RN and Corson compact, then it is Eberlein compact.*

*Proof.* Let $\{f_\gamma : \gamma \in \Gamma\}$ be a family of continuous maps $f_\gamma : K \to [0, 1]$ that separates points of $K$ and, for each $x \in K$, $\{\gamma \in \Gamma : f_\gamma(x) > 0\}$ is countable. Let $H$ denote the set of all rationals in $(0, 1)$ and, for each $\alpha = (q, \gamma) \in H \times \Gamma$, let

$$U_\alpha = f_\gamma^{-1}((q, 1]), \quad A_\alpha = f_\gamma^{-1}([q, 1]) \text{ and } \quad V_\alpha = f_\gamma^{-1}((0, 1]).$$

Then for each $\alpha$, $A_\alpha$ is closed, $V_\alpha$ open and $A_\alpha \subset V_\alpha$. Moreover $\{V_\alpha : \alpha \in H \times \Gamma\}$ is point-countable. Hence by Corollary 5, $\{A_\alpha : \alpha \in H \times \Gamma\}$ is $\sigma$-point-finite and hence $\{U_\alpha : \alpha \in H \times \Gamma\}$ is also $\sigma$-point-finite. Clearly $\{U_\alpha : \alpha \in H \times \Gamma\}$ is a family of open $F_\sigma$-subsets of $K$, and, for $x, y \in K$ with $x \neq y$, $\{x, y\} \cap U_\alpha$ is a singleton for a suitable $\alpha$. Hence by Rosenthal's characterization of Eberlein compact spaces, $K$ is Eberlein compact. $\square$

**Remark.** The following result, presumably by Reznichenko, is stated in [2]: each strongly fragmentable Corson compact space is Eberlein compact. On account of Theorem 1, this result is equivalent to Theorem 6 above. However, the result cited in [2] has never been published as far as we know.

## References

[1] K. Alster, *Some remarks on Eberlein compacts*, Fund. Math., **104** (1979), 43–46.

[2] A. V. Arhangel'skii, *General Topology II*, Encyclopaedia of Mathematical Science, Vol. 50, Springer, 1995.

[3] Alexander D. Arvanitakis, *Some remarks on Radon Nikodým compact spaces*, Fund. Math., **172** (2002), 41–60.

[4] I. Namioka, *Eberlein and Radon-Nikodým compact spaces*, Lecture Notes, University College London, Autumn term, 1985.

[5] I. Namioka, *Radon-Nikodým compact spaces and fragmentability*, Mathematika, **34** (1987), 258–281.

[6] J. Orihuela, W. Schachermayer and M. Valdivia, *Every Radon-Nikodým Corson compact space is Eberlein compact*, Studia Math., **98** (1991), 157–174.

[7] N. K. Ribarska, *Internal characterization of fragmentable spaces*, Mathematika, **34** (1987), 243–257.

[8] Haskell P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math., **28** (1974), 83–111.

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