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CONCERNING METRIZABLE CONTINUA OF CONVERGENCE

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Abstract. We consider compact Hausdorff spaces with the property that each continuum of convergence is metrizable. We first conduct a comprehensive study of this property. We then investigate the relationship of this property to the Hahn-Mazurkiewicz Problem in the class of locally connected continua. In so doing, we find analogues to theorems of J. Cornette, J. Simone, and L. B. Treybig, respectively.

1. Introduction

We wish to study apparently essential questions regarding metrizable continua of convergence and their relationship to the Hahn-Mazurkiewicz Problem. It has long been known that metric continua play a large role in the non-metric Hahn-Mazurkiewicz Problem. Results addressing the role of metric subcontinua in the Hahn-Mazurkiewicz Problem include [3], [5], [7], [9], [12], and [19].

Additionally, J. Nikiel [11] has shown that each hereditarily locally connected continuum is the continuous image of an ordered continuum, and J. Simone [14] has shown that each continuum that is not hereditarily locally connected contains a non-degenerate continuum of convergence. It is therefore somewhat natural to consider those continua which have the property that each continuum of convergence is metrizable.

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Herein, we study that property and explore results analogous to those cited above, primarily in the class of locally connected continua.

2. Definitions and Notation

Definition 2.1. Suppose X is a topological space and let $\{S_n\}_{n=1}^{\infty}$ be a sequence of subsets of X. Then

$\limsup S_n = \{x \in X$: each open set containing x intersects infinitely many $S_n\}$, and

$\liminf S_n = \{x \in X$: each open set containing x intersects all but finitely many $S_n\}$.

In case $\limsup S_n = \liminf S_n = L$, we say that $\{S_n\}_{n=1}^{\infty}$ is said to be convergent and L is said to be the limit of $\{S_n\}_{n=1}^{\infty}$. We may write $\lim S_n = L$. (It is well known that if X is compact and each S_n is connected, then $L = \lim S_n$ is a continuum.)

We will say that a continuum C is a continuum of convergence of X provided that there exists a countable collection $\{C_i\}_{i=1}^{\infty}$ of continua in X such that $\{C_i\}_{i=1}^{\infty}$ converges, C is the limit of $\{C_i\}_{i=1}^{\infty}$, and no continuum $C_n \in \{C_i\}_{i=1}^{\infty}$ intersects C.

Definition 2.2. A compact Hausdorff space X is said to have the metrizable continua of convergence property (MCC) if and only if each continuum of convergence of X is metrizable. We will also say simply that X is MCC.

Some additional definitions are also needed. A continuum is a compact connected Hausdorff space. A continuum X is said to be hereditarily locally connected provided that each subcontinuum of X is locally connected. A Hausdorff space X is said to be an IOK provided there exists a compact totally ordered space K and a continuous onto map $f : K \to X$. If K is connected, X is said to be an IOC. A space S is rim-P if and only if S admits a basis of open sets so that each open set has boundary with property P.

All mappings herein are continuous. For spaces X and Y, the onto mapping $f : X \to Y$ is monotone if for every $y \in Y$, $f^{-1}(y)$ is connected. For compact space X and space Y, the onto mapping $f : X \to Y$ is confluent if for every subcontinuum K in Y and every component C of $f^{-1}(K)$, $f(C) = K$.

For space X and $U \subseteq X$, the closure of U in X is denoted $\mathrm{Cl}(U)$ and the boundary of U in X is denoted $\mathrm{Bd}(U)$.

3. Fundamental Properties of MCC Compacta

We first investigate the nature of MCC spaces. In particular, we study whether MCC is preserved under continuous maps, under products, etc.

Lemma 3.1. *Suppose X is a continuum and C is a continuum of convergence of X. Then there exists a countable collection $\{C_i\}_{i=1}^{\infty}$ of mutually disjoint continua in X, and C is the continuum of convergence of $\{C_i\}_{i=1}^{\infty}$.*

Proof: Let $S = \{C'_i\}_{i=1}^{\infty}$ denote a countable collection of continua in X such that C is the continuum of convergence of $\{C'_i\}_{i=1}^{\infty}$. Notice that, for each fixed $j \in N$, there exists $m \in N$ such that for all $n > m$, $C'_j \cap C'_n = \emptyset$. If not, there exists a sequence $C'_{j_1}, C'_{j_2}, C'_{j_3}, \ldots$ of elements of S such that $C'_j \cap C'_{j_n} \neq \emptyset$ for each n. For each $n \in N$, select $x_n \in (C'_j \cap C'_{j_n})$. Then there exists a limit point x of the sequence $\{x_n\}$ such that $x \in C'_j$ and $x \in C$, a contradiction.

Consider C'_1. Let S_1 denote the collection of all elements of S that meet C'_1. Let $C_1 = \cup S_1$. Let k_2 denote the maximum of all subscripts of elements of S that meet some element of S_1. Consider C'_{k_2+1}. Let S_2 denote the collection of all elements of S that meet C'_{k_2+1}. Let $C_2 = \cup S_2$. Let k_3 denote the maximum of all subscripts of elements of S that meet some element of S_2. Consider C'_{k_3+1}. Let S_3 denote the collection of all elements of S that meet C'_{k_3+1}. Let $C_3 = \cup S_3$. Let k_4 denote the maximum of all subscripts of elements of S that meet some element of S_3.

We continue inductively to generate a sequence $\{C_i\}_{i=1}^{\infty}$ whose elements are mutually disjoint continua by construction. □

Theorem 3.2. *Suppose X is an* MCC *continuum and X fails to be locally connected at $x \in X$. Then, for each open set O containing x, there exists a non-degenerate metric subcontinuum M of X such that $M \subseteq O$.*

Proof: Let $x \in O$ with O open in X. Since X fails to be locally connected at $x \in X$, X is not connected im kleinen at x. As such (see [18, Theorem 2]), there exist open sets U and U' and mutually disjoint continua $C'_1, C'_2, \ldots$ so that $x \in U \subset Cl(U) \subset U' \subset Cl(U') \subset O$ and each $C'_i \subseteq O$ meets both U and $(X - U')$.

For each i, select a component C_i of $C'_i \cap Cl(U')$ such that C_i meets both U and $(X - U')$. The continuum of convergence M of $\{C_i\}_{i=1}^{\infty}$ is metrizable and, by construction, is contained in O. □

The proof of the following is straightforward and is left to the reader.

Theorem 3.3. *A space X is* MCC *if and only if each closed subspace of X is* MCC*.*

Definition 3.4. A subset C of a locally connected continuum X is a cyclic element of X if and only if C is maximal with respect to the property of having no cut point.

We next show a natural analogue to a classical result of J. Cornette [1] - a locally connected continuum X is an IOC if and only if each cyclic element of X is an IOC. K. Kuratowski ([6, pp. 317-318]) has shown that every continuum of convergence C of a continuum X is a continuum of convergence of some cyclic element (containing C) of X. Combining this result with 3.3 above yields the following.

Theorem 3.5. *A locally connected continuum X is* MCC *if and only if each cyclic element of X is* MCC*.*

Theorem 3.6. *Let X be a compact Hausdorff space. Suppose $f : X \to Y$ is a monotone mapping onto the Hausdorff space Y. If X is* MCC *then Y is* MCC*.*

Proof: Suppose C is a non-degenerate continuum of convergence of Y and $\{C_i\}_{i=1}^{\infty}$ is a sequence of mutually disjoint continua in Y for which $C = \lim C_i$. By a result of R. Engelking ([4, Theorem 6.1.29, p. 441]), the inverse $f^{-1}(C_i)$ is connected for each i. Then there exists a sequence $\{D_i\}_{i=1}^{\infty}$ of mutually disjoint continua in X so that $f(D_i) = C_i$ for each i. By R. L. Moore ([10, Theorem 59, p. 24]) and by the confluence of f, there exists a subsequence $\{C_k\}_{k=1}^{\infty}$ such that the continuum of convergence D of $\{D_k\}_{k=1}^{\infty}$ is non-degenerate and metrizable. It is straightforward, using continuity, to show that $f(D) \supseteq C$ and therefore C is metrizable. □

Corollary 3.7. *Let X be a compact Hausdorff* MCC *space and G an upper semi-continuous decomposition of X into continua. Then X/G is* MCC*.*

There does exist an example of a continuous map of a compact MCC Hausdorff space such that the image is not MCC. Let X denote $[0,1] \cup \{\{\frac{1}{n}\} \times C\} \cup \{\{0\} \times [a,b]\}$, where C denotes the Cantor set and $[a,b]$ denotes a non-separable arc. Similarly, let Y denote $[0,1] \cup \{\{\frac{1}{n}\} \times [0,1]\} \cup \{\{0\} \times [a,b]\}$. Let $g : C \to [0,1]$ be a continuous map of the Cantor set onto $[0,1]$ and define $f : X \to Y$ by

(3.1)
$$f(x,t) = \begin{cases} (x, g(t)) & \text{if } t \in \{\{\frac{1}{n}\} \times C\} \text{ for some } n = 1, 2, \ldots \\ (x,t) & \text{otherwise} \end{cases}$$

Then X trivially is MCC and Y is not.

H. M. Tuncali [21] has shown that rim-metrizability is also preserved under monotone mappings of such spaces. However, rim-metrizability and MCC do not coincide in the class of continua. There exist rim-metrizable continua that are not MCC and vice versa. Let L and C denote the long line, $L = ([0, \omega_1) \times [0,1)) \cup \{(\omega_1, 0)\}$ ordered lexicographically, and the Cantor set, respectively. Then $X = (\{0\} \times [0,1]) \cup (\cup\{L \times \{t\} : t \in C\})$ is a rim-metrizable continuum that clearly is not MCC. This example is due to Tuncali and is studied in detail in [20]. Similarly, $X = L \cup \{\Omega \times [0,1]\}$ is an MCC continuum which is not rim-metrizable. The fact that X is not rim-metrizable follows from Corollary 2.2 of [21].

The following simple theorem is an analogue to an important result of L. B. Treybig [16] - if the product $X \times Y$ of two infinite Hausdorff spaces X and Y is an IOK then each of X and Y is metrizable.

Theorem 3.8. *Let X and Y be non-degenerate continua. $X \times Y$ is* MCC *if and only if each of X and Y is metrizable.*

Proof: Suppose $\{x_k\}_{k=1}^{\infty}$ is a countable sequence in X with limit point x_0. Then the product $\{x_k\}_{k=1}^{\infty} \times Y$ is a sequence of disjoint continua in $X \times Y$ with metrizable limiting continuum $\{x_0\} \times Y$. Therefore, $\pi_Y(\{x_0\} \times Y) = Y$ is metrizable. A similar argument shows that X is metrizable. □

It is straightforward to show that both X and Y must be continua in the preceding theorem. Additionally, the product of locally connected MCC continua need not be MCC. Let $A = [a,b]$ denote a non-separable arc and A trivially is MCC. However, there exists a sequence of disjoint continua in the product $A \times A$ such that the limiting continuum of the sequence is a copy of A.

4. Applications to the Hahn-Mazurkiewicz Problem

Our interest in MCC spaces was initially spurred by the following result; we show that certain continuous images of ordered compacta are MCC. The reader is referred to a closely related (unpublished) result by A. J. Ward [22], quoted by Simone [15].

Theorem 4.1. *If X is a first countable IOK, X is* MCC.

Proof: Let C be a continuum of convergence in X and let $\{C_i\}_{i=1}^{\infty}$ be a sequence of mutually exclusive continua in X such that $C = \lim C_i$. Suppose also that $\{x_i\}_{i=0}^{\infty}$ is a sequence of elements of X such that $x_i \in C_i$ for all $i \geq 1$, $x_0 \in C$, and x_0 is the unique limit point of $\{x_i\}_{i=1}^{\infty}$ in X. Then $C \cup (\cup_{i=1}^{\infty} C_i)$ is closed so that it is itself an IOK. Let $[a, b]$ denote a real arc and $\{a_i\}_{i=0}^{\infty}$ be a sequence of distinct points in $[a, b]$ such that $a = a_1$, $b = a_0$ and a_0 is the only limit point of $\{a_i\}_{i=0}^{\infty}$. Let Z denote the disjoint union $[a, b] \cup \{g_i\}_{i=0}^{\infty}$. Now let H be the decomposition of Z into $\{a_n, x_n\}$ for each $n = 0, 1, 2, 3, \ldots$ and points of $Z - \{\{a_n, x_n\} : n = 0, 1, 2, 3, \ldots\}$. The proof that H is an upper semi-continuous decomposition of Z is straightforward and is left to reader. Note that Z/H is an IOK since Z is. Since C is closed in Z/H, the set $N = [a, b) \cup \{C_i\}_{i=1}^{\infty}$ is open F_σ in Z/H. Then $C = Bd_Z N$ is metrizable by [8]. □

In [13] and [15], Simone investigates the "metric" components of a Hausdorff space X where X is a Suslinian IOK. Define a relation R on X by xRy if and only if there exists a metric continuum in X containing x and y. For each $x \in X$, define $M_x = \{y \in X : xRy\}$. M_x is called the metric component of x. Simone shows that R is an equivalence relation, and if X is also first countable then M_x is a continuum for each $x \in X$. Simone's construction in [13] and a result of Treybig [18, Theorem 3], respectively, motivate the following theorem.

Theorem 4.2. *Suppose X is a first countable locally connected* MCC *continuum. Then there exists an upper semi-continuous decomposition G of X into continua such that X/G is hereditarily* *locally connected and thus an IOC. Furthermore, if x and y are distinct points in $g \in G$ there exists a separable subcontinuum of X which contains both x and y.*

Proof: For each $x \in X$, define $M_x = \{y \in X$: there exists a separable subcontinuum of X containing x and $y\}$. Let $G = \{M_x : x \in X\}$.

We first show that M_x is a continuum for each $x \in X$. M_x is clearly connected so we show that it is closed. Let p be a limit point of M_x and let $\{B_n\}$ be a countable basis at p. For each n, let $x_n \in (B_n \cap M_x)$. For each n, select a separable subcontinuum M_n in X such that M_n contains x_n and x. Then $Cl(\cup M_n)$ is a separable continuum in X containing both x and p. Therefore, $p \in M_x$ and M_x is closed.

Now, suppose G is not upper semi-continuous. Then there exists $g \in G$, $x \in g$, a countable basis $\{B_n\}$ at x, an open set U such that $g \subset U$, and a sequence $\{h_n\}$ of mutually disjoint separable subcontinua of X such that

i) $h_n \cap g = \emptyset$ for all n,

ii) $h_n \cap B_n \neq \emptyset$ for each n, and

iii) $h_n \cap (X - U) \neq \emptyset$ for each n.

For each n, select $x_n \in (B_n \cap h_n)$ and $u_n \in (h_n \cap \mathrm{Bd}(U))$. Then $x_n \to x$ and $u_n \to u$ for some $u \in \mathrm{Bd}(U)$. Then the continuum of convergence h of $\{h_n\}$ is a metrizable continuum containing x and u. Therefore, $u \in g$ which is a contradiction.

We finally show that X/G is hereditarily locally connected. Simone has shown [14] that a continuum is hereditarily locally connected if and only if it contains no continuum of convergence. We then assume that there exists a sequence $\{C_n\}_{n=1}^{\infty}$ of mutually exclusive non-degenerate continua with continuum of convergence C. Let $\phi : X \to X/G$ denote the natural map. Set $D_n = \phi^{-1}(C_n)$ for each n; each D_n is a subcontinuum of X by the confluence of ϕ. By the aforementioned result of Moore, there exists a subsequence $\{D_k\}_{k=1}^{\infty}$ of $\{D_n\}_{n=1}^{\infty}$ so that $D = \lim D_k$ is a metrizable continuum M such that C is contained in $\phi(M)$, contradicting the maximality of the points of C. Therefore, X/G is hereditarily locally connected and is an IOC by Nikiel [11]. □

We note that, with X and G as in the previous theorem, if X/G is separable then X/G is metrizable by Treybig [17]. The collection

N of non-degenerate elements of G is then dense. If not, there exist connected open sets U and V in X such that $U \subseteq Cl_X(U) \subseteq V$, $\phi(Cl_X(U)) = Cl_{X/G}(U)$, $Cl_X(U)$ is a non-degenerate metric continuum, and $V \cap (\cup N) = \emptyset$. This involves a contradiction since $Cl_X(U)$ is then contained in a single non-degenerate element of G.

With the aid of Theorem 9 of [2], the following is immediate.

Corollary 4.3. *Let X and G be as in the previous theorem. Suppose further that there exists a finite-to-one map of a first countable arc onto X/G and $Bd(g)$ is totally disconnected for each $g \in G$. Then X is an IOC if and only if and each $g \in G$ is an IOK.*

5. Concluding Remarks

Theorem 4.1 and Corollary 4.3 above motivate the following natural question. Is an MCC continuum necessarily an IOK? Consider again the long line L and let X denote the subspace of $Y = L \times [0, 1]$ such that $X = L \cup (\Omega \times [0, 1])$. Then X clearly is MCC but X is not an IOK by a result of Treybig [16].

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