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THE UNIFORMITY INVARIANTS OF THE IDEAL OF NULL SETS IN SEPARABLE METRIC SPACES

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ABSTRACT. P. Komjath has asked the following question: If every set of reals of size $\aleph_1$ has measure zero does it follow that the union of $\aleph_1$ lines in the plane also has measure zero. In order to obtain a model establishing the negative result it is necessary to consider models where all sets of reals of size $\aleph_1$ have measure zero yet the analogous statement for the plane fails.

P. Komjath has asked the following question: If every set of reals of size $\aleph_1$ has measure zero does it follow that the union of $\aleph_1$ lines also has measure zero. The following observation, while simple in itself, significantly narrows the range of possibilities for the models of set theory one might hope to use to provide a negative answer to Komjath's question. To begin, the notion of a null set can be generalized to an arbitrary metric space.

Definition 1. If $\mathcal{X} = (X, d)$ is a metric space and $S \subseteq X$ then S will be said to be a *$\mathcal{X}$-null set* if for every $\epsilon > 0$ there is a sequence $\{(x_i, \epsilon_i)\}_{i=0}^{\infty} \subseteq X \times \mathbb{R}^+$ such that $\sum_{i=0}^{\infty} \epsilon_i < \epsilon$ and for every $s \in S$ there is some i such that $d(x_i, s) < \epsilon_i$.

Let $\mathbb{R}^n$ represent n-dimensional space with the Euclidean metric. Observe that if $X \subseteq \mathbb{R}^n$ then X being a $\mathbb{R}^n$-null set is the same as being a n-dimensional Lebesgue null set only in the case when $n = 1$.

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Proposition 1. *If every subset of the plane of size $\aleph_1$ is a $\mathbb{R}^2$-null set then the union of any $\aleph_1$ lines in the plane has Lebesgue measure zero.*

Proof. To begin, observe that it suffices to show that the union of any $\aleph_1$ lines segments spanning the unit square has Lebesgue measure zero. Except for those which pass through just a corner, thus clearly having null union, any such set of lines can decomposed in 6 sets depending on which pair of sides of the unit square they intersect. The argument is similar in all 6 cases, so assume that $\mathcal{L}$ is a set of line segments whose endpoints are on the top and bottom of the unit square and that $\epsilon > 0$. Let $L = \{(x,y) \in [0,1]^2 : (\exists L \in \mathcal{L})\{(x,0),(y,1)\} \subseteq L\}$ and choose points $\{(x_i,y_i)\}_{i=0}^{\infty} \subseteq [0,1]^2$ such that

$$(\forall (x,y) \in L)(\exists i)\sqrt{(x_i - x)^2 + (y_i - y)^2} < \epsilon_i$$

and $\sum_{i=0}^{\infty} \epsilon_i < \epsilon/2$. Let Q_i be the parallelogram whose corners are $(x_i - \epsilon_i, 0)$, $(x_i + \epsilon_i, 0)$, $(y_i - \epsilon_i, 1)$ and $(y_i + \epsilon_i, 1)$ and note that the planar Lebesgue measure of Q_i is $2\epsilon_i$. Moreover, for every $L \in \mathcal{L}$ there is some i such that $L \subseteq Q_i$. Therefore $\bigcup_{i=0}^{\infty} Q_i$ has planar Lebesgue measure no greater than ϵ and it covers the union of the line segments in $\mathcal{L}$. □

In light of Proposition 1, in order to obtain a negative answer to Komjath's question it would be necessary to find a model of set theory in which ever subset of $\mathbb{R}$ size $\aleph_1$ is a $\mathbb{R}$-null set but there is subset of $\mathbb{R}^2$ of size $\aleph_1$ which is not a $\mathbb{R}^2$-null set. The first step would be to find a model in which there are two separable metric spaces $\mathcal{X}$ and $\mathcal{Y}$ such that any subset of $\mathcal{X}$ of size $\aleph_1$ is $\mathcal{X}$-null but there is a subset of $\mathcal{Y}$ of size $\aleph_1$ which is not $\mathcal{Y}$-null. This is the point of the following definition and proposition.

Definition 2. If f and g are functions from $\mathbb{N}$ to $\mathbb{N}$ define $\text{non}(f,g)$ to be the least cardinal κ such that there exists $A \subseteq \prod_{i=0}^{\infty} f(i)$ of cardinality κ such that for each sequence of finite sets $\{B_i\}_{i=0}^{\infty}$ such that $B_i \in \prod_{j=0}^{i} [f(j)]^{g(i)}$ there is $a \in A$ such that $a \restriction m+1 \notin B(m)$ for all integers m. In order to avoid the clumsy notation $a \restriction m+1$, unless mentioned otherwise, the notation $a \restriction m$ will be used to denote the restriction of a function to the initial segment $m+1$.

Definition 3. If f and g are functions from $\mathbb{N}$ to $\mathbb{N}$ define a two metric spaces $\mathcal{X}(f,g) = (\prod_{i=0}^{\infty} f(i), d_f)$ and $\mathcal{Y}(f,g) = (\prod_{i=0}^{\infty} f(i), d_g)$ where

$$d_g(h, h') = \sum_{i=0}^{\infty} \frac{\delta(h(i), h'(i))}{g(i)}$$

$$d_f(h, h') = \sum_{i=0}^{\infty} \frac{\delta(h(i), h'(i))}{f(i)}$$

and where $\delta(n,m) = 1$ if $n = m$ and $\delta(n,m) = 0$ otherwise.

Observe that $\mathcal{X}(f,g)$ and $\mathcal{Y}(f,g)$ are homeomorphic to the Cantor space provided that $\sum_{i=0}^{\infty} 1/f(i) < \infty$ and $\sum_{i=0}^{\infty} 1/g(i) < \infty$ respectively.

Proposition 2. *Let f and g be functions from $\mathbb{N}$ to $\mathbb{N}$ such that*

$$\sum_{i=0}^{\infty} \frac{g(i)}{f(i)} < \infty \tag{0.1}$$

$$\frac{1}{f(i)} > \sum_{m=i+1}^{\infty} \frac{1}{f(m)} \tag{0.2}$$

Then any subset of $\mathcal{X}(f,g)$ of cardinality less than non(f,g) *is $\mathcal{X}(f,g)$-null.*

Proof. Let $A \subseteq \prod_{i=0}^{\infty} f(i)$ be of size less than non(f,g). From Definition 3 it follows that there are $B_i \in \left[\prod_{j=0}^{i} f(j)\right]^{g(i)}$ such that for each $a \in A$ there is some i such that $a \restriction i \in B_i$. Since it may as well be assumed that A is closed under finite modifications it follows that it may also be assumed that for each $a \in A$ there are infinitely many i such that $a \restriction i \in B_i$. Given $\epsilon > 0$ use Hypothesis 0.1 to find k such that $\sum_{i=k}^{\infty} \frac{g(i)}{f(i)} < \epsilon$. Let $\{(x_n, \epsilon_n)\}_{n=0}^{\infty} \subseteq \prod_{i=0}^{\infty} a_i \times \mathbb{R}^+$ be a sequence chosen so that

$$(\forall m \geq k)(\forall b \in B_m)(\exists j) \sum_{i=k}^{m-1} g(i) \leq j < \sum_{i=k}^{m} g(i) \text{ and } b = x_j \restriction m$$

(where $\sum_{i=k}^{k-1} g(i)$ is defined to be 0) and $\epsilon_j = \frac{1}{f(m)}$. Then

$$\sum_{i=0}^{\infty} \epsilon_i = \sum_{j=k}^{\infty} \frac{g(j)}{f(j)} < \epsilon$$

by the choice of k. Moreover, for each $a \in A$ there is some $m > k$ such that $a \restriction m \in B_m$ and, hence, $a \restriction m = x_j \restriction m$ for some j such that $\sum_{i=k}^{m-1} g(i) \leq j < \sum_{i=k}^{m} g(i)$. It follows that $d_f(x_j, a) \leq \sum_{i=m+1}^{\infty} 1/f(i) < 1/f(m) = \epsilon_j$ by Hypothesis 0.2. Hence A is a $\mathcal{X}(f,g)$-null set. □

Proposition 3. *If f and g are monotonic functions from $\mathbb{N}$ to $\mathbb{N}$ then there is a subset of $\mathcal{Y}(f,g)$ of cardinality* $\text{non}(f,g)$ *which is not $\mathcal{Y}(f,g)$-null.*

Proof. Let $A \subseteq \prod_{i=0}^{\infty} f(i)$ be the set of size $\text{non}(f,g)$ whose existence is implied by Definition 3. If $\{(y_i, \epsilon_i)\}_{i=0}^{\infty}$ are such that $\sum_{i=0}^{\infty} \epsilon_i < 1$ and $y_i \in \prod_{i=0}^{\infty} f(i)$ then let

$$B_m = \left\{ y_i \restriction m : \frac{1}{g(m-1)} > \epsilon_i \geq \frac{1}{g(m)} \right\}$$

and note that $|B_m| < g(m)$ since otherwise $\sum_{i=0}^{\infty} \epsilon_i \geq \sum_{i \in B_m} \epsilon_i \geq |B_m|/g(m) \geq 1$. It follows that there is some $a \in A$ such that $a \restriction m \notin B_m$ for all $m \in \mathbb{N}$. Hence, if $y_i \restriction m \in B_m$ then there is some $j < m$ such that $y_i(j) \neq f(j)$ and so $d_g(y_i, a) \geq 1/g(j) \geq 1/g(m-1) > \epsilon_i$; in other words, A is not a $\mathcal{Y}(f,g)$-null set. □

Propositions 2 and 3 point to the fact that it would be of interest to find models of set theory where there are monotonic functions f, g, F and G such that $\text{non}(f,g) = \aleph_1$ and $\text{non}(F,G) = \aleph_2$ and

$$\sum_{i=0}^{\infty} 1/g(i) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 1/F(i) < \infty \tag{0.3}$$

(so that the associated metric spaces $\mathcal{X}(F,G)$ and $\mathcal{Y}(f,g)$ are separable) and such that

$$\sum_{i=0}^{\infty} F(i)/G(i) < \infty \quad \text{and} \quad 1/F(i) > \sum_{m=i+1}^{\infty} 1/F(m) \tag{0.4}$$

so that the hypotheses of Proposition 2 are satisfied. This would allow the conclusion that any subset of cardinality $\aleph_1$ is $\mathcal{X}(F,G)$-null while, on the other hand, then there is a subset of $\mathcal{Y}(f,g)$ of cardinality $\aleph_1$ which is not $\mathcal{Y}(f,g)$-null.

The construction of such models can be accomplished using forcing partial orders, invented by Shelah, consisting of trees whose successors have norms attached to them — such partial orders are described in Sections 7.3 and 7.4 of [1]. A countable support product of these can be used along lines similar to those described in [2]. The appendix will sketch the argument.

Of course, finding the two metric spaces $\mathcal{X}(F,G)$ and $\mathcal{Y}(f,g)$ does not admit the conclusion that it is consistent that any subset of cardinality $\aleph_1$ is $\mathbb{R}$-null while there is a subset of $\mathbb{R}^2$ of cardinality $\aleph_1$ which is not $\mathbb{R}^2$-null. But is there some indication of how far off the mark this result is? To answer this, for any compact metric space $\mathcal{X} = (X,d)$ let $\Delta_{\mathcal{X}} : \mathbb{N} \to \mathbb{N}$ be the function defined by the fact that $\Delta_{\mathcal{X}}(k)$ is the least cardinal such that there is $A \in [X]^{\Delta_{\mathcal{X}}(k)}$ such that for every $x \in X$ there is $a \in A$ such that $d(a,x) < 1/k$. An examination of the proof in the Appendix will reveal that $\Delta_{\mathcal{Y}(f,g)}$ grows much more quickly than $\Delta_{\mathcal{X}(F,G)}$ whereas $\Delta_{\mathbb{R}^2}$ grows only as the square of $\Delta_{\mathbb{R}}$. Hence the following is of interest.

Question 1. *Given metric spaces $\mathcal{X}$ and $\mathcal{Y}$ such that every subset of cardinality $\aleph_1$ is $\mathcal{X}$-null while there is a subset of cardinality $\aleph_1$ which is not $\mathcal{Y}$-null, what can be said about the growth rates of $\Delta_{\mathcal{Y}}$ and $\Delta_{\mathcal{X}}$? Is it consistent that*

$$\frac{\Delta_{\mathcal{Y}}(n)}{\Delta_{\mathcal{X}}(n)}$$

be bounded by a linear function of n? While this would be required in order to deal with the case of $\mathbb{R}$ and $\mathbb{R}^2$ it is not even known if this ratio could be bounded by an exponential function.

Appendix

Definition 4. If k and n are integers and $\mathcal{A} \subseteq \mathcal{P}(\mathcal{P}(k))$ then define $\nu_{k,n}(\mathcal{A})$ as follows:

- if $\cup\mathcal{A} = k$ then $\nu_{k,n}(\mathcal{A}) \geq 0$
- if for every $P : \mathcal{A} \to n$ there is some $j \in n$ such that $\nu_{k,n}(P^{-1}\{j\}) \geq i$ then $\nu_{k,n}(\mathcal{A}) \geq i+1$
- otherwise $\nu_{k,n}(\mathcal{A})$ is defined to be -1.

Lemma 1. *If k and n are integers then $\nu_{k,n}([k]^{n^p}) \geq p$.*

Proof. This is elementary. $\square$

For the rest of this section, fix a function $g : \mathbb{N} \to \mathbb{N}$ such that $\sum_{i=0}^{\infty} 1/g(i) < \infty$. Define functions f, F, and G and a sequence $\{b_i\}_{i=0}^{\infty}$ by setting $b_0 = 1$ and inductively defining

$$G(n) = (b_n)^n \tag{0.5}$$

$$F(n) = 2^n G(n) \tag{0.6}$$

$$f(n) = g(n)b_n + 1 \tag{0.7}$$

$$b_{n+1} = \left(\prod_{i=0}^{n} \binom{f(i)}{g(i)}\right)^{\prod_{i=0}^{n} \binom{F(i)}{G(i)}} \tag{0.8}$$

and observe that conditions 0.3 and 0.4 are satisfied. Then define $\nu_i^* = \nu_{F(i),G(i)}$ and note that $\nu_i^*([F(i)]^{G(i)}) \geq i$. Therefore it is possible to define the partial order $\mathbb{P}$ to consist of all sequences s such that

- $s(k) \subseteq [F(k)]^{G(k)}$ for each k
- for each j there is some k such that $\nu_k^*(s(k)) > j$

with the ordering $s \leq t$ if and only if $s(i) \subseteq t(i)$ for each i. Let $\mathbb{P}_\kappa$ be the countable support product of κ copies of $\mathbb{P}$. Elements of $\mathbb{P}_\kappa$ will be thought of as functions with domain $\kappa \times \mathbb{N}$

If $p \in \mathbb{P}_\kappa$ then it is possible to find an extension $q \leq p$ such that for each integer k there is at most one $\eta \in \kappa$ such that $q(\eta, k)$ has more than one element of $[F(k)]^{G(k)}$ in it — such conditions are known as skew conditions. If p is a skew condition define k to be a non-trivial level if there is some $\eta \in \kappa$ such that $p(\eta, k)$ has more than one element of $[F(k)]^{G(k)}$. It will henceforth be assumed that all $p \in \mathbb{P}_\kappa$ are skew conditions.

Lemma 2. *If $\Gamma \subseteq \mathbb{P}_\kappa$ is generic over V then* $\mathrm{non}(F,G) \geq \kappa$ *in $V[\Gamma]$.*

Proof. This is elementary — see Section 7.3 of [1] for details. $\square$

Lemma 3. *If $\Gamma \subseteq \mathbb{P}_\kappa$ is generic over V and $2^{\aleph_0} = \aleph_1$ holds in V then* $\mathrm{non}(f,g) = \aleph_1$ *in $V[\Gamma]$.*

Proof. It will be shown that for every sequence of finite sets $\{a_i\}_{i=1}^{\infty}$ in $V[\Gamma]$ such that $a_i \in [f(j)]^{g(i)}$ there is $x \in V \cap \prod_{i=0}^{\infty} f(i)$ such that $x(m) \notin a(m)$ for all m. This is more than is required to show that $\text{non}(f,g) = \aleph_1$ in $V[\Gamma]$.

Assume that the sequence of names $\{\mathring{a}_i\}_{i=1}^{\infty}$ is forced to be a counterexample by the condition $p \in \mathbb{P}_\kappa$. Let $\mathfrak{M} \prec (H(\kappa), \in)$ be a countable elementary submodel containing p and $\{\mathring{a}_i\}_{i=1}^{\infty}$. Let $\{\eta_i\}_{i=1}^{\infty}$ enumerate $\kappa \cap \mathfrak{M}$ in such a way that every ordinal in $\kappa \cap \mathfrak{M}$ is listed infinitely often. Now construct, in $\mathfrak{M}$, a sequence $\{(p_i, Y_i, L_i)\}_{i=0}^{\infty}$ such that

- $L_0 = 0$
- $p_0 \leq p$
- $Y_i \subseteq [f(i)]^{g(i)}$ has size b_i
- $p_{i+1} \Vdash_{\mathbb{P}_\kappa}$ "$\mathring{a}_j \in \check{Y}_j$" for each j such that $L_i \leq j < L_{i+1}$
- $\nu^*_{L_i}(p_i(\eta_i, L_i)) \geq i$
- $p_{i+1}(\eta_j) \restriction L_i = p_i(\eta_j) \restriction L_i$ for all $j \leq i$

and note that in the last condition, and henceforth, the convention on the use of "$\restriction$" introduced in Definition 2 has been dropped.

Assuming that this can be done, it follows that there is some $p_\omega \in \mathbb{P}_\kappa$ such that $p_\omega \leq p_i$ for each i. Moreover $p_\omega \Vdash_{\mathbb{P}_\kappa}$ "$(\forall j)\ \mathring{a}_j \in Y_j$" and it follows from Induction Hypothesis 0.6 that $f(i) \setminus \cup Y_i \neq \emptyset$ for each i and so it is possible to choose $x \in \prod_{i=0}^{\infty} f(i)$ in V as required.

To establish that the induction can be completed suppose that p_n and L_n have been constructed. Choose $L_{n+1} > b_{L_n+1}$ such that $\nu^*_k(p_n(\eta_{n+1}, L_{n+1})) > n+1$. By extending p_n, if necessary, it may be assumed that p_n has no non-trivial levels between L_n and L_{n+1}. Let Q be the set of all $q \restriction (\kappa \times L_{n+1})$ such that $q \leq p_n$ and q has no non-trivial levels below L_{n+1}. Notice that the cardinality of Q is no greater than $\prod_{i=0}^{L_n} \binom{F(i)}{G(i)}$ which is the exponent in the definition of b_{L_n+1} and, hence, is less than or equal to the exponent used in the definition of $b_{L_{n+1}}$.

By considering each $x \in p_n(\eta_{n+1}, L_{n+1})$ and each element of Q in turn, it is possible to find $q^* \leq p_n$ such that $q^* \restriction \kappa \times L_{n+1} = p_n \restriction \kappa \times L_{n+1}$ and if q^*_x is defined by

$$q^*_x(\mu, j) = \begin{cases} x & \text{if } (\mu, j) = (\eta_{n+1}, L_{n+1}) \\ q^*(\mu, j) & \text{if } (\mu, j) \neq (\eta_{n+1}, L_{n+1}) \end{cases}$$

then there is a function

$$\Psi_x : Q \to \prod_{j=0}^{L_{n+1}-1} [f(n)]^{g(n)}$$

such that for any $r \leq q^*_x$ such that $\bar{r} = r \restriction \kappa \times L_{n+1} \in Q$,

$$q^* \Vdash_{\mathbb{P}_\kappa} \text{“} \mathring{a}_j = \Psi_x(\bar{r})(j)\text{”}$$

for each $j < L_{n+1}$. The function defined on $p_n(\eta_{n+1}, L_{n+1})$ which sends x to Ψ_x has range no greater in cardinality than $b_{L\lceil n+1}$ and, therefore, there is $X \subseteq p_n(\eta_{n+1}, L_{n+1})$ such that

$$\nu^*_{L_{n+1}}(p_n(\eta_{n+1}, k)) \geq n + 1.$$

Let p_{n+1} be defined by

$$p_{n+1}(\mu, j) = \begin{cases} X & \text{if } (\mu, j) = (\eta_{n+1}, L_{n+1}) \\ q(\mu, j) & \text{if } (\mu, j) \neq (\eta_{n+1}, L_{n+1}) \end{cases}$$

and observe that this is the desired condition. If $\Psi_x = \Psi$ for each $x \in X$ let $Y_j = \{\Psi(q)\}_{q \in Q}$. □

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