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THE CONNECTIVITY STRUCTURE OF THE HYPERSPACES $C_\epsilon(X)$

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Abstract. This paper reports the first results of the investigation of the hyperspaces $C_\epsilon(X)$ of a continuum X. We focus our attention on the connectivity structure of $C_\epsilon(X)$, proving results regarding aposyndesis, local connectedness, arcwise connectedness, cyclic connectedness, and cut points.

1. Introduction

A hyperspace of a set, S, is a specified collection of nonempty closed subsets of S with the Vietoris topology [2, p. 3]. The study of hyperspaces began in the early 20th century and continues to be an area of active research; a brief history of hyperspace theory can be found in S. B. Nadler's standard 1978 reference on the subject [1].

A variety of hyperspaces have been investigated. For compact, connected Hausdorff spaces X, the most extensively studied hyperspaces are those whose points are the compact subsets of X and those whose points are the subcontinua of X. Numerous results regarding these and many other hyperspaces can be found in [1] and [2]. The purpose of this article is to investigate a class of hyperspaces that was defined in [7], but which has not been independently studied. Informally, the hyperspaces of this investigation are those whose points are continua with arbitrarily small diameter. We think of these hyperspaces as being approximations of their

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underlying sets in much the same way that the mark of a pencil on paper approximates a point. This notion of approximation was the initial motivation for our study.

A *compactum* is a nonempty compact metric space, and a continuum is a connected compactum. If Z is a compactum, then $C(Z)$ denotes the space of all subcontinua of Z with the Hausdorff metric. We recall that for any compactum Z, an *order arc* α is a subset of $C(Z)$ for which there exists a homeomorphism $h : [0,1] \to \alpha$ with the property that $a < b \implies h(a) \subset h(b)$. We will refer to $h(0)$ and $h(1)$ as the *endpoints* of α and recall that if X is a continuum and $A \subset B \subset X$, then there exists an order arc in $C(X)$ with endpoints A and B. For convenience, we will use $[A, B]$ to denote an order arc with endpoints A and B, and if $A \subset C \subset B$, we will use $[A, C, B]$ to denote an order arc with endpoints A and B that contains C.

For any metric space (Z, ρ), we will let $\mathrm{diam}_\rho : C(Z) \to [0, \infty)$ denote the diameter mapping on $C(Z)$. We will write diam for diam_ρ when the metric on Z is understood.

We are now prepared to define the object of our investigation.

Definition 1.1. Let Z be a compactum with metric ρ. For any $\epsilon \geq 0$, let

$$C_{\rho,\epsilon}(Z) = \{A \in C(Z) : \mathrm{diam}_\rho(A) \leq \epsilon\},$$

where $\mathrm{diam}_\rho(A) = \sup\{\rho(x, y) : x, y \in A\}$. We will write $C_\epsilon(Z)$ when the metric on Z is understood.

We note that $C_0(Z)$ is the space of singleton subsets of Z; $C_0(Z)$ is easily seen to be homeomorphic to Z.

We also remark that $C_\epsilon(Z) = \mathrm{diam}^{-1}[0, \epsilon]$ for each $\epsilon > 0$. This observation is of great importance to us because the diameter map is continuous on $C(Z)$ for every metric space (Z, ρ) [6, p. 55]. This fact will be used in many of the results throughout this paper.

2. Preliminary Results

Proposition 2.1. *$C_\epsilon(Z)$ is compact for every compactum Z.*

Proof: Since $C_\epsilon(Z) = \mathrm{diam}^{-1}[0, \epsilon]$ and $[0, \epsilon]$ is a closed subset of $[0, \infty)$, it follows from the continuity of diam that $C_\epsilon(Z)$ is a closed subset of $C(Z)$. Thus, since $C(Z)$ is compact, it follows that $C_\epsilon(Z)$ is compact. □

Proposition 2.2. *$C_\epsilon(X)$ is connected for every continuum X.*

Proof: Let $\mathcal{A}$ denote the union over the collection of all order arcs of the form $[\{p\}, E]$ where $p \in X$ and $\mathrm{diam}(E) = \epsilon$. Then $\mathrm{diam}^{-1}[0, \epsilon] = \mathcal{A}$, for if $A \subseteq X$ with $\mathrm{diam}(A) \leq \epsilon$, and $p \in A \subseteq E$ where $\mathrm{diam}(E) = \epsilon$, then there exists an order arc in $C(X)$ of the form $[\{p\}, A, E]$. Thus, $A \in \mathcal{A}$. To see the converse, note that if $A \in \mathcal{A}$, then $\mathrm{diam}(A)$ is clearly less than or equal to ϵ. Thus, $A \in C_\epsilon(X)$. So, since $\mathcal{A}$ is the union over a collection whose members are each connected and intersect the collection of singleton subsets of X, it follows that $\mathcal{A}$ and, hence, $\mathrm{diam}^{-1}[0, \epsilon]$ are connected. □

Corollary 2.3. *$C_\epsilon(X)$ is a continuum for every continuum X.*

Proof: This result follows immediately from propositions 2.1 and 2.2. □

The following result appears as an exercise in [1], but does not seem to appear elsewhere in the literature. Because of its importance to the results of this paper, we include a proof for completeness.

Proposition 2.4. *The diameter map on $C(X)$ for any continuum X is a monotone map.*

Proof: By arguments similar to those given in propositions 2.1 and 2.2, it follows that $\mathrm{diam}^{-1}[\epsilon, \mathrm{diam}(X)]$ is a continuum. Thus, $\mathrm{diam}^{-1}[0, \epsilon]$ and $\mathrm{diam}^{-1}[\epsilon, \mathrm{diam}(X)]$ are subcontinua of $C(X)$ whose union is $C(X)$. Therefore, $\mathrm{diam}^{-1}(\epsilon)$ is connected by the unicoherence of $C(X)$ [1, 1.176]. □

Corollary 2.5. *For any continuum X and any $0 \leq \epsilon \leq \mathrm{diam}(X)$, $\mathrm{diam}^{-1}(\epsilon)$ is a continuum.*

Proof: Note first that for any $p \in X$, we have that $\{p\} \in \mathrm{diam}^{-1}(0)$. Moreover, $X \in \mathrm{diam}^{-1}(\mathrm{diam}(X))$. Thus, $\mathrm{diam}^{-1}(\epsilon)$ is nonempty for $\epsilon = 0$ and $\epsilon = \mathrm{diam}(X)$. Therefore, for any $0 < \epsilon < \mathrm{diam}(X)$, we must have that $\mathrm{diam}^{-1}(\epsilon) \neq \emptyset$, for otherwise $\mathrm{diam}^{-1}([0, \epsilon))$ and $\mathrm{diam}^{-1}((\epsilon, \mathrm{diam}(X)])$ would be nonempty, open subsets of $C(X)$ whose union was $C(X)$. Thus, $\mathrm{diam}^{-1}(\epsilon)$ is nonempty for all $0 \leq \epsilon \leq \mathrm{diam}(X)$. By the continuity of the diameter map, we have that $\mathrm{diam}^{-1}(\epsilon)$ is compact. Morever, $\mathrm{diam}^{-1}(\epsilon)$ is connected by Proposition 2.4. Therefore, $\mathrm{diam}^{-1}(\epsilon)$ is a continuum. □

3. Results Regarding the Connectivity Structure of $C_\epsilon(X)$

Proposition 3.1. *A continuum X is locally connected if and only if $C_\epsilon(X)$ is locally connected for every $\epsilon > 0$.*

Proof: Let X be a continuum. We will prove the proposition by showing that for any $\epsilon > 0$, X is c.i.k. at each of its points if and only if $C_\epsilon(X)$ is also. Suppose first that X is c.i.k. at every point. Let $A \in C_\epsilon(X)$, and suppose that $C_\epsilon(X)$ is not c.i.k. at A. It follows that for some $\delta > 0$, there is no closed, connected neighborhood of A in $C_\epsilon(X)$ of diameter $\leq \delta$. Note that since X is c.i.k. at each point of A, and since A is compact, we can cover A with finitely many closed, connected neighborhoods in X of diameter less than δ, say $K_1, K_2, \cdots, K_n$, such that A belongs to the interior of $\langle K_1, K_2, \cdots, K_n\rangle \subseteq C(X)$. Let $\mathcal{K}_1 = \langle K_1, K_2, \cdots, K_n\rangle$ and observe that since $A \cap K_i \neq \emptyset$ and K_i is a continuum for each $i = 1, 2, \cdots, n$, we have that $\bigcup_{i=1}^n K_i \in C(X)$; thus, any two members of $\mathcal{K}_1$ can be joined by an arc in $\mathcal{K}_1$ through $\bigcup_{i=1}^n K_i$. Therefore, $\mathcal{K}_1$ is a subcontinuum of $C(X)$ with $\operatorname{diam}(\mathcal{K}_1) < \delta$ that contains A in its interior. Hence, there exist two nonempty, closed, mutually separated subsets, $\mathcal{E}_1$ and $\mathcal{F}_1$, of $\mathcal{K}_1 \cap C_\epsilon(X)$ such that $\mathcal{K}_1 \cap C_\epsilon(X) = \mathcal{E}_1 \cup \mathcal{F}_1$. We will assume without loss of generality that $A \in \mathcal{E}_1$.

Repeat the above procedure for each $i = 2, 3, \cdots$ to construct a nested sequence of continua, $\mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \cdots$, such that $\operatorname{diam}(\mathcal{K}_i) < \delta i^{-1}$ and A belongs to the interior of each $\mathcal{K}_i$. For each $i = 2, 3, \cdots$, define $\mathcal{E}_i = \mathcal{E}_{i-1} \cap \mathcal{K}_i$ and $\mathcal{F}_i = \mathcal{F}_{i-1} \cap \mathcal{K}_i$. Then $\mathcal{K}_i \cap C_\epsilon(X) = \mathcal{E}_i \cup \mathcal{F}_i$, and $A \in \mathcal{E}_i$ for every $i = 1, 2, \cdots$. Note that since $\mathcal{E}_i$ and $\mathcal{F}_i$ are mutually separated for every i, j, we have that $\bigcap_{i=1}^\infty (\mathcal{E}_i \cup \mathcal{F}_i) = (\bigcap_{i=1}^\infty \mathcal{E}_i) \cup (\bigcap_{i=1}^\infty \mathcal{F}_i)$. Hence, we have that

$$\{A\} = \bigcap_{i=1}^\infty (\mathcal{K}_i \cap C_\epsilon(X)) = \bigcap_{i=1}^\infty (\mathcal{E}_i \cup \mathcal{F}_i) = \bigcap_{i=1}^\infty \mathcal{E}_i \cup \bigcap_{i=1}^\infty \mathcal{F}_i = \{A\} \cup \bigcap_{i=1}^\infty \mathcal{F}_i.$$

However, since each $\mathcal{F}_i$ is a subcontinuum of $C_\epsilon(X)$, it follows that $\bigcap_{i=1}^\infty \mathcal{F}_i \neq \emptyset$. This is a contradiction since $A \notin \mathcal{F}_i$ for all i. Therefore, we must have that $C_\epsilon(X)$ is c.i.k. at every $A \in C_\epsilon(X)$. Thus, $C_\epsilon(X)$ is locally connected.

Now suppose that $C_\epsilon(X)$ is locally connected, and let $p \in X$. It follows that $C_\epsilon(X)$ is c.i.k. at $\{p\}$. Thus, if $\delta > 0$ and U is an open subset of X with $p \in U$ and $\operatorname{diam}(U) < \delta$, then $\langle U\rangle$ contains a closed connected neighborhood $\mathcal{V} \subseteq \langle U \rangle$ that contains $\{p\}$ in its interior. Thus, $\bigcup \mathcal{V}$ is a closed connected neighborhood in X [1, 1.43] that contains p in its interior and, since $\mathcal{V} \subseteq \langle U \rangle$, we have that $\bigcup \mathcal{V} \subseteq U$. Hence, $\text{diam}(\bigcup \mathcal{V}) < \delta$. Therefore, X is c.i.k. at every $p \in X$ and so X is locally connected. $\square$

In [3], J. T. Goodykoontz investigates the aposyndetic properties of $C(X)$ and 2^X using techniques that appeal to the unicoherence of these hyperspaces. In particular, he uses the unicoherence of $C(X)$ to prove that $C(X)$ is countable closed set aposyndetic. In Theorem 3.2, we generalize Goodykoontz's result that $C(X)$ is aposyndetic by proving that $C_\epsilon(X)$ is aposyndetic for every $\epsilon > 0$. Because $C_\epsilon(X)$ is not necessarily unicoherent, our proof is necessarily different from that given in [3], being more reminiscent of the classic proof by F. B. Jones that the product of any two nondegenerate continua is aposyndetic [4, Theorem 7].

Theorem 3.2. *Let X be a continuum. Then $C_\epsilon(X)$ is aposyndetic for every $\epsilon > 0$.*

Proof: Let $0 < \epsilon \leq \text{diam}(X)$, and let $A, B \in C_\epsilon(X)$ with $A \neq B$. We will show that $C_\epsilon(X)$ is aposyndetic at A with respect to B.

If $A \subset B$, let U be an open subset of X such that $A \subseteq U$ and $B \not\subseteq \overline{U}$. Let $\mathcal{U} = \langle U \rangle \cap C_\epsilon(X)$. Clearly, $B \notin \overline{\mathcal{U}}$. For each $E \in \overline{\mathcal{U}}$, let $p_E \in E$ and let α_E be an order arc of the form $[\{p_E\}, E]$. It is easy to see that $\overline{\mathcal{U}} = \bigcup_{E \in \overline{\mathcal{U}}} \alpha_E$ since each $\alpha_E \subseteq \overline{\mathcal{U}}$. Therefore, since $C_0(X)$ is a subcontinuum of $C_\epsilon(X)$ that meets each α_E and $\text{diam}(B) > 0$, it follows that $\overline{\mathcal{U}} \cup C_0(X)$ is a subcontinuum of $C_\epsilon(X) - \{B\}$ that contains A in its interior.

If $A \not\subset B$, let U be an open subset of X such that $U \cap A \neq \emptyset$ and $\overline{U} \cap B = \emptyset$. If $A \subset U$, define $\mathcal{U} = \langle U \rangle \cap C_\epsilon(X)$; otherwise, let V be an open subset of X such that $A \subset U \cup V$ and $A \not\subset V$, and define $\mathcal{U} = \langle U, V \rangle \cap C_\epsilon(X)$. For each $E \in \overline{\mathcal{U}}$, let $p_E \in E \cap \overline{U}$, let E^* be a member of $\text{diam}^{-1}(\epsilon)$ that contains E, and define α_E to be an order arc of the form $[\{p_E\}, E, E^*]$. Clearly, we have that $A \in \overline{\bigcup_{E \in \overline{\mathcal{U}}} \alpha_E}$. Furthermore, since every element of $\overline{\bigcup_{E \in \overline{\mathcal{U}}} \alpha_E}$ is a subset of X that meets $\overline{U}$, it follows that $B \notin \overline{\bigcup_{E \in \overline{\mathcal{U}}} \alpha_E}$. Thus, if $0 \leq \lambda \leq \epsilon$ with $\lambda \neq \text{diam}(B)$, it follows that $\overline{\bigcup_{E \in \overline{\mathcal{U}}} \alpha_E} \cup \text{diam}^{-1}(\lambda)$ is a subcontinuum of $C_\epsilon(X) - \{B\}$ that contains A in its interior. $\square$

Question 3.3. For what continua X is it true that $C_\epsilon(X)$ is countable closed set aposyndetic for all $\epsilon > 0$?

Corollary 3.4. *Let X be a continuum. Then $C_\epsilon(X)$ is decomposable for every $\epsilon > 0$.*

Proof: We have from [4, Theorem 8] that a continuum is decomposable if and only if it contains two points for which it is aposyndetic at one with respect to the other. Therefore, the corollary follows from Theorem 3.2. □

Proposition 3.5. *If Z is an arcwise connected compactum, then $C_\epsilon(Z)$ is arcwise connected for all $\epsilon \geq 0$.*

Proof: Let $A, B \in C_\epsilon(Z)$ and select any $a \in A$ and $b \in B$. Since $C(A)$ and $C(B)$ are arcwise connected, there exist arcs α and β from A to $\{a\}$ and $\{b\}$ to B, respectively. Furthermore, there is an arc, γ, from $\{a\}$ to $\{b\}$ in $C_0(Z)$ since Z is arcwise connected. Since $\alpha \cup \gamma \cup \beta$ contains an arc from A to B, it follows that $C_\epsilon(Z)$ is arcwise connected. □

Example 3.6. We now proceed to constuct a continuum X in $\mathbf{R}^3$ that fails to be arcwise connected, but for which $C_\epsilon(X)$ is arcwise connected for every $\epsilon > 0$. Let $S_0 = \{(x, 0, \sin\frac{1}{x}) : 0 < x \leq 1\}$, and define S_k for each $k = 1, 2, \cdots$ as follows:

$$S_k = \{(x, \frac{x}{k}, \sin\frac{1}{x}) : \frac{1}{k\pi} \leq x \leq 1\} \cup \{(x, \frac{x}{k}, \frac{1}{k}\sin\frac{1}{x}) : 0 < x \leq \frac{1}{k\pi}\}.$$

Let $I = \{(0, 0, z) : -1 \leq z \leq 1\}$ and $W = \cup_{k=0}^{\infty} S_k$. Then, let C be an arc in $\mathbf{R}^3$ such that $C \cap (W \cup I) = \{(1, 0, \sin 1)\} \cup \{(1, \frac{1}{k}, \sin 1) : k = 1, 2, \cdots\}$. Define $X = W \cup I \cup C$.

Clearly, X fails to be arcwise connected since it contains no arc from points of I to points of $W \cup C$. However, I and $W \cup C$ are each arcwise connected and so we have from Proposition 3.5 that both $C_\epsilon(I)$ and $C_\epsilon(W \cup C)$ are arcwise connected for every $\epsilon \geq 0$. Let $\epsilon > 0$. From the discussion above, it is enough to show that for any $A, B \in C_\epsilon(X)$ with $A \cap I \neq \emptyset$ and $B \cap (W \cup C) \neq \emptyset$, there exists an arc in $C_\epsilon(X)$ from A to B. To see this, choose a positive integer k for which $1/k < \epsilon/2$; then, let $J = \{(0, 0, z) : -\frac{1}{k} \leq z \leq \frac{1}{k}\}$ and observe that J is the remainder of S_k in I. Let $a \in A \cap I$ and use the arcwise connectedness of $C(A)$ to choose an arc α_1 from A to $\{a\}$. Then, use the arcwise connectedness of $C_\epsilon(I)$ to choose an arc α_2 from $\{a\}$ to J. Now, since $C(S_k \cup J)$ is arcwise connected and $\mathrm{diam}(J) < \epsilon$, there is an arc α_3 in $C_\epsilon(S_k \cup J)$ from J to some $Y \in C_\epsilon(S_k)$. Let $b \in B$, and use the arcwise connectedness of $C_\epsilon(W \cup C)$ to choose an arc α_4 from Y to $\{b\}$. Then, use the arcwise connectedness of $C(B)$ to choose an arc α_5 from $\{b\}$ to B. It follows that $\cup_{i=1}^5 \alpha_i$ contains an arc in $C_\epsilon(X)$ from A to B. Therefore, $C_\epsilon(X)$ is arcwise connected.

Theorem 3.7. *Every point of $C_\epsilon(X)$ is a non-cut point of $C_\epsilon(X)$ for all $\epsilon > 0$.*

Proof: Let $A \in C_\epsilon(X)$. We will show that $C_\epsilon(X) - \{A\}$ is connected by finding a connected set $\mathcal{K} \subset C_\epsilon(X) - \{A\}$, with the property that every member of $C_\epsilon(X) - \{A\}$ belongs to a connected subset of $C_\epsilon(X) - \{A\}$ that meets $\mathcal{K}$.

If $\mathrm{diam}(A) = 0$, let $\mathcal{K} = \mathrm{diam}^{-1}(\epsilon)$. Then $\mathcal{K}$ is connected by Corollary 2.5. For each $E \in C_\epsilon(X) - \{A\}$, let $E^* \in \mathcal{K}$ such that $E \subseteq E^*$, and let α_E be an order arc of the form $[E, E^*]$. Then $E \in \alpha_E \subset C_\epsilon(X) - \{A\}$ and $\alpha_E \cap \mathcal{K} \neq \emptyset$.

If $\mathrm{diam}(A) > 0$, choose $0 < \lambda < \mathrm{diam}(A)$ and let $\mathcal{K} = \mathrm{diam}^{-1}(\lambda)$. Then $\mathcal{K}$ is connected by Corollary 2.5. If $\mathrm{diam}(E) < \lambda$, let $E^* \in \mathcal{K}$ such that $E \subset E^*$ and define α_E to be an order arc of the form $[E, E^*]$. If $\mathrm{diam}(E) \geq \lambda$, then define α_E as follows: If $E \not\subset A$, let $p \in E - A$; otherwise, let p be any member of E. Let α_E be an order arc of the form $[\{p\}, E]$. Then, $E \in \alpha_E \subset C_\epsilon(X) - \{A\}$ and $\alpha_E \cap \mathcal{K} \neq \emptyset$. □

Question 3.8. Is $C_\epsilon(X) - \mathcal{B}$ always connected when $\mathcal{B} \subset C_\epsilon(X)$ is countable?

Remark 3.9. We do not know the answer to 3.8 even in the case when $\mathcal{B}$ is nondegenerate and finite. We point out that in [5], J. Krasinkiewicz proves that no zero-dimensional subset of $C(X)$ separates $C(X)$. His proof uses the fact that every map from $C(X)$ to S^1 is homotopic to a constant, a property which $C_\epsilon(X)$ does not necessarily have (e.g., $C_\epsilon(S^1)$ with $\epsilon < 2$). Therefore, we ask the following more general question:

Question 3.10. Is $C_\epsilon(X) - \mathcal{B}$ always connected when $\mathcal{B}$ is zero-dimensional?

We recall that a space S is said to be *cyclicly connected* provided that every two points of S belong to some simple closed curve in S [8, p. 77]. G. T. Whyburn proves in [8] that a locally connected continuum is cyclicly connected if and only if it has no cut point. Thus, the proposition below follows immediately from Proposition 3.1 and Theorem 3.7:

Theorem 3.11. *If X is a locally connected continuum, then $C_\epsilon(X)$ is cyclicly connected for every $\epsilon > 0$.*

Example 3.12. Let $S = \{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$, $I = \{0\} \times [-1, 1]$, and $X = S \cup I$. Choose any $\epsilon < 2$. Then, for any $A \in C_\epsilon(I)$ and $B \in C_\epsilon(S)$, no arc exists in $C_\epsilon(X)$ between A and B. Therefore, $C_\epsilon(X)$ fails to be arcwise connected and, hence, fails to be cyclicly connected.

Definition 3.13. We define a topological space S to be *weakly cyclicly connected* provided it is true that if x, y, and z are three points of S, then S contains a continuum K containing x and y but not z.

Example 3.14. It is easy to see that any cyclicly connected continuum is weakly cyclicly connected. However, if X denotes the $\sin\frac{1}{x}$ continuum defined in Example 3.12, then $X \times [0, 1]$ is a weakly cyclicly connected continuum that is not cyclicly connected. An example of such a continuum in the plane is described in Example 3.17.

By a *chain* in a space S, we will mean a set $\mathcal{C} = \{C_1, C_2, \cdots, C_n\}$ of closed connected neighborhoods in S for which $C_i \cap C_{i+1} \neq \emptyset$ for each $i = 1, \cdots, n-1$. Each member of $\mathcal{C}$ is called a *link* of $\mathcal{C}$. We say that a chain in S *joins a point x to a point y* provided that C_1 is a neighborhood of x and C_n is a neighborhood of y. We note if $\mathcal{C}$ is a chain in a compact set, then $\bigcup \mathcal{C}$ is a continuum.

Proposition 3.15. *If X is a nondegenerate, aposyndetic continuum that contains no cut point, then X is weakly cyclicly connected.*

Proof: Suppose that x, y, and z are three points of such a continuum X. Then each point of $X - \{z\}$ belongs to a closed connected neighborhood which misses z. Let A denote the set of all points of $X - \{z\}$ that can be joined to x by a chain of closed connected neighborhoods whose links are contained in $X - \{z\}$. Then A is closed in $X - \{z\}$. To see this, suppose that a point $\{p\}$ of $X - \{z\}$ is a limit point of A, and let C be a closed connected neighborhood of p that misses z. Then the interior of C contains a point q of A, and there is a chain $\{C_1, \cdots, C_k\}$ in $X - \{z\}$ joining x to q. Since $q \in C_k \cap C$ and p is in the interior of C, we have that $\{C_1, \cdots, C_k, C\}$ is a chain in $X - \{z\}$ that joins x to p. Hence, $p \in A$.

Now, suppose that $y \notin A$ and let $B = (X - \{z\}) - A$. We will show that no point of A is a limit point of B. For purpose of contradiction, suppose that there is some $a \in A$ such that a is a limit point of B, and let $\{C_1, \cdots, C_n\}$ be a chain in $X - \{z\}$ that joins x to a. Then C_n is a neighborhood of some $b \in B$. However, this implies that $\{C_1, \cdots, C_n\}$ joins a to b. This means that b is a member of A, contradicting the definition of B. Therefore, A and B are nonempty, mutually separated sets for which $X - \{z\} = A \cup B$, contradicting the assumption that X has no cut point. Hence, X is weakly cyclicly connected. □

Corollary 3.16. *If X is a nondegenerate continuum, then $C_\epsilon(X)$ is weakly cyclicly connected for every $\epsilon > 0$.*

Proof: This result follows immediately from theorems 3.2 and 3.7 and Proposition 3.15. □

Example 3.17. We note that an arc is an aposyndetic continuum that is not weakly cyclicly connected. We now construct a planar continuum that is weakly cyclicly connected and has no cut point, but which is neither cyclicly connected nor aposyndetic: Let $R = [-1, 0] \times [-2, 1]$, and let W be a copy of the Warsaw circle with limit arc $\{0\} \times [-1, 1]$ such that $W \cap R = \{0\} \times [-2, 1]$. Then $W \cup R$ is a continuum with the desired properties.

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