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HYPERCONVEX SEMI-METRIC SPACES

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ABSTRACT. We examine the close analogy which exists between Helly graphs and hyperconvex metric spaces, and propose the *hyperconvex semi-metric* space as an unifying concept. Unlike the metric spaces, these semi-metric spaces have a rich theory in the discrete case. Apart from some new results on Helly graphs, the main results concern: fixed point property of contractible semi-metric spaces (for nonexpansive multifunctions); and the injectivity of hyperconvex semi-metric spaces.

1. INTRODUCTION

The notion of hyperconvex spaces was introduced by Aronszajn and Panitchpakdi in [1], where it is shown that a metric space is hyperconvex if and only if it is injective (with respect to the nonexpansive mappings). Since every metric space has an injective hull [7], it follows that every metric space is isometric with a subspace of a (minimal) hyperconvex superspace. Also it is known that a real Banach space is hyperconvex if and only if it is isometrically isomorphic to a space of continuous real-valued functions on a Stone space. Classical examples of hyperconvex spaces include the well-known spaces ℓ^∞ and L^∞.

Quilliot concludes his note [14] with some brief but intriguing speculations about the application of graph theory to topology. The

body of the paper is a study of Helly graphs, where a connected graph G is *Helly* if the balls of G have the (2-)*Helly property*: if C is any collection of pairwise intersecting balls (where a ball is taken with respect to the usual hop-distance metric d_G), then there is a vertex common to all the members of C. Quilliot suggests that this definition be adapted to metric spaces in general, so that a "Helly space" is a compact metric space satisfying the Helly condition for its family of closed balls (and satisfying also a certain convexity condition, which we consider later). He then states some results and a conjecture about Helly spaces, analogous to previous results about Helly graphs.

As it happens, these Helly spaces (minus the compactness restriction) are exactly the hyperconvex spaces from the functional analysis literature. This means that, besides trying to extend ideas about Helly graphs to "Helly spaces", one may consider mining the (by now) fairly extensive literature on hyperconvex spaces for ideas to transfer back to Helly graphs. We give a couple of examples of this in Section 2. This kind of exercise is not, however, our main concern in the present paper. Rather, we are motivated by the following problem. The concept of a Helly graph is defined via the simple graph metric. It might seem, therefore, that the Helly graph should be just a special case of the "Helly" (or hyperconvex) metric space. This, however, is not so. Due to the convexity condition (see Section 4), only uncountably infinite metric spaces can be hyperconvex (apart from the singleton space). Our proposal for finding a common generalization, and explaining the overwhelmingly strong analogies which exist between Helly graphs and hyperconvex metric spaces, is to weaken the notion of metric involved. The weak metrics which are relevant here are those which have been found useful before in domain theory [6, 8, 10] and digital topology [17]: partial metrics and semi-metrics. These "metrics" are the topic of Section 3.

Section 4 contains our key definition: the hyperconvex semi-metric space. In fact it is exactly the usual definition, except that "metric" is weakened to "semi-metric". We then give a sample of results about discrete (in fact, finite) spaces of this kind. These are inspired by (Helly) graph theory, and lead to a fixed point theorem

which generalizes several existing fixed point results from graph theory/digital topology. The theme here is that "discrete hyperconvex space" is not an oxymoron.

Section 5 presents the extension (= injectivity) result for our structures. In fact there are two versions: the unrestricted version, which generalizes the extension property of hyperconvex metric spaces, and the discrete version, which generalizes the corresponding Helly graph property.

The last main section (6) is concerned with the idea that the (usual) hyperconvex metric spaces are expressible as inverse limits of our discrete hyperconvex spaces. Our results here are incomplete, and involve a conjecture which we have been unable to prove.

2. Helly Graph Results Suggested by Hyperconvexity Theory

The purpose of this section is to illustrate the transfer of ideas from hyperconvex theory to Helly graph theory by showing that, in analogy with results in [16], we can develop good notions of *power graph* and *function graph* for Helly graphs. Nearly all of this section may be skipped by the reader who is not interested in graph theory.

By a *graph G* we mean a set $V(G)$ of *vertices* (or *points*), with a reflexive and symmetric relation $E(G)$ the *edges*, also known as a *tolerance graph* in [12]. For graphs G_1 and G_2, a mapping $f : G_1 \to G_2$ is a *graph homomorphism* if f maps vertices of G_1 to vertices of G_2, preserving the tolerance relation, i.e., $(f(x), f(y)) \in E(G_2)$ whenever $(x, y) \in E(G_1)$. By a *multifunction* $g : G_1 \to G_2$ we understand a function that assigns to a vertex x of G_1 a nonempty subset $g(x)$ in G_2. The multifunction g is said to be *strong* if $(x, y) \in E(G_1)$ implies that every vertex in $g(x)$ is adjacent to some vertex in $g(y)$, and vice versa. For the multifunction $f : G_1 \to G_2$, a *selection* of f is a graph homomorphism $\dot{f} : G_1 \to G_2$ such that $\dot{f}(x) \in f(x)$ for any vertex $x \in V(G_1)$. An (induced) subgraph G of a graph H is a *retract* of H if there exists a graph homomorphism $r : H \to G$ called *retraction* such that $r(x) = x$ for each vertex x of G. The *strong product* of graphs G_1 and G_2 is the graph $G_1 \times G_2$ which has the vertex set $V(G_1) \times V(G_2)$ the cartesian product of the vertex sets of G_1 and G_2, and the edge set $E(G_1 \times G_2)$ such that $((x_1, x_2), (y_1, y_2)) \in E(G_1 \times G_2)$ if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$.

For any connected graph G, the *r-ball*, for any $r \in \mathbb{N}$, centered at a vertex $x \in V(G)$, is the set $\overline{B}_G(x, r) = \{y \in V(G) \mid d_G(x, y) \leq r\}$ (where d_G is the usual graph distance). If G is finite, then G is said to be *Helly* if every subfamily F of the closed balls of G is intersection nonempty whenever any two elements of F are pairwise intersecting. If G is infinite, graph theorists usually use the term "strong Helly" for this property. It is well-known that the class of Helly graphs is closed under retracts and strong products (moreover, every Helly graph is a retract of a finite strong product of m-paths I_m) [9, 14]. We shall state following results for finite graphs, though the restriction to the finite case can easily be removed.

In [22] we introduced the "neighbourhood convexity" of a Helly graph. The neighbourhood convex sets of the Helly graph G are the sets of the form

$$\bigcap_i \overline{B}_G(x_i, r_i).$$

Thus they are exactly the "admissible sets" of hyperconvex theory. A useful result about neighbourhood convexity is that every strong multifunction $f : H \to G$ which maps each vertex of any finite graph H to a nonempty neighbourhood convex set of the Helly graph G has a selection [22].

For any Helly graph G, denote by $\mathcal{D}(G)$ the set of all nonempty neighbourhood convex sets of G. Let $\mathcal{P}_s(G)$ be the graph with vertex set $V(\mathcal{P}_s(G)) = \mathcal{D}(G)$, and edge set $E(\mathcal{P}_s(G))$, where $(A, B) \in E(\mathcal{P}_s(G))$ if and only if

$$\forall x \in A, \exists y \in B, (x, y) \in E(G) \ \& \ \forall y \in B, \exists x \in A, (x, y) \in E(G).$$

Notice that the "strong" relation $E(\mathcal{P}_s(G))$ is the conjunction of the two "weak power" relations in Brink [2]. Also it is easy to see that the graph distance on $\mathcal{P}_s(G)$ is the Hausdorff metric derived from d_G [21].

Proposition 2.1. $\mathcal{P}_s(G)$ *is a Helly graph if G is Helly.*

Proof. Let $H = \mathcal{P}_s(G)$. For any subset A of $V(G)$, denote $N_G(A)$ to be the subset $\{x \in V(G) \mid \exists y \in A, (x, y) \in E(G)\}$. Since $E(G)$ is reflexive, for any $A \in \mathcal{D}(G)$, it is clear that A is a ball, or the intersection of some balls of G; hence A can be represented by $\bigcap_\alpha \overline{B}_G(x_\alpha, r_\alpha)$. We show that $(N_G)^n(A) \in \mathcal{D}(G)$, and

it is enough to consider the case $n = 1$ only: clearly $N_G(A) \subseteq \bigcap_\alpha N_G(\overline{B}_G(x_\alpha, r_\alpha))$. If $s \in \bigcap_\alpha N_G(\overline{B}_G(x_\alpha, r_\alpha))$, then $\overline{B}_G(s, 1) \cap \overline{B}_G(x_\alpha, r_\alpha) \neq \emptyset$ for all α. By the Helly property of G, there exists a vertex, say z, such that $z \in \bigcap_\alpha \overline{B}_G(x_\alpha, r_\alpha)$ and $(z, s) \in E(G)$. Therefore we have $s \in N_G(A)$.

Now let $\mathcal{B} = \{\overline{B}_H(A_i, r_i) \mid i \in I\}$ be a collection of balls in H such that $\overline{B}_H(A_i, r_i) \cap \overline{B}_H(A_j, r_j) \neq \emptyset$ for any $i, j \in I$. We claim that $\bigcap_{i \in I} \overline{B}_H(A_i, r_i) \neq \emptyset$: it is clear that $(N_G)^{r_i}(A_i)$ is the maximal element of $\overline{B}_H(A_i, r_i)$ (under inclusion in $V(G)$). For any $(N_G)^{r_i}(A_i)$ and $(N_G)^{r_j}(A_j)$, since $\overline{B}_H(A_i, r_i) \cap \overline{B}_H(A_j, r_j) \neq \emptyset$, there must exist $\Delta \in \mathcal{D}(G)$ such that $\Delta \subseteq (N_G)^{r_i}(A_i) \cap (N_G)^{r_j}(A_j)$; hence $(N_G)^{r_i}(A_i) \cap (N_G)^{r_j}(A_j) \neq \emptyset$. By the Helly property of G, we have $\bigcap_{i \in I}(N_G)^{r_i}(A_i) \neq \emptyset$.

We still have to show that $\bigcap_{i \in I}(N_G)^{r_i}(A_i) \in \bigcap_{i \in I} \overline{B}_H(A_i, r_i)$. It is clear that, for any vertex $z \in \bigcap_{i \in I}(N_G)^{r_i}(A_i)$, z is r_j adjacent to some vertex y_j of A_j for any $j \in I$. We show the converse: for any $j \in I$ and any vertex $y \in A_j$, we consider the collection $L = \{\overline{B}_G(y, r_j)\} \cup \{(N_G)^{r_i}(A_i) \mid i \in I\}$. Clearly $\overline{B}_G(y, r_j) \subseteq (N_G)^{r_j}(A_j)$. Furthermore, we have $\overline{B}_G(y, r_j) \cap (N_G)^{r_i}(A_i) \neq \emptyset$, since $\overline{B}_H(A_j, r_j) \cap \overline{B}_H(A_i, r_i) \neq \emptyset$ for any $i \in I, j \neq i$. Thus $L \subseteq \mathcal{D}(G)$ is pairwise intersecting. By the Helly property of G, we have $\overline{B}_G(y, r_j) \cap \bigcap_{i \in I}(N_G)^{r_i}(A_i) \neq \emptyset$, which implies that y is r_j adjacent to some vertex $z \in \bigcap_{i \in I}(N_G)^{r_i}(A_i)$. $\qquad\square$

For any pair of finite graphs H and G, we define a "function graph" G^H such that, $V(G^H)$ is the set of all graph homomorphisms from H to G, and $(f, g) \in E(G^H)$ if and only if $(f(x), g(x)) \in E(G)$ for all $x \in V(H)$. Then

Proposition 2.2. G^H *is a Helly graph if G is Helly.*

Remark 2.3. It suffices to prove this for the case that $G = I_m$, due to the fact that Helly graphs are exactly the retracts of (finite) strong products of m-paths I_m. In more detail:

(1) If Proposition 2.2 holds for $G = K_1, G = K_2$, it holds for $K_1 \times K_2$, since $(K_1 \times K_2)^H \cong K_1^H \times K_2^H$;
(2) If Proposition 2.2 holds for $G = K$, it holds for any retract L of K, since L^H is a retract of K^H.

Proof of Proposition 2.2. Consider any collection of balls in G^H, $\mathcal{B} = \{\overline{B}_{G^H}(f_i, r_i) \mid i \in I\}$, such that $\overline{B}_{G^H}(f_i, r_i) \cap \overline{B}_{G^H}(f_j, r_j) \neq \emptyset$ for any $i, j \in I$. We have to show that $\bigcap \mathcal{B} = \bigcap_{i \in I} \overline{B}_{G^H}(f_i, r_i) \neq \emptyset$.

For any $g \in V(G^H), r \in \mathbb{N}$, and $x \in V(H)$, let $\Phi(g, r, x) = \bigcup \{\phi(x) \mid \phi \in \overline{B}_{G^H}(g, r)\}$. We claim that $\Phi(g, r, x) = \overline{B}_G(g(x), r)$. It is clear that $\Phi(g, r, x) \subseteq \overline{B}_G(g(x), r)$, since $d_G(g(x), \phi(x)) \leq r$ for any $\phi \in \overline{B}_{G^H}(g, r)$. For the reverse inclusion, we shall for convenience make the assumption that $G = I_m$. Let $y \in \overline{B}_G(g(x), r)$; without loss of generality, we can assume that $y = g(x) - k, 0 \leq k \leq r$. (Here we have assumed that the vertices of I_m are denoted $0, 1, \ldots, m$.) Define the "shift" mapping $s_k : I_m \to I_m$ (that is, $G \to G$) by:

$$s_k(z) = z \dot{-} k,$$

for each $z \in V(G)$. Then s_k is a graph homomorphism, $s_k \circ g$ is a graph homomorphism such that $d_{G^H}(g, s_k \circ g) \leq r$, and $s_k \circ g(x) = y$. Thus $y \in \Phi(g, r, x)$.

We define a multifunction $h : H \to G$ by

$$h(x) = \bigcap_i \Phi(f_i, r_i, x),$$

for all $x \in V(H)$. In the following we would like to show that h is a strong multifunction which maps each vertex of H to a neighbourhood convex set of G. (Notice that, invoking the assumption that $G = I_m$, a neighbourhood convex set is just a subinterval of I_m. Moreover, the selection property is trivial, since we can for example always select the leftmost vertex of an interval. These observations do not, however, seem to lead to any significant simplification of this part of the proof.) For then, by the selection property of G, we can conclude that h has a selection $f : H \to G$, and clearly, for any such selection, $f \in \bigcap \mathcal{B}$: we have already shown that, for any $x \in V(H)$, $h(x)$ is a neighbourhood convex set of G. (*h is well-defined:*) It is easy to check that, for any $x \in V(H)$ and any $i, j \in I$, we have $\Phi(f_i, r_i, x) \cap \Phi(f_j, r_j, x) \neq \emptyset$ since $\overline{B}_{G^H}(f_i, r_i) \cap \overline{B}_{G^H}(f_j, r_j) \neq \emptyset$. Thus, by the Helly property of G, we have $h(x) = \bigcap_{i \in I} \Phi(f_i, r_i, x) \neq \emptyset$. (*h is a strong multifunction:*) Let x, y be any vertices of H such that $(x, y) \in E(H)$. We claim that $h(x)$ and $h(y)$ are strongly adjacent: let z be any vertex of $h(x)$. It is clear that we have $z \in \Phi(f_i, r_i, x)$ for all $i \in I$. Furthermore, for any $j \in I$, we have $d_G(z, f_j(y)) \leq d_G(z, f_j(x)) + d_G(f_j(x), f_j(y)) \leq r_j + 1$. Thus, we

have $\overline{B}_G(z, 1) \cap \Phi(f_j, r_j, y) \neq \emptyset$. By the Helly property of G, we have $\overline{B}_G(z, 1) \cap \bigcap_{i \in I} \Phi(f_i, r_i, y) \neq \emptyset$. Hence there exists $w \in h(y)$ such that $(z, w) \in E(G)$. Similarly, we can prove that, for any $w \in h(y)$, there exists $z \in h(x)$ such that $(z, w) \in E(G)$. $\qquad\square$

3. Semi-metrics, Partial Metrics

The distance functions we shall work with are weaker than metrics, in that the triangle inequality is relaxed, or even dispensed with altogether. For most of our results, including those in the next section, we require distance functions satisfying just the axioms

(1) $d(x, x) = 0, \forall x \in X$,
(2) $d(x, y) = d(y, x)(\geq 0), \forall x, y \in X$.

These are the *semi-pseudo metrics* of Čech [3]. We shall generally abbreviate this to *smetric*. A *weighted* smetric space is a triple (X, d, w) where d is an smetric on X, and $w : X \to \mathbb{R}^{+0}$ is an assignment of non-negative "weights" to the points of X, satisfying

(3) $d(x, z) \leq d(x, y) + w(y) + d(y, z), \forall x, y, z \in X$.

A suitable intuition for these structures is that the elements are "approximate" elements of some space (say, intervals in $\mathbb{R}$ or, more generally, balls in some metric space), the smetric distance is the inf of the distances between points of the two balls, and the weight is the diameter. One may also think of weighted graphs, where the smetric distance is the sum of the weights along a path of least weight between two vertices. (Both of these types of example will be discussed in a more precise fashion a little later.)

The requirement that a weighting function should exist is not too restrictive, at least in bounded spaces; in fact, we have:

Proposition 3.1. *Every bounded smetric space is weightable.*

Proof. Let (X, d) be an smetric space. The (relaxed) triangle axiom requires that the weight of an element $x \in X$ satisfies:

$$w(x) \geq d(y, z) - d(x, y) - d(x, z),$$

for all $y, z \in X$. Hence we obtain a weighting function by putting

$$w(x) = \sup\{d(y, z) - d(x, y) - d(x, z) \mid y, z \in X\},$$

for any $x \in X$. $\qquad\square$

Weighted smetrics may be compared with the partial metrics (pmetrics) of Matthews [8]. Incorporating modifications due to O'Neill [10] and Heckmann [6], we have that a *pmetric* on a set X is a map $p : X \times X \to \mathbb{R}$ satisfying:

(p1) $p(x,y) = p(y,x)$,
(p2) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p3) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$,

for all $x, y, z \in X$. These axioms are due to Matthews; the change made by O'Neill was to allow the distance function to take negative as well as non-negative values. Following Heckmann, we have omitted the axiom of "small self-distances" (so that what we have defined is actually a *weak* pmetric). Given a pmetric p on X, O'Neill defines the *dual* pmetric by:

$$p^*(x,y) = p(x,y) - p(x,x) - p(y,y).$$

An elementary property of the dual is that the sum $p + p^*$ is a metric.

We are particularly interested in pmetrics derived from weighted graphs. Let $w : V(G) \to \mathbb{R}^+$ be a vertex weighting on the graph G. Then the *length* of a path $\pi = \langle v_0, \ldots, v_n \rangle \subseteq V(G)$, $\mathrm{lgth}(\pi)$, is the sum

$$\sum_{i=0}^{n} w(v_i).$$

The distance function p is defined by $p(x,y) = \inf\{\mathrm{lgth}(\pi) \mid \pi = \langle v_0, \ldots, v_n \rangle \subseteq V(G), x = v_0, y = v_n\}$. It is easy to check that p is a pmetric on $V(G)$. Consider the dual pmetric, p^*. Since $p(x,x) = w(x)$ (in which case $p^*(x,x) = -w(x)$), p^* is in effect calculated by deducting the weights of the end points from the path length given above. Notice that $p^*(x,y)$ is negative if and only if $x = y$, and $p^*(x,y) \le 0$ if and only if (x,y) is an edge of G. In general, it is this "dual" pmetric p^* that we shall take as the pmetric associated with a weighted graph G. The pmetric p^* obtained by assigning weight 1 to each vertex is related to the usual "distance" d_G defined on G by:

$$p^*(x,y) = d_G(x,y) - 1 = \frac{1}{2}(p + p^*)(x,y) - 1.$$

Finally, by "forgetting" the negative self-distances (that is, by setting each $p^*(x, x)$ to 0), we obtain a weighted smetric s; thus we take s as the smetric associated with the weighted graph G. A pmetric or smetric associated with a graph G is called *uniform* if the weight function of G itself is constant.

It would be possible to develop the results below in terms of (O'Neill) pmetrics. The negative distances, however, are unfamiliar, and cause some technical complications. Hence we have preferred to work with the very rudimentary structure of smetric spaces.

Given a pmetric or smetric d on a set X, the *induced graph* has X as vertex set, and (x, y) as an edge if and only if $d(x, y) \leq 0$.

4. Hyperconvex Semi-metric Spaces

Given an smetric space (X, d), $x \in X$, and $r \geq 0$, the *closed ball* of $x \in X$ with radius r, $\overline{B}(x, r)$, is defined in the usual way. The basic definitions concerning hyperconvexity extend straightforwardly to smetric spaces:

Definition 4.1. An smetric space (X, d) is called *hyperconvex* if for any indexed collection of closed balls in X, $\{\overline{B}(x_i, r_i)\}$, satisfying

$$(4.1) \qquad d(x_i, x_j) \leq r_i + r_j,$$

it follows that $\bigcap_i \overline{B}(x_i, r_i) \neq \emptyset$.

An smetric space (X, d) will be called *totally convex* if:

(C) For any two points $x, y \in X$, such that $d(x, y) \leq r_1 + r_2$ $(r_1 \geq 0, r_2 \geq 0)$, there exists a point $z \in X$ such that $d(x, z) \leq r_1$ and $d(y, z) \leq r_2$.

Recall that a family F of subsets of X is said to satisfy the (2-)*Helly property* if for every subfamily F' of F such that any two elements of F' are intersection nonempty, F' is intersection nonempty.

Clearly, an smetric space (X, d) is totally convex if and only if the closed balls $\overline{B}(x_i, r_i), \overline{B}(x_j, r_j)$ with Property 4.1 are intersection nonempty. Hence it is easy to check that Definition 4.1 is equivalent to total convexity plus the Helly property for closed balls.

Let G be a connected weighted graph, and s the associated smetric defined in Section 3. Then it is easy to verify that $(V(G), s)$ is totally convex. Notice that total convexity fails, in general, if one tries to use the metric $p + p^*$ (or d_G) in place of s. (If the distance function d that we are working with is a metric, then the inequalities $d(x, z) \le r_1$ and $d(y, z) \le r_2$ are actually equalities in the case $d(x, y) = r_1 + r_2$; this forces us to have a non-denumerable infinity of points between any two distinct points.) A further evident fact is that, if the weighting w is uniform, the set of (closed) balls with respect to s coincides with the set of balls with respect to the standard graph distance d_G except that some d_G-balls of radius 0 (i.e. singletons) may fail to be s-balls. We thus have:

Proposition 4.2. *A finite graph G is Helly if and only if, giving each vertex the (constant) weight k, $(V(G), s)$ is a hyperconvex smetric space.*

An *admissible* subset of an smetric space (X, d) is a set of the form

$$\bigcap_i \overline{B}(x_i, r_i).$$

Proposition 4.3. *The admissible subsets in any finite hyperconvex smetric space are connected (in the induced graph).*

Proof. Let (X, d) be an smetric space, and $S \subseteq X$ an admissible set of X. Suppose S is not connected. Let $x, y \in S$ be points lying in distinct components of S, such that the distance $d(x, y)$ is as small as possible. Then $d(x, y) = r > 0$. By the property of total convexity, $\exists z \in X, d(x, z) \le \frac{r}{2}$ and $d(z, y) \le \frac{r}{2}$. Then the (admissible) subsets $\overline{B}(x, \frac{r}{2}), \overline{B}(y, \frac{r}{2}), S$ intersect pairwise. However, there is no point common to all of them. Contradiction. $\qquad\square$

Recall that a *graph convexity* [4, 5] on a finite connected graph G is a collection $\mathcal{C}$ of subsets of $V(G)$, called the *convex* sets, such that

(c1) the set $V(G)$ is convex,
(c2) the intersection of convex sets is convex,
(c3) any convex set induces a connected subgraph of graph G.

It follows at once that the admissible sets of a finite hyperconvex smetric space constitute a graph convexity. Thus Proposition 4.3 generalizes Theorem 3.1 of [22] (where we showed that the closed balls of a Helly graph generate a graph convexity). More properties of Helly graphs can in fact be generalized to hyperconvex smetrics. We provide a few examples here, leaving a more systematic investigation for another occasion.

As morphisms of smetric spaces we take the *nonexpansive* maps, that is, maps $f : (X, d) \to (X', d')$ such that

$$d'(f(x), f(y)) \leq d(x, y),$$

for all $x, y \in X$. Then we have

Proposition 4.4. *Any retract of a hyperconvex smetric space is hyperconvex.*

In case the smetric spaces in question are graphs with their standard metrics or smetrics (d_G or s), these morphisms reduce to the usual graph homomorphisms. Moreover, the following recursive definition of "contractible" for smetrics provides a straightforward generalization of contractible (or dismantlable) graphs:

Definition 4.5. A finite smetric space (X, d) is *contractible* if X is either a singleton, or else has distinct points $x, y \in X$ such that
   (1) for any $r \geq 0, \overline{B}(x, r) \subseteq \overline{B}(y, r)$ (when this holds the point x is said to be *dominated* by y);
   (2) $(X \backslash \{x\}, d)$ is contractible.

Notice that every contraction mapping (that is, a function which maps some point x to a point y which dominates x, and leaves all other points fixed) is a retraction, but the converse is not true even for retractions which have at most one non-fixed point. For example, let $X = \{a, b\}$ be the two-point metric space with $d(a, b) = 1$. Then the map $f : a \mapsto a, b \mapsto a$ is a retraction, but it is not a contraction since $\overline{B}(b, \frac{1}{2}) \not\subseteq \overline{B}(a, \frac{1}{2})$. In fact, a necessary condition for x to be dominated by y is that $d(x, y) = 0$. There are many non-trivial contractible smetric spaces (including of course all the contractible graphs), but the only contractible finite metric space is the singleton.

**Theorem 4.6.** *Every finite hyperconvex smetric space is contractible.*

Proof. Let (X, d) be finite hyperconvex. We have to find a dominated vertex. Let Δ be the diameter of X (that is, $\sup\{d(x, y) \mid x, y \in X\}$, and let a, b be points such that $d(a, b) = \Delta$. We may assume that $\Delta > 0$ (since X is obviously contractible if all distances are zero). Let r be the greatest number such that $r < \Delta$ and there exists a point z such that $d(a, z) = r$. Thus, every point in $X \backslash \overline{B}(a, r)$ is at distance Δ from a. Let u be any point in $X \backslash \overline{B}(a, r)$. Given any ε with $0 < \varepsilon < \Delta$, we have (by total convexity) a point v such that $d(a, v) \leq \Delta - \varepsilon, d(v, u) \leq \varepsilon$. Then $v \in \overline{B}(a, r)$. Since this holds for arbitrarily small $\varepsilon > 0$, there must (by finiteness) be a point w in $\overline{B}(a, r)$ such that $d(u, w) = 0$. Fix any point u_0 in $X \backslash \overline{B}(a, r)$ (for example, let $u_0 = b$). For each point $x \in X$, let B_x be the ball $\overline{B}(x, d(x, u_0))$. By the construction, the family $\{\overline{B}(a, r)\} \cup \{B_x \mid x \in X\}$ is pairwise intersecting; let y be a point common to all the balls. Then u_0 is dominated by y. For clause (2) (of Definition 4.5), use Proposition 4.4 and the remark following Definition 4.5. $\square$

The "fixed point theorem" we shall consider is actually an *almost* fixed point property of nonexpansive multifunctions. Given a map $f : X \to \mathcal{P}(X)$ ((X, d) smetric), we say that x is an *almost fixed point* of f if there exists $y \in f(x)$ such that $d(x, y) = 0$. Provided that X is bounded, the Hausdorff distance on $\mathcal{P}(X)$ (defined in exactly the usual way) is trivially an smetric, say $d_{\mathcal{H}}$, and this is the one we assume when we ask that f be nonexpansive. A bounded smetric space X has the *nonexpansive almost fixed point property* (nonexpansive AFPP, for short) if every nonexpansive multifunction on X has an almost fixed point.

Theorem 4.7. *Every contractible smetric space has the nonexpansive AFPP.*

Proof. Let (X, d) be a contractible smetric space. Since X is contractible, there exist $x, y \in X$ such that $\overline{B}(x, r) \subseteq \overline{B}(y, r)$ for any $r \geq 0$ (Condition 1), and the subspace $(X \backslash \{x\}, d)$ is contractible (Condition 2).

Let $f : X \to \mathcal{P}(X)$ be any nonexpansive multifunction. We prove by induction on $\#X$ that f has an almost fixed point. *Initial step:* It is clear that X has the nonexpansive AFPP if X is a singleton.

$$f(y)$$
$$w$$

$$z \qquad y \qquad x$$
$$g(z)$$
$$f(z)$$
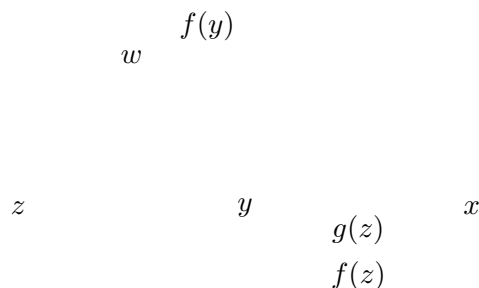
FIGURE 1. $x, y, z, w, g(z), f(z)$ and $f(y)$

Inductive step: We assume that X has the nonexpansive AFPP when $\#X \le m \in \mathbb{N}, m \ge 1$, and we claim that X has the nonexpansive AFPP when $\#X = m + 1$. Define the multifunction $g :$ $X\backslash\{x\} \to X\backslash\{x\}$ by

$$g(u) = \begin{cases} f(u), & x \notin f(u) \\ (f(u)\backslash\{x\}) \cup \{y\}, & x \in f(u). \end{cases}$$

We claim that g is nonexpansive. Indeed, let $a, b \in X\backslash\{x\}$ with $d(a, b) = k$. Since f is nonexpansive, $d_{\mathcal{H}}(f(a), f(b)) \le k$. Now $g(a), g(b)$ are obtained from $f(a), f(b)$ by replacing any occurrences of x by y. Since $\overline{B}(x, r) \subseteq \overline{B}(y, r)$ for any $r \ge 0$, it is clear that any true statement of the form $d_{\mathcal{H}}(A, B) \le m$ remains true after the replacement of any occurrences of x by y; more precisely, $d_{\mathcal{H}}(A, B) \le m \Rightarrow d_{\mathcal{H}}((A\backslash\{x\}) \cup \{y\}, (B\backslash\{x\}) \cup \{y\}) \le m$. In particular, $d_{\mathcal{H}}(f(a), f(b)) \le k \Rightarrow d_{\mathcal{H}}(g(a), g(b)) \le k$. Thus g is nonexpansive.

Therefore by our inductive assumption and Condition 2, there exists a point $z \in X\backslash\{x\}$ such that z is an almost fixed point of g, i.e. $d(z, z') = 0$ for some $z' \in g(z)$. Then (since $f(z)$ differs from $g(z)$ by, at most, having x in place of y) z must be an almost fixed point also of f, except in the following case: y is the only point of $g(z)$ such that $d(y, z) = 0$, $x \in f(z)$, but $y \notin f(z)$, see Figure 1. Since $d(y, z) = 0$, we have $d_{\mathcal{H}}(f(y), f(z)) = 0$. So $f(y)$ contains a point, say w, such that $d(w, x) = 0$, hence $d(w, y) = 0$ ($\because \overline{B}(x, 0) \subseteq \overline{B}(y, 0)$). Thus y is an almost fixed point of f. $\qquad\square$

This result generalizes various graph-theoretic almost fixed point theorems in the literature, such as Poston [12], Rosenfeld [15], and our own "AFPP for strong multifunctions" in [21]. For example, in relation to [15]:

(1) single-valued functions are replaced by multifunctions;
(2) the n-dimensional grid is replaced by an arbitrary contractible graph;
(3) graphs are replaced by smetric spaces.

5. Extension Property and Injectivity

We seek to characterize those smetric spaces M such that, given an smetric space X and isometric extension Y of X, any morphism $f : X \to M$ can be extended to $f' : Y \to M$. For a satisfactory result of this kind we need to work with weighted smetrics. Moreover, we need to place a bound on the weights involved.

Definition 5.1. A weighted smetric space (X, d, w) is *m-bounded* $(m \geq 0)$ if $w(x) \leq m$ for all $x \in X$.

For useful examples of m-bounded smetric spaces, we may consider the ball-sets of hyperconvex spaces. Given a metric space (X, d), $B(X)$ shall denote the set of closed balls of X, taken with the "inf" smetric d_s:

$$d_s(B_1, B_2) = \inf\{d(x, y) \mid x \in B_1, y \in B_2\}.$$

The *diameter* of a bounded set S in X is $\sup\{d(x, y) \mid x, y \in S\}$.

Proposition 5.2.

(1) *Let X be totally convex. Then, taking the weight of any ball to be its diameter, $B(X)$ is a weighted smetric space.*
(2) *With (X, d) hyperconvex, let $B_m(X)$ $(m \geq 0)$ be the set of closed balls of X with diameter $\leq m$. Then $B_m(X)$ is hyperconvex (and m-bounded).*

Proof. We show (2) only: convexity of $B_m(X)$ is evident. For the Helly property, let $\overline{B}(B_i, \rho_i)_{i \in I}$ be a collection of pairwise intersecting balls of $B_m(X)$, where $B_i = \overline{B}(x_i, r_i)$. Then the balls $\overline{B}(x_i, r_i + \rho_i + \frac{m}{2})$ are pairwise intersecting in X, hence have a common point x. Clearly, $d_s(\overline{B}(x, \frac{m}{2}), B_i) \leq \rho_i$, for all $i \in I$. $\square$

With regard to the bound m in Proposition 5.2, it is worth mentioning that the unbounded case (with $B(X)$ in place of $B_m(X)$) is of little interest. For, if X is any bounded metric space, then $B(X)$ is trivially hyperconvex. (Any set of balls of $B(X)$ has a nonempty intersection; indeed, any large enough ball of X belongs to every ball of $B(X)$.) On the other hand, $B(X)$ fails to be hyperconvex when X is unbounded. For example, taking X as $\mathbb{R}$, the balls $\overline{B}(\overline{B}(k, 1), 1), k \in \mathbb{N}$, are pairwise intersecting, but have no common element.

An observation relevant to the extension theorem is that, when X is totally convex, $B_m(X)$ has a convexity property stronger than the ordinary smetric convexity (C). Namely, we have

(C_m) If $d(x, y) \leq r_1 + r_2 + m$, then there exists u (with weight $\leq m$) such that $d(x, u) \leq r_1$ and $d(u, y) \leq r_2$.

Theorem 5.3. *Let (M, d, w) be m-bounded hyperconvex satisfying (C_m). Let X, Y be m-bounded smetric spaces, Y an isometric extension of X, and $f : X \to M$ nonexpansive. Then f has a (nonexpansive) extension $f' : Y \to M$.*

Proof. We suppose in the first instance that Y is an one-element extension $X \cup \{y\}$; the result will then follow by a standard maximality argument. For each pair x_1, x_2 of elements of X, we have

$$d(x_1, x_2) \leq d(x_1, y) + w(y) + d(y, x_2).$$

Hence $d_M(f(x_1), f(x_2)) \leq d(x_1, y) + m + d(y, x_2)$. It follows by the (C_m) property that the balls $\overline{B}(f(x_1), d(x_1, y)), \overline{B}(f(x_2), d(x_2, y))$ intersect. Hence the extension of f to $X \cup \{y\}$ may be achieved by mapping y to any common element of the balls $\overline{B}(f(x), d(x, y)), x \in X$. Returning to the general case $X \subseteq Y$, let us call an extension of f to a domain X' with $X \subseteq X' \subseteq Y$ a *partial* extension. Clearly, the union of any chain of partial extensions is a partial extension. By Zorn's lemma there is a maximal partial extension, and by the preceding argument this must in fact be an extension to the whole of Y. □

Thus we have that, for the class of m-bounded spaces, the hyperconvex spaces satisfying (C_m) are injective.

For a discrete (and constructive) version of these ideas, we may proceed as follows. Restrict attention to countable spaces in which all the distances and weights are natural numbers (or more generally, multiples of a constant $\varepsilon > 0$); for convenience, we call these spaces *discrete*. The discrete version (C_m^D) of (C_m), is the modification of (C_m) in which the variables r_1, r_2 are restricted to have natural number values. Then we can prove the discrete version of Theorem 5.3, in which each notion is replaced by (or restricted to) its discrete counterpart, by the same proof as before. The significance of this is twofold:

(1) We do not actually need Zorn's lemma, since the required extension can be accomplished by a sequence of one-element extensions (as in Quilliot [13] for the graph case).

(2) We now have an abundance of countable, and indeed finite, injective spaces. For example, M may be taken as the space of sub-intervals of length m of a fixed (bounded or unbounded) interval (k, l) of $\mathbb{Z}$ $(l - k > m + 1)$, with obvious weight and distance functions.

Notice that these injectives are not, in general, representable as (weighted) Helly graphs, whose injectivity properties are already well understood from the work of Quilliot and others. They may, perhaps, be regarded as only a fairly modest generalization of the Helly graphs. The generalization involved would be more noteworthy, were we to admit negative distances (where, in an interval model, a negative distance measures the degree of overlap of two intervals). However, we shall not here investigate the version with negative distances.

6. Approximation of Hyperconvex Metric Spaces

It was shown in [17] that every compact metric space can be represented as an inverse limit of finite smetric spaces. The construction, briefly, is as follows. Suppose that (M, d) is compact metric. Let $\Sigma = C_1, C_2, \ldots$ be a sequence of finite covers of M by closed balls such that each C_{k+1} refines C_k, and $\text{mesh}(C_n) \to 0$ as $n \to \infty$. Each C_n is considered as an smetric space in the usual way, that is by putting

$$d_s(B, B') = \inf\{d(x, x') \mid x \in B, x' \in B'\}.$$

For each k, we define $f_k : C_{k+1} \to C_k$ so that $B \subseteq f_k(B)$ for all $B \in C_{k+1}$. Then f_k is nonexpansive, and we may consider the inverse limit $\varprojlim(C_k, f_k) = L$. It is easy to see that L is a pseudo-metric space which is isometric with M.

This construction is extremely simple, but notice that it is essential that we use the smetric rather than, say, the Hausdorff metric in each C_k. With the Hausdorff metric we do not get nonexpansive bonding maps. (To see this, it suffices to consider the case that M is the unit interval I, and C_k is the cover of I by 2^k subintervals of length 2^{-k}.)

Approximation by discrete structures in this style is not limited to the compact case, although that is the case that has been most studied. Indications for the locally compact metrizable case have been given in [23], and for Polish spaces in [18]. In the present context the interesting case to consider is the (bounded but non-compact) hyperconvex metric space I^∞. The covering C_k of I by 2^k subintervals of length 2^{-k} gives us a covering C_k^∞ of I^∞ by elementary cubes of size 2^{-k} (actually balls of I^∞ of diameter 2^{-k}). The intersection graph of C_k^∞, say G_k, is the product of countably many copies of the intersection graph of C_k, and is therefore (strongly) Helly. Give each vertex of G_k the weight 2^{-k}. Each uniformly weighted graph G_k may thus be considered as a hyperconvex smetric space (Proposition 4.2), and, under the same construction as before, the inverse limit of these smetric spaces is a pseudo-metric equivalent of I^∞. In brief, we have:

Proposition 6.1. *The hyperconvex metric space I^∞ is approximable by discrete hyperconvex smetric spaces (specifically, by uniformly weighted Helly graphs).*

This kind of approximation of continuous structures by discrete structures has been applied, in related cases, to transfer fixed point results from one type of structure to the other: see [20]. Fixed point (or fixed clique) properties of infinite graphs have been considered to a limited extent (see [11]). But the finite case is far better understood, and we should prefer to set about developing this application by comparing compact hyperconvex metric spaces with finite smetric spaces. Every compact hyperconvex metric space is, up to a constant multiple of the distance, a (nonexpansive) retract of I^∞, and the finite Helly graphs are exactly those finite graphs which are retracts of some G_k.

Such observations suggest that the following is plausible:

Conjecture 6.2. Every compact hyperconvex metric space is approximable by finite hyperconvex smetric spaces (indeed, by uniformly weighted Helly graphs).

But so far this remains unproven.

7. Conclusion

We have tried to show that the theories of Helly graphs and of hyperconvex metric spaces can be unified by means of the concept of the hyperconvex semi-(pseudo) metric space. Using this idea, many more individual results of the kind illustrated in this paper can be developed. Some more systematic topics needing investigation are:

(1) Whether it would be advantageous to use partial metrics throughout the development;
(2) A more complete development of the inverse limit approach (Section 6), including if possible settling the conjecture mentioned there.

In a wider perspective, the work is intended as a case study in the geometric aspect of smetrics (or pmetrics), for we think that these weak metrics are the appropriate ones for digital topology and geometry.

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