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COMPACTIFICATIONS OF BAIRE SPACES κ^ω

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ABSTRACT. We show that the space of irrationals can be compactified in such a way that the remainder is the union of, apriori prescribed, countably many compact spaces each of weight not exceeding ω_1. We show that any Baire space of an uncountable weight has a compactification such that its remainder is a σ-discrete space.

1. Compactifying Baire spaces of uncountable weight

The Cartesian product of countably many copies of an infinite discrete space of cardinality κ is called the *Baire space of weight* κ. The Baire space of weight ω is homeomorphic to the space of irrational numbers.

No Baire space of any uncountable weight can have a compactification whose remainder is going to be the union of finitely many metrizable subspaces. We shall show that there is a one whose remainder is the union of countably many discrete (just metrizable) subspaces.

Throughout our discussion, we treat cardinals as von Neumann ordinals endowed with the discrete topology. Let κ be an uncountable cardinal. The symbol ${}^{\leq\omega}\kappa$ denotes the complete tree of height $\omega+1$, i.e.,

$$^{\leq\omega}\kappa = {}^{<\omega}\kappa \cup {}^{\omega}\kappa,$$

where

$$^{<\omega}\kappa = \{s : s \text{ is a function and } Dom\,(s) \in \omega \text{ and } Rng\,(s) \subseteq \kappa\}$$

and

$$^{\omega}\kappa = \{s : s \text{ is a function and } Dom\,(s) = \omega \text{ and } Rng\,(s) \subseteq \kappa\}.$$

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If $s \in^{<\omega} \kappa$ and $\alpha \in \kappa$, then $s^\frown\alpha$ denotes the concatenation of s by α.

For each $n \in \omega$, let $L_n = \{t \in^{<\omega} \kappa : |t| = n\}$ and $T_n = \{t \in^{<\omega} \kappa : |t| \leq n\}$.

For each $s \in^{<\omega} \kappa$, let $Cone(s) = \{t \in^{\leq\omega} \kappa : s \subseteq t\}$.

Let X_κ be the space whose underlying set is ${}^{\leq\omega}\kappa$ which is endowed with the *tree topology*, i.e., topology generated by sets of the form

$$Cone(s) - (Cone(s^\frown\alpha_1) \cup Cone(s^\frown\alpha_2) \cup ... \cup Cone(s^\frown\alpha_k)),$$

where $s \in^{<\omega} \kappa$ and $\alpha_i \in \kappa$ for each $i = 1, 2, .., k$. In the series of simple lemmas that follows we will verify the required properties for the space X_κ to be a required compactification of the Baire space of the uncountable weight κ.

Lemma 1. *If $s, t \in^{<\omega} \kappa$, $s \neq t$, and*

$$t \in Cone(s) - (Cone(s^\frown\alpha_1) \cup Cone(s^\frown\alpha_2) \cup ... \cup Cone(s^\frown\alpha_k))$$

then

$$Cone(t) \subseteq Cone(s) - (Cone(s^\frown\alpha_1) \cup Cone(s^\frown\alpha_2) \cup ... \cup Cone(s^\frown\alpha_k)).$$

Lemma 2. *If $s, t \in^{<\omega} \kappa$, $s \not\subseteq t$, and $s \not\supseteq t$, then $Cone(t) \cap Cone(s) = \emptyset$.*

Lemma 3. *If $s \in^{<\omega} \kappa$, then $Cone(s) \cap^\omega \kappa = \prod\{C_i : i \in \omega\}$, where $C_i = \{s(i)\}$ for each $i \in Dom(s)$, and $C_i = \kappa$ for each $i \notin Dom(s)$. Thus the subspace ${}^\omega\kappa$ of X_κ is the Baire space of weight κ.*

Lemma 4. *For each $n \in \omega$, L_n is a discrete subspace of X_κ (and T_n is a closed subspace of X_κ).*

Proof. If $s \in L_n$, then $L_n \cap Cone(s) = \{s\}$. □

Theorem 5. *X_κ is a compactification of the Baire space ${}^\omega\kappa$.*

Proof. The space X_κ is Hausdorff (use Lemma 2). By Lemma 3, the Baire space ${}^\omega\kappa$ is a dense subspace of the space X_κ.

Suppose to the contrary that X_κ is not a compact space. Thus there exists an open cover $\mathcal{P}$ of X_κ without a finite subcover. Without loss of generality we may assume that $\mathcal{P}$ consists of the basic open sets. Let $U_0 \in \mathcal{P}$ be a basic set containing $\emptyset \in X_\kappa$. Since $U_0 = Cone(\emptyset)\backslash(Cone(\emptyset^\frown\alpha_1) \cup Cone(\emptyset^\frown\alpha_2) \cup ... \cup Cone(\emptyset^\frown\alpha_k))$, one of $Cone(\emptyset^\frown\alpha_i)$, $i = 1, 2, .., k$, cannot be covered by finitely many elements of the cover $\mathcal{P}$. Thus there exists a sequence s_1 of length 1 such that $Cone(s_1)$ cannot be covered by finitely many elements of the cover $\mathcal{P}$.

Suppose that we have defined sequences $s_1, s_2, .., s_n$ satisfying the following conditions:

(i) For each $k \leq n$, $s_k \in^{<\omega} \kappa$ and $Dom(s_k) = k$;

(ii) $s_1 \subset s_2 \subset ... \subset s_n$;

(iii) For each $k \leq n$, the set $Cone(s_k)$ cannot be covered by finitely many elements of the cover $\mathcal{P}$.

Let $U_n \in \mathcal{P}$ be a basic set containing $s_n \in^{<\omega} \kappa$. Since $U_n = Cone(t)\backslash(Cone(t^\frown\beta_1) \cup Cone(t^\frown\beta_2) \cup ... \cup Cone(t^\frown\beta_k))$, s_n must be equal to t, by virtue of Lemma 1. Hence one of $Cone(s_n^\frown\beta_j)$, $j = 1, 2, .., k$, cannot be covered by finitely many elements of the cover $\mathcal{P}$. Thus there exists a sequence s_{n+1} of length $n+1$ such that $s_n \subset s_{n+1}$ and $Cone(s_{n+1})$ cannot be covered by finitely many elements of the cover $\mathcal{P}$.

By induction, there exists a sequence $s_0, s_1, .., s_n, ...$ satisfying the following conditions:

(i) For each $k \in \omega$, $s_k \in^{<\omega} \kappa$ and $Dom(s_k) = k$;

(ii) $s_1 \subset s_2 \subset ... \subset s_n \subset ....$

(iii) For each $k > 0$, the set $Cone(s_k)$ cannot be covered by finitely many elements of the cover $\mathcal{P}$.

Let $x = \bigcup\{s_k : k \in \omega\}$. Since $x \in^\omega \kappa$, there exists U in $\mathcal{P}$ that contains the point x. Thus $x \in Cone(t) - (Cone(t^\frown\beta_1) \cup Cone(t^\frown\beta_2) \cup ... \cup Cone(t^\frown\beta_k))$. It follows that $t \subset x$ and thus $t = s_n$ for some $n \in \omega$. Since $x \in Cone(t) - (Cone(t^\frown\beta_1) \cup Cone(t^\frown\beta_2) \cup ... \cup Cone(t^\frown\beta_k))$, $s_{n+1} \in Cone(t) - (Cone(t^\frown\beta_1) \cup Cone(t^\frown\beta_2) \cup ... \cup Cone(t^\frown\beta_k))$ too. By lemma 1, $Cone(s_{n+1}) \subseteq Cone(t) - (Cone(t^\frown\beta_1) \cup Cone(t^\frown\beta_2) \cup ... \cup Cone(t^\frown\beta_k))$, a contradiction. $\square$

2. Compactifying irrationals

We begin by proving an easy fact.

Lemma 6. *Let Y be a compact Hausdorff space and let $p \in Y$ be a non-isolated point. Suppose that X is a compactification of the space $Y - \{p\}$ with remainder Z. Let U be an open neighborhood of the point p in the space Y and let V be an open neighborhood of a point $x \in Z$. Then $U \cap V \neq \emptyset$.*

Proof. The set $F = X - U$ is a compact subset of the space $Y - \{p\} \subset X$. So $V - F$ is an open neighborhood of the point x. Hence $\emptyset \neq (V - F) \cap (Y - \{p\}) \subseteq V \cap U$. □

Let $\mathcal{R}$ be the class of all compact Hausdorff spaces that can be used as a remainder of some compactification of the discrete countable space ω. According to Parovičenko's theorem (cf. [1]), any compact Hausdorff space of weight not exceeding ω_1 is in $\mathcal{R}$.

Lemma 7. *Let $Y = \oplus\{X_n : n \in \omega\}$ be the topological sum of compact Hausdorff spaces X_n. If $Z \in \mathcal{R}$, then there exists a compactification X of the space Y such that the remainder $X - Y$ is homeomorphic to Z.*

Proof. Without loss of generality, we may assume that Y and Z are disjoint. Let $\widetilde{X}$ be a compactification of the discrete space ω such that the remainder $\widetilde{X} - \omega$ is homeomorphic to Z. For any open set U of the space $\widetilde{X}$ such that $U \cap Z \neq \emptyset$, let $e(U) = \oplus\{X_n : n \in U \cap \omega\} \cup (U \cap Z)$. We take X to be the set $Y \cup Z$ with topology generated by the sets that are open subsets of the space Y or of the form $e(U)$. □

Lemma 8. *Let Y be a compact Hausdorff space and let $p \in Y$ be a non-isolated point that has a countable base of closed-open subsets of Y. If $Z \in \mathcal{R}$, then there exists a compactification X of the space $Y - \{p\}$ such that the remainder $X - (Y - \{p\})$ is homeomorphic to Z.*

Let C be a compact Hausdorff space and let $\{d_n : n \in \omega\}$ be an enumeration of a countable subset of C. Suppose further that each point d_n is non-isolated and has a countable base of closed-open subsets of C. Let $Z_n \in \mathcal{R}$ for each $n = 1, 2, ...$ By induction, we define a sequence of spaces $\{C_n : n \in \omega\}$ as follows:

$C_0 = C$;

$C_{n+1} =$ a compactification of the space $C_n - \{d_n\}$ such that the remainder $C^*_{n+1} = C_{n+1} - (C_n - \{d_n\})$ is homeomorphic to the space Z_{n+1} (such a compactification exists by virtue of Lemma 8).

For $n = 1, 2, ...,$ let p_n be the natural projection from C_n to C_{n-1}, i.e.,

$$p_n(x) = \begin{cases} d_{n-1}, \ if\ x \in C^*_n \\ x, \ if\ x \notin C^*_n \end{cases} .$$

Lemma 9. *For $n = 1, 2, ...,$ $p_n : C_n \to C_{n-1}$ is continuous.*

Proof. Proof. Let U be an open neighborhood of the point d_{n-1} in the space C_{n-1}. The set $F = C_{n-1} - U$ is a compact subset of the space C_n. Clearly $p_n^{-1}(U) = (U - \{d_{n-1}\}) \cup Z = C_n - F$. In consequence, the set $p_n^{-1}(U)$ is open in the space C_n. □

Let us consider the inverse sequence

$$C_0 \leftarrow^{p_1} C_1 \leftarrow^{p_2} C_2 \leftarrow^{p_3} .... \leftarrow C_{n-1} \leftarrow^{p_n} C_n \leftarrow ...$$

and its limit X, i.e.,

$$X = \left\{(x_i) \in \prod\{C_i : i \in \omega\} : p_n(x_n) = x_{n-1} \; for \; n = 1, 2, ...\right\}.$$

Lemma 10. *X is a compact Hausdorff space.*

Let $M_0 =$

$$\left\{(x_i) \in \prod\{C_i : i \in \omega\} : x_i = x \; for \; i \in \omega \text{ and } x \in C - \{d_n : n \in \omega\}\right\};$$

If $n > 0$, $M_n = \{(x_i) \in \prod\{C_i : i \in \omega\} : x_i = d_{n-1} \; for \; i = 0, 1, 2, ..., n-1 \text{and } x_i = x \text{ for } i \geq n \text{ and } x \in C_n^*\}$. The sets M_n, $n \in \omega$, are pairwise disjoint.

Lemma 11. *M_0 and $C - \{d_n : n \in \omega\}$ are homeomorphic.*

Lemma 12. *For each $n = 1, 2, ...,$ M_n and Z_n are homeomorphic.*

Both lemmas, above, follow immediately from the following one:

Lemma 13. *Let $\prod\{X_\alpha : \alpha \in S\}$ be the product of spaces X_α, where $X_\alpha = X$ for each $\alpha \in S$. Then the diagonal $\Delta = \{(x_\alpha) \in \prod\{X_\alpha : \alpha \in S\} : x_\alpha = x$ for each $\alpha \in S$ and $x \in X\}$ and the space X are homeomorphic.*

Proof. Let $h : X \to \Delta$ be defined as follows:

$$h(x) = (x_\alpha), \text{ where } x_\alpha = x \; for \; each \; \alpha \in S.$$

One can easily see that if $A \subseteq X$ and $\alpha \in S$ and $\pi_\alpha : \prod\{X_\alpha : \alpha \in S\} \to X_\alpha$ is a natural projection, then $h(A) = \Delta \cap \pi_\alpha^{-1}(A)$. □

Lemma 14. $X = \bigcup\{M_n : n \in \omega\}$.

Proof. Let $(x_i) \in X$. Consider the following two cases:
Case (a) $\forall i\ x_i = x_{i+1}$;
Case (b) $\exists i\ x_i \neq x_{i+1}$.
In case (a), let $x = x_i$ for each i. Since $x_0 = x$, $x \in C$. Clearly, $x \neq d_n$ for each $n \in \omega$ (for if $x = d_n$, then $p_{n+1}(x_{n+1}) = d_n = x$ and $x = x_{n+1} \in C^*_{n+1}$; a contradiction). Hence $(x_i) \in M_0$.
In case (b), since $p_{i+1}(x_{i+1}) = x_i$, $x_{i+1} \in C^*_{n+1}$ and $x_i = d_i$. Thus $x_j = d_i$ for each $j \leq i$, and $x_j = x_{i+1}$ for each $j \geq i+1$. Hence $(x_i) \in M_{i+1}$. □

Lemma 15. *Let $U = U_0 \times U_1 \times ... \times U_n \times C_{n+1} \times C_{n+2} \times ...$ be an open basic subset of the product $\prod\{C_i : i \in \omega\}$. If $U \cap X \neq \emptyset$, then $(U_0 \cap U_1 \cap ... \cap U_n) \cap (C - \{d_0, d_1, ..., d_{n-1}\}) \neq \emptyset$.*

Proof. By Lemma 14, $U \cap M_k \neq \emptyset$ for some $k \in \omega$. If $k = 0$, then there exists $x \in C - \{d_n : n \in \omega\}$ such that $(x_i) \in U$ and $x_i = x$ for each $i \in \omega$. Hence $x \in U_0 \cap U_1 \cap ... \cap U_n$. Thus $(U_0 \cap U_1 \cap ... \cap U_n) \cap (C - \{d_0, d_1, ..., d_{n-1}\}) \neq \emptyset$. Let $k > 0$ and let $(x_i) \in U \cap M_k$. Thus there exists $x \in C^*_k$ such that

$$x_i = \begin{cases} d_{k-1},\ if\ i < k \\ x,\ if\ i \geq k \end{cases}.$$

If $k > n$, then $d_{k-1} \in U_0 \cap U_1 \cap ... \cap U_n$. Assume that $k \leq n$. The set $U = \bigcap\{U_i : i \leq k-1\} \cap (C - \{d_0, d_1, ..., d_{k-2}\})$ is an open neighborhood of the point d_{k-1} in the subspace $(C - \{d_0, d_1, ..., d_{k-2}\})$. The set $V = \bigcap\{U_i : k \leq i \leq n\}\}$ an open neighborhood of the point x in the space C_k. By Lemma 6, $U \cap V \neq \emptyset$. □

Lemma 16. *M_0 is a dense subset of X.*

Proof. Let $U = U_0 \times U_1 \times ... \times U_n \times C_{n+1} \times C_{n+2} \times ...$ be an open basic subset of the product $\prod\{C_i : i \in \omega\}$ such that $U \cap X \neq \emptyset$. By Lemma 15, $(U_0 \cap U_1 \cap ... \cap U_n) \cap (C - \{d_0, d_1, ..., d_{n-1}\}) \neq \emptyset$. In consequence, $(U_0 \cap U_1 \cap ... \cap U_n) \cap (C - \{d_n : n \in \omega\}) \neq \emptyset$. If $x \in (U_0 \cap U_1 \cap ... \cap U_n) \cap (C - \{d_n : n \in \omega\})$ and (x_i) is such that $x_i = x$ for $i \in \omega$, then $(x_i) \in U \cap M_0$. □

Lemma 17. *If $\{d_n : n \in \omega\}$ is a dense subset of C, then $\{M_n : n \geq 1\}$ is a $\pi - net$ in X.*

Proof. Let $U = U_0 \times U_1 \times ... \times U_n \times C_{n+1} \times C_{n+2} \times ...$ be an open basic subset of the product $\prod\{C_i : i \in \omega\}$ such that $U \cap X \neq \emptyset$.

By Lemma 15, $(U_0 \cap U_1 \cap ... \cap U_n) \cap (C - \{d_0, d_1, ..., d_{n-1}\}) \neq \emptyset$. In consequence, $U_0 \cap U_1 \cap ... \cap U_n$ contains infinitely many elements among $\{d_n : n \in \omega\}$. Pick any m such that $m > n$ and $d_m \in U_0 \cap U_1 \cap ... \cap U_n$. Then $M_m \subseteq U$. □

Theorem 18. *There exists a compactification X of the space of irrational numbers ω^ω such that: (i) $X - \omega^\omega = \bigcup\{M_n : n \geq 1\}$, (ii) $\{M_n : n \geq 1\}$ is a $\pi-$net in X, (iii) For each $n = 1, 2, ...$, M_n and Z_n are homeomorphic.*

3. Applications

The metrizability number $m(X)$ of a space X is the smallest cardinal number κ such that X can be represented as a union of κ many metrizable subspaces. In [2] we showed that compact Hausdorff spaces with finite metrizability number can be represented as follows:

Theorem 19. *If X is a (locally) compact Hausdorff space with $m(X) = n$, $2 \leq n < \omega$, then X can be represented as $X = G \cup F$, where G is an open dense metrizable subspace of X, $F \cap G = \emptyset$, and $m(F) = n - 1$.*

A similar representation theorem may not hold for compact Hausdorff spaces with countable metrizability number.

Theorem 20. *There exists a compact Hausdorff space X with a countable $\pi-$base such that $m(U) = \omega$ for each non-empty open subset U of X.*

Proof. Let X be the space constructed from the Cantor set C, an arbitrary countable dense subset $\{d_n : n \in \omega\}$, and from Z_n that is e.g., the one-point compactification of a discrete space of cardinality $\aleph_1$, for each $n = 1, 2, ...$. □

Theorem 21. *If M is a zero-dimensional metrizable space, then M has a compactification Y such that $Y \backslash M$ is a union of countably many discrete subspaces of Y.*

Proof. The space M can be embedded into a Baire space κ^ω. Let X be the compactification of the Baire space κ^ω as given in Theorem 5. Then the closure of M in the space X gives the required compactification Y. □

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