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THE HYPERSPACES $C(p, X)$

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Abstract. Let $C(X)$ denote the hyperspace of subcontinua of a continuum X. For $p \in X$, define the hyperspaces $C(p, X) = \{A \in C(X) : p \in A\}$ and $\mathcal{K}(X) = \{C(p, X) : p \in X\}$. Let I denote the unit interval. The class of continua X for which $\mathcal{K}(X)$ coincides with $\mathcal{K}(I)$ (the class of the so-called arc-similar continua) is characterized as the class of continua having two end-points and arcs as proper nondegenerate subcontinua. Other classes of continua are characterized as well, in terms of the hyperspaces $\mathcal{K}(X)$.

1. Introduction

Throughout this paper $C(X)$ will denote the hyperspace of subcontinua of a continuum X equipped with the Hausdorff metric (see definitions 1.6 and 2.1 in [5]). Also, for $D \in C(X)$ define the hyperspace $C(D, X) = \{A \in C(X) : D \subset A\}$. For convenience, we shall denote $C(\{p\}, X)$ simply by $C(p, X)$. Finally, we define $\mathcal{K}(X) = \{C(p, X) : p \in X\}$.

The hyperspace $C(X)$ has been largely studied and now we know that it is extremely useful in the study of continuum theory; more precisely, several properties of a continuum X can be determined in terms of the topological properties of $C(X)$, and vice versa. For more information on this subject we refer the reader to [5]. Following this idea, the aim of this paper is to investigate and present

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some relations between topological properties of a continuum X and those of its hyperspaces $C(p, X)$.

The hyperspaces $C(p, X)$ have not been largely investigated. Nevertheless, there are some known results about them; among the most important is that they are absolute retracts (see [3, Theorem 2]).

One can also find in the literature conditions under which $C(p, X)$ is a Hilbert cube (see [1] and [3, theorems 4, 6, 8]).

In this paper we study relations between some particular continua X and their hyperspaces $\mathcal{K}(X)$. Several examples and counterexamples are also given.

First, we give a characterization of the class of continua X for which $\mathcal{K}(X)$ coincides with $\mathcal{K}(I)$, where I denotes the unit interval. We call it the class of *arc-similar* continua, and we characterize it as the class of continua having two end-points and arcs as proper nondegenerate subcontinua.

Next, we describe the class of continua X for which $\mathcal{K}(X)$ coincides with $\mathcal{K}(S)$, where S is a simple closed curve. This class is determined as the class of continua having arcs as proper nondegenerate subcontinua and no end-points. Finally, a particular class of continua (class $\mathcal{P}$) is characterized as well, in terms of the hyperspaces $\mathcal{K}(X)$.

2. Preliminaries

In this paper, a *continuum* means a compact, connected metric space, and a *mapping* means a continuous function. We denote by I the unit interval, by $\mathbb{N}$ the set of all positive integers, by $\mathbb{C}$ the set of all complex numbers (equipped with the natural topology), and by S^1 the unit circle, i.e., $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Further, for a continuum X, and $A \subset B \subset X$, we denote by $\mathrm{cl}_B(A)$, $\mathrm{int}_B(A)$, $\mathrm{ext}_B(A)$, and $\mathrm{bd}_B(A)$ the closure, the interior, the exterior, and the boundary of A with respect to B. In case $B = X$, we shall simply omit the subindex. Also, $\dim(X)$ will denote the dimension of the continuum X, and $\mathrm{diam}(X)$, its diameter. Finally, if the continuum X has a metric d, $x \in X$, and A is a closed subset of X, let $d(x, A) = \inf\{d(x, a) : a \in A\}$. Moreover, $N(\varepsilon, A)$ denotes the set $\{x \in X : d(x, A) < \varepsilon\}$.

Let $A, B \in C(X)$. An *order arc* from A to B is a continuous function $\alpha : I \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$, and $\alpha(r) \subsetneq \alpha(s)$ whenever $r < s$ (see [12, 1.2–1.8]).

We also say that an order arc α, from A to B is *unique,* if for any order arc β, from A to B, we have that $\alpha(I) = \beta(I)$.

A *Whitney map for* $C(X)$ is a mapping $\mu : C(X) \to [0, \infty)$ such that $\mu(X) = 1$, $\mu(\{p\}) = 0$ for each $p \in X$, and $\mu(A) < \mu(B)$ whenever $A \subsetneq B$ (see [5, p. 105]).

Similarly, we define a *Whitney map for* $C(p, X)$ as a mapping $\mu : C(p, X) \to [0, \infty)$ such that $\mu(X) = 1$, $\mu(\{p\}) = 0$, and $\mu(A) < \mu(B)$ whenever $A \subsetneq B$.

3. General Properties

Definition 3.1. Let X, Y be continua and $f : X \to Y$ a mapping. The *induced mapping* $C(f) : C(X) \to C(Y)$ is given by $C(f)(A) = f(A)$.

To know more about these mappings we refer the reader to the paper [4].

Definition 3.2. Let X, Y be continua and $f : X \to Y$ a mapping. f is said to be *confluent* provided that for any $B \in C(Y)$ and any component A of $f^{-1}(B)$ we have that $f(A) = B$.

Lemma 3.3. *Let X and Y be continua. A mapping $f : X \to Y$ is confluent if and only if for any $p \in X$, $C(f)\big(C(p, X)\big) = C(f(p), Y)$.*

Proof: It is not difficult to see that $C(f)\big(C(p, X)\big) \subset C(f(p), Y)$. Now, if f is confluent, $B \in C(f(p), Y)$, and A is the component of $f^{-1}(B)$ containing p, then $A \in C(p, X)$ and $f(A) = B$. We therefore obtain that $C(f(p), Y) \subset f\big(C(p, X)\big)$. Hence, $C(f)\big(C(p, X)\big) = C(f(p), Y)$.

Conversely, if f is not confluent, there exist $B \in C(Y)$ and a component A of $f^{-1}(B)$ such that $f(A) \subsetneq B$. Let $p \in A$. If there exists $D \in C(p, X)$ with $f(D) = B$, then $A \cup D \in C(p, X)$, so we have that $A \cup D$ is a connected set contained in the component A of $f^{-1}(B)$, and is such that $B = f(A \cup D)$. This contradicts the choice of A. Hence, $C(f)\big(C(p, X)\big) \subsetneq C(f(p), Y)$. □

Note that $C(p, X)$ is a closed subset of $C(X)$, so it is compact. Also, $C(p, X)$ is arcwise connected, so the hyperspace $C(p, X)$ is a continuum too.

The following is a natural result.

Lemma 3.4. *Let X and Y be continua and let $h : X \to Y$ be a homeomorphism. Then $C(p, X) \approx C\big(h(p), Y\big)$.*

Recall that a *cut point* of a topological space X is a point $p \in X$ such that $X \setminus \{p\}$ is not connected. The following result is related to the main theorem in [6].

Lemma 3.5. *Let X be a continuum and let $p \in X$. Then neither $\{p\}$ nor X is a cut point of $C(p, X)$.*

Proof: Let $A, B \in C(p, X) \setminus \big\{X, \{p\}\big\}$. Taking order arcs from $\{p\}$ to A and $\{p\}$ to B, it is easy to see that $C(p, X) \setminus \{X\}$ is arcwise connected. Similarly, $C(p, X) \setminus \big\{\{p\}\big\}$ is arcwise connected. □

Definition 3.6. Let X be a continuum, let $p \in X$, and let $A \in C(p, X)$. We say that A is *terminal at p* if for each $B \in C(p, X)$ we have that either $A \subset B$ or $B \subset A$. We say that A is *terminal* provided it is terminal at a for every $a \in A$.

Lemma 3.7. *Let X be a continuum and let $p \in X$. Suppose $A \in C(p, X)$ is such that $\{p\} \subsetneq A \subsetneq X$. Then A is terminal at p if and only if A is a cut point of $C(p, X)$.*

Proof: Suppose A is terminal at p and consider the following sets: $\mathcal{A} = \{B \in C(p, X) : B \subset A\}$ and $\mathcal{B} = \{B \in C(p, X) : A \subset B\}$. Then both $\mathcal{A}$ and $\mathcal{B}$ are closed, $\mathcal{A} \cap \mathcal{B} = \{A\}$, and $\mathcal{A} \setminus \{A\} \neq \emptyset \neq \mathcal{B} \setminus \{A\}$. Moreover, since A is terminal at p, we get $\mathcal{A} \cup \mathcal{B} = C(p, X)$. Therefore, A is a cut point of $C(p, X)$.

Conversely, if A is not terminal at p, then there exists $K \in C(p, X)$ such that $K \setminus A \neq \emptyset$ and $A \setminus K \neq \emptyset$. Let $\mathcal{A} = \{B \in C(p, X) : A \not\subset B\}$. Thus, $K \in \mathcal{A}$. Moreover, $\mathcal{A}$ is arcwise connected: each element of $\mathcal{A}$ can be connected by an order arc with $\{p\}$ in $\mathcal{A}$.

Take now $B \in C(p, X) \setminus \{A\}$, then it is enough to connect B with K by a path contained in $C(p, X) \setminus \{A\}$. Since $\mathcal{A}$ is arcwise connected, we may assume that $B \notin \mathcal{A}$; thus, $A \subsetneq B$. Consider order arcs α and β, from K to X and from B to X, respectively. Then $\alpha \cup \beta$ is a path in $C(p, X) \setminus \{A\}$ joining B and K. □

Lemma 3.8. *Let X be a continuum and let $A \in C(X) \setminus \{X\}$. Suppose $\alpha_1 : I \to C(X)$ and $\alpha_2 : I \to C(X)$ are two order arcs from A to X such that $\alpha_1(I) \neq \alpha_2(I)$. Then there exist $s, t \in I$ in such a way that $\alpha_1(s) \setminus \alpha_2(t) \neq \emptyset$ and $\alpha_2(t) \setminus \alpha_1(s) \neq \emptyset$.*

Proof: Let $\mu : C(X) \to I$ be a Whitney map. Take $s \in I$ such that $\alpha_1(s) \notin \alpha_2(I)$ and let $r' = \mu(\alpha_1(s))$. Consider also $t \in I$ such that $\mu(\alpha_2(t)) = r'$. Then, according to the definition of a Whitney map, neither of the continua $\alpha_1(s)$ nor $\alpha_2(t)$ is contained in the other. □

Lemma 3.9. *Let A' be a subcontinuum of a continuum X. Then $C(A', X)$ is an arc if and only if any two elements of $C(A', X)$ are comparable.*

Proof: If $C(A', X)$ is an arc, then there exists a unique order arc α, from A to X, such that $\alpha(I) = C(A', X)$. Because of the monotoneity of order arcs, we obtain that any two elements of $C(A', X)$ are comparable.

Conversely, let α_1 be an order arc from A' to X. If $C(A', X)$ is not an arc, then there exists $K \in C(A', X) \setminus \alpha_1(I)$. Take now an order arc α_2 from A' to X containing K. Then $\alpha_1(I) \neq \alpha_2(I)$, whence by Lemma 3.8 we can find $s, t \in I$ such that $\alpha_1(s) \setminus \alpha_2(t) \neq \emptyset$ and $\alpha_2(t) \setminus \alpha_1(s) \neq \emptyset$, i.e., $\alpha_1(s)$ and $\alpha_2(t)$ are not comparable. □

Lemma 3.10. *Let X be a continuum and let $A' \in C(X)$ be such that $C(A', X)$ is an arc. If $A, B \in C(A', X)$, then $B \setminus A$ is connected.*

Proof: Suppose $B \setminus A$ is not connected, and consider two components K and K' of $B \setminus A$. Then $A \cup K$ and $A \cup K'$ are connected (see [13, Corollary 5.9]). Moreover, $A \cup K$ and $A \cup K'$ are two noncomparable elements of $C(A', X)$ This contradicts Lemma 3.9. □

The following is an easy result and the proof is left to the reader.

Lemma 3.11. *Let $n \in \mathbb{N}$. Suppose there exist two families of subcontinua $\{K_1, K_2, \dots, K_n\}$ and $\{C_1, C_2, \dots, C_n\}$, such that $K_1, K_2, \dots, K_n$ are pairwise disjoint and $K_i \subset C_i$ for every $i \in \{1, 2, \dots\}$. For each i take an order arc $\alpha_i : I \to C(X)$ from K_i to C_i. Then there exists $\delta > 0$ such that if $|s| \leq \delta$ and $j \neq i$, then $\alpha_i(s) \cap \alpha_j(s) = \emptyset$.*

Lemma 3.12. *Let X be a continuum and let $A' \in C(X)$ be such that $C(A', X)$ is an arc. If $A \in C(A', X) \setminus \{X\}$, then $\mathrm{bd}(A) \in C(X)$.*

Proof: It is enough to prove that $\mathrm{bd}(A)$ is connected.

Suppose that H_1 and H_2 are two distinct components of $\mathrm{bd}(A)$. By Lemma 3.10, $\mathrm{cl}(X \setminus A) \in C(X)$, so for each $i \in \{1, 2\}$ we can take an order arc α_i, from H_i to $\mathrm{cl}(X \setminus A)$. In particular, for each $s \in (0,1]$, $A \cup \alpha_i(s) \in C(X)$ and $\alpha_i(s) \setminus A \neq \emptyset$. Moreover, by Lemma 3.11, there exists $\delta > 0$ such that $\alpha_1(\delta) \cap \alpha_2(\delta) = \emptyset$. We therefore obtain that $A \cup \alpha_1(\delta)$ and $A \cup \alpha_2(\delta)$ are two noncomparable elements of $C(A', X)$. However, this leads to a contradiction with Lemma 3.9. Hence, $\mathrm{bd}(A)$ is connected. □

Definition 3.13. Let $n \in \mathbb{N}$. A continuum Y is an *n-od* if there exists $K \in C(Y)$ such that $Y \setminus K$ has at least n components. Further, we will say that K is a *core* of the n-od. If $n = 3$, Y is called a *triod.*

Definition 3.14. Let $n \in \mathbb{N}$. A continuum X is an *n-cell* provided that X is homeomorphic to I^n.

It is known that if the continuum X contains n-ods, then $C(X)$ contains n-cells (see [14, Theorem 1]). Proceeding in a similar way to that of [14, Theorem 1], it is not difficult to prove the following result.

Lemma 3.15. *Let X be a continuum, let $p \in X$, and let $n \in \mathbb{N}$. If p is contained in the core of an n-od, then $C(p, X)$ contains an n-cell.*

A continuum X is said to be *decomposable* if it can be written as the union of two of its proper subcontinua; otherwise, X is said to be *indecomposable.* X is *hereditarily decomposable (indecomposable)* provided each of its proper, nondegenerate subcontinua is decomposable (indecomposable). Further, the *composant of p in X* is defined by $\Sigma_p = \bigcup\{A \in C(X) \setminus \{X\} : p \in A\}$.

Theorem 3.16. *Let X be a continuum and let $N \in \mathbb{N}$ be such that the set $\{p \in X : C(p, X)$ has cut points $\}$ is at most countable and for each $p \in X$, $\dim\big(C(p, X)\big) < N$. Then every proper and nondegenerate subcontinuum of X is decomposable.*

Proof: Suppose that X has a proper, nondegenerate, indecomposable subcontinuum Y. Then Y has uncountably many composants (see [13, Theorem 11.15]). Let $x_1, \ldots, x_N$ be N points in Y chosen in such a way that they lie in different composants of Y and $C(x_i, X)$ does not have cut points for any $i \in \{1, \ldots, N\}$. Then Y is not a cut point of $C(x_i, X)$ and thus, by Lemma 3.7, for each i we can choose a subcontinuum $K_i \in C(x_i, X)$ such that $Y \setminus K_i \neq \emptyset$

and $K_i \setminus Y \neq \emptyset$. For each i let L_i be the component of $K_i \cap Y$ containing x_i; in particular, $L_i \cap L_j = \emptyset$ whenever $i \neq j$.

For each i let α_i be an order arc from L_i to K_i. Then by Lemma 3.11, there exists $\delta > 0$ such that $\alpha_i(\delta) \cap \alpha_j(\delta) = \emptyset$ whenever $i \neq j$.

Let $Z = Y \cup \bigcup_{i=1}^{N} \alpha_i(\delta)$. It is easy to see that $Z \in C(X)$ and $\alpha_i(\delta) \setminus Y \neq \emptyset$, for each i. On the other hand, by construction $Z \setminus Y$ has at least N components; therefore, Z is an N-od with core Y. However, by Lemma 3.15, $C(p, X)$ contains an N-cell for each $p \in Y$, a contradiction with our hypotheses. □

Remark. A well-known theorem by Mazurkiewicz states that any compact metric space of dimension ≥ 2 contains a nondegenerate indecomposable continuum (see [13, 13.57]). According to this, the continuum X in the previous theorem must be 1-dimensional.

We shall now present some examples which illustrate the structure of some basic hyperspaces $C(p, X)$.

Theorem 3.17. *Let X be an arc with end points a and b. Then $C(p, X)$ is an arc if $p \in \{a, b\}$; otherwise, $C(p, X)$ is a 2-cell.*

Proof: It suffices to prove the case $X = I$. Let $p \in I$.

Case 1. Suppose that $p = 0$ and consider the function $g : I \to C(0, I)$ given by $g(t) = [0, t]$. Then it is not difficult to see that g is a homeomorphism. Therefore, $C(0, I)$ is an arc. Similarly, $C(1, I)$ is an arc.

Case 2. Let $p \in I \setminus \{0, 1\}$ and let a function $g : [0, p] \times [0, 1-p] \to C(p, I)$ be given by $g(r, s) = [p-r, p+s]$. Again, it is not difficult to see that g is a continuous bijection. Hence, $C(p, I)$ is a 2-cell. □

Proceeding in a way similar to that in Theorem 3.17, one can prove the following result.

Theorem 3.18. *If X is a simple closed curve, then $C(p, X)$ is a 2-cell for each $p \in X$.*

Lemma 3.19. *The following conditions are equivalent for a continuum X:*

i) *X is hereditarily indecomposable.*
ii) *$C(p, X)$ is an arc for each $p \in X$.*

Proof: Suppose that there exists $p \in X$ such that $C(p, X)$ is not an arc. Then, according to Lemma 3.9, one can find $K_1, K_2 \in$

$C(p,X)$, such that $K_1 \setminus K_2 \neq \emptyset$ and $K_2 \setminus K_1 \neq \emptyset$. However, $K_1 \cup K_2(t) \in C(X)$, and it is decomposable, so X is not hereditarily indecomposable.

Conversely, suppose that $C(p,X)$ is an arc for each $p \in X$. Let $K \in C(X)$ and assume that $K = A \cup B$ for some $A, B \in C(X)$. Take $p \in A \cap B$. By Lemma 3.9, A and B are comparable; thus, K is indecomposable and, therefore, X is hereditarily indecomposable. □

A question which arises naturally is whether the structure of the hyperspaces $C(p,X)$ characterizes the continuum X. In order to answer this question, we introduce the concept of *arc-similar* continua.

Definition 3.20. Let X be a continuum and let a, b be two distinct points in X. We say that (X, a, b) is *arc-similar* if $C(a,X)$ and $C(b,X)$ are arcs and $C(p,X)$ is a 2-cell, whenever $p \notin \{a, b\}$.

We have seen in Theorem 3.17 that arcs are arc-similar continua; however, the converse is not true, as the following example shows.

Example 3.21. Consider the Knaster continuum X with two end points a and b (see [8, p. 205]). Then, by Lemma 3.9, $C(a,X)$ and $C(b,X)$ are arcs. Proceeding in a similar way as we did in the proof of Theorem 3.17, it is not difficult to see that $C(p,X)$ is a 2-cell if $a \neq p \neq b$. However, for a formal proof of this, see Theorem 5.21.

Nevertheless, the example presented above is a rather complicated continuum. Therefore, a question that arises naturally is whether there exists a decomposable, arc-similar continuum which is not an arc. A natural candidate could be the following:

Example 3.22. Let X be the continuum in Example 3.21, and suppose a and b are the end points of X. Take also an arc A with endpoints c and d. Let Y be the continuum obtained by identifying a and c. It might seem that Y is arc-similar, but it is not, which we show as follows. Observe that the subcontinuum X is terminal at any point p which does not belong to the composant of a in X. Therefore, applying Lemma 3.7, we obtain that $C(p,Y)$ has cut points for uncountably many $p \in Y$. Since 2-cells do not have cut points, we can conclude that Y is not arc-similar.

In Theorem 5.15, we shall actually prove that the arc is the *only* decomposable arc-similar continuum.

4. Main tools

Lemma 4.1. *Let X be a continuum and suppose $n \in \mathbb{N}$ is such that* $\dim(C(p,X)) < n$ *for each $p \in X$. If $A, B \in C(X)$, then both $B \setminus A$ and $A \cap B$ have at most $n-1$ components.*

Proof: If $B \setminus A$ has at least n components, then $A \cup B$ is an n-od with core A. Thus, by Lemma 3.15, $C(p,X)$ contains an n-cell for each $p \in A$, which contradicts our hypotheses. The fact that $A \cap B$ has at most $n-1$ components follows from [11, Theorem 4] □

The following are easy lemmas, and we omit the proofs.

Lemma 4.2. *Let X be a decomposable continuum, say $X = A \cup B$ where A and B are proper subcontinua of X. If K is a component of $A \cap B$, then $K \cap \operatorname{bd}(A) \neq \emptyset$ and $K \cap \operatorname{bd}(B) \neq \emptyset$.*

Lemma 4.3. *Let X be a continuum and let $K \in C(X)$ be such that $K = \operatorname{cl}(U)$ for some open subset U of X. Then $K = \operatorname{cl}\big(\operatorname{int}(K)\big)$.*

Recall that a continuum X is *unicoherent* provided that whenever A and B are subcontinua of X, satisfying $A \cup B = X$, then $A \cap B$ is connected.

Lemma 4.4. *Let X be a non-unicoherent continuum satisfying the condition* $\dim(C(p,X)) < 3$ *for each $p \in X$. Then there exist two proper subcontinua A and B of X such that i)$A \cup B = X$, ii)$A \cap B$ is not connected, iii)$A = \operatorname{cl}\big(\operatorname{int}(A)\big)$, $B = \operatorname{cl}\big(\operatorname{int}(B)\big)$, and iv) $\operatorname{int}(A) = A \setminus B$, $\operatorname{int}(B) = B \setminus A$.*

Proof: Since X is not unicoherent, we can take two proper subcontinua A' and B' of X, such that $A' \cup B' = X$, and $A' \cap B'$ is not connected.

Define $B = \operatorname{cl}(X \setminus A')$ and $A = \operatorname{cl}(X \setminus B)$. Note that A and B are proper subsets of X and that $A \cup B = X$. Moreover, applying Lemma 4.1, we have that $A' \cap B'$ has exactly two components H_1 and H_2.

Claim. $A, B \in C(X)$.

We will first prove that $B \in C(X)$, for which it is enough to show that $X \setminus A'$ is connected.

Suppose $X\setminus A'$ is not connected. Then, by Lemma 4.1, $X\setminus A'$ has exactly two components, one of which–say W– is such that $\mathrm{cl}(W)$ intersects both H_1 and H_2 (see [13, 11.52 (a)]).

Let L_1 and L_2 be components of $\mathrm{cl}(W)\cap H_1$ and $\mathrm{cl}(W)\cap H_2$, respectively, and note that $L_1\cap L_2=\emptyset$. Take now, for each $i\in\{1,2\}$, an order arc α_i from L_i to $\mathrm{cl}(W)$, and $\delta>0$ such that $\alpha_1(\delta)\cap\alpha_2(\delta)=\emptyset$. Let Z be the component of $X\setminus A'$ which is not W; we may assume that $Z\not\subseteq\alpha_1(\delta)\cup\alpha_2(\delta)$. Consider $T=A'\cup\left(\bigcup_{i=1}^{2}\alpha_i(\delta)\right)\cup Z$. Then T is a triod with core A'; thus, by Lemma 3.15, $C(p,X)$ contains a 3-cell for every $p\in A'$, which contradicts our hypotheses. Hence, B is connected. In a similar way it can be shown that A is connected, and the claim is proved.

Observe now that $A=\mathrm{cl}\Big(X\setminus\mathrm{cl}(X\setminus A')\Big)=\mathrm{cl}\big(\mathrm{int}(A')\big)$, and therefore, $\mathrm{bd}(A)=A\cap\mathrm{cl}(X\setminus A)=A\cap\mathrm{cl}\Big(X\setminus\mathrm{cl}\big(\mathrm{int}(A')\big)\Big)=A\cap\mathrm{cl}\big(\mathrm{ext}(A')\big)=A\cap\mathrm{cl}(X\setminus A')$. Hence, $\mathrm{bd}(A)=A\cap B=\mathrm{cl}(X\setminus B)\cap B=\mathrm{bd}(B)$.

As a consequence of this we obtain that $B\setminus A=B\setminus(B\cap A)=B\setminus\mathrm{bd}(B)=\mathrm{int}(B)$. And, similarly, $A\setminus B=\mathrm{int}(A)$. Also note that $B=\mathrm{cl}\big(\mathrm{int}(B)\big)$ and $A=\mathrm{cl}\big(\mathrm{int}(A)\big)$, by means of Lemma 4.3.

Finally, it remains to show that $A\cap B$ is not connected.

Let $i\in\{1,2\}$. By Lemma 4.2, we have that $\emptyset\neq H_i\cap\mathrm{bd}(A')\subset H_i\cap B$, and that $H_i\cap\mathrm{bd}(B')\neq\emptyset$. Now, it is easy to see that $B\subset B'$. Therefore, we can conclude that $H_i\cap\mathrm{bd}(B)\neq\emptyset$; thus, $H_i\cap(A\cap B)\neq\emptyset$. Since $A\cap B\subset H_1\cup H_2$, we deduce that $A\cap B$ is not connected. □

Definition 4.5. Let X be a continuum and let $A_1,A_2,A_3\in C(X)$. We will say that A_1,A_2 and A_3 form a *weak triod* if $A_1\cap A_2\cap A_3\neq\emptyset$ and $A_i\setminus(A_j\cup A_k)\neq\emptyset$ whenever $\{i,j,k\}=\{1,2,3\}$.

Recall that a *noose* is the one-point union of an arc and a simple closed curve in such a way that the arc intersects the simple closed curve in one of its end points. It is easy to see that a noose is a weak triod which is not a triod. Also, a continuum with the shape of the Greek letter θ is a weak triod which is not a triod. Moreover, one can easily prove that a triod is always a weak triod.

For more information on triods and weak triods, we refer the reader to [15].

Theorem 4.6. [15, Theorem 1.8] *Let X be a continuum and let $A,B,C \in C(X)$ be such that they form a weak triod. Then X contains a triod.*

Theorem 4.7. *Let X be a continuum such that* $\dim(C(p,X)) < 3$ *for each $p \in X$. Then X contains neither triods nor weak triods.*

Proof: This is a direct consequence of Theorem 4.6 and Lemma 3.15. □

Lemma 4.8. *Let X be a continuum and let $W,Y,Z \in C(X)$ be such that $Y \cap Z$ is not connected, $W \subsetneq Y$, and $W \cap Z = Y \cap Z$. Then X contains a triod.*

Proof: Let L_1 and L_2 be two distinct components of $Y \cap Z$. For $i \in \{1,2\}$ take an order arc α_i from L_i to Z, and $\delta > 0$ in such a way that $\alpha_1(\delta) \cap \alpha_2(\delta) = \emptyset$. Let $T = Y \cup \alpha_1(\delta) \cup \alpha_2(\delta)$. Then one can prove that T is a triod with core W. □

Theorem 4.9. *Let X be a continuum such that* $\dim\big(C(p,X)\big) < 3$ *for each $p \in X$ and the set $\{p \in X : C(p,X)$ has cut points$\}$ is at most countable. If $Y \in C(X) \setminus \{X\}$, then Y is unicoherent.*

Proof: Suppose that Y is not unicoherent. Then by Lemma 4.4, we can take two proper subcontinua A and B of Y in such a way that *i)* $A \cup B = Y$, *ii)* $\mathrm{int}_Y(A) = A \setminus B$, $\mathrm{int}_Y(B) = B \setminus A$, *iii)* $A = \mathrm{cl}\big(\mathrm{int}_Y(A)\big)$, $B = \mathrm{cl}\big(\mathrm{int}_Y(B)\big)$, and *iv)* $A \cap B$ is not connected.

Let $p \in Y$ be such that $C(p,X)$ has no cut points. Then, according to Lemma 3.7, there exists $K \in C(p,X)$ such that $Y \setminus K \neq \emptyset$ and $K \setminus Y \neq \emptyset$. We shall suppose that $B \setminus K \neq \emptyset$ and define $C = A \cup K$. Note from *iii)*, that $B \setminus C \neq \emptyset$.

Claim 1. $A \cap K \neq \emptyset$. In particular, $C \in C(X)$.

Suppose that $A \cap K = \emptyset$. Since $p \in Y \cap K$, we obtain that $p \in B \cap K$. Hence, $B \cup K \in C(X)$. Note that the set $(B \cup K) \cap A = B \cap A$ is not connected. Thus, applying Lemma 4.8 to the subcontinua $B \cup K, B$ and A, we obtain that X contains a triod. This yields a contradiction with Theorem 4.7

Claim 2. If $A \cap B$ is contained in a component of $B \cap C$, then Y contains a triod.

Let W be the component of $B \cap C$ containing $A \cap B$. Since $B \setminus C \neq \emptyset$, we have $W \subsetneq B$. On the other hand, it is not difficult to see that $W \cap A = B \cap A$. Therefore, applying Lemma 4.8, we obtain that Y contains a triod.

Claim 3. If $A \cap B$ intersects more than one component of $B \cap C$, then X contains a triod.

Let C_1 and C_2 be the two components of $B \cap C$ (see Lemma 4.1).

Note that $A \cup C_1 \cup C_2 \in C(X)$ and $K \setminus (A \cup C_1 \cup C_2) \supset K \setminus Y \neq \emptyset$, whence $A \cup C_1 \cup C_2 \subsetneq A \cup K = C$. Finally, it is not difficult to see that $(A \cup C_1 \cup C_2) \cap B = B \cap C$, so we can apply Lemma 4.8 to the subcontinua $A \cup C_1 \cup C_2$, C, and B, to obtain that X contains a triod.

As a result of the claims above, we obtain a contradiction with Theorem 4.7. □

Lemma 4.10. *Let X be a continuum such that* $\dim\big(C(p,X)\big) < 3$ *for each $p \in X$, and the set $\{p \in X : C(p,X)$ has cut points$\}$ is at most countable. Take $Y \in C(X) \setminus \{X\}$. If A_1 and A_2 are two proper subcontinua of Y such that $A_1 \cup A_2 = Y$, then $A_1 \cap A_2 \in C(Y)$ and there exists a unique order arc from $A_1 \cap A_2$ to A_i, for each $i \in \{1, 2\}$.*

Proof: By means of Theorem 4.9, $A_1 \cap A_2 \in C(Y)$; thus, there exists an order arc α_1 from $A_1 \cap A_2$ to A_1.

Suppose that we have an order arc α_2 from $A_1 \cap A_2$ to A_1, such that $\alpha_1(I) \neq \alpha_2(I)$. Then, applying Lemma 3.8, we can choose $s, t \in I$ satisfying $\alpha_1(s) \setminus \alpha_2(t) \neq \emptyset$ and $\alpha_2(t) \setminus \alpha_1(s) \neq \emptyset$.

Consider the set $Z = A_2 \cup \alpha_1(s) \cup \alpha_2(t)$. Then it is easy to see that Z is a weak triod, but this contradicts Theorem 4.7. Proceeding similarly, one can prove the result for A_2. □

Recall that a continuum X is *irreducible between the points* $a, b \in X$ provided that, for any $A \in C(X)$ containing a and b, we have that $A = X$.

Definition 4.11. Let X be a continuum and let $A, B \subset X$. We say that X is *irreducible between A and B* provided that

(*) X is irreducible between the points a and b if and only if $a \in A$ and $b \in B$.

Definition 4.12. For a continuum X, irreducible between two points a and b, define the family $\mathbb{D}_{(X,a)} = \{A \in C(a,X) : A = \mathrm{cl}\big(\mathrm{int}(A)\big)\}$.

Lemma 4.13. *Let Y be a continuum irreducible between the subcontinua A' and B'. Let $A \in C(Y)$ be such that $A' \cap A \neq \emptyset$ and*

$A \setminus A' \neq \emptyset$. Then $A' \subset \mathrm{int}(A)$. In particular, A' is a terminal subcontinuum of Y.

Proof: Let $p \in A \setminus A'$. Then there exists $B \in C(p, X) \setminus \{X\}$ such that $B \cap B' \neq \emptyset$. Since Y is irreducible between A' and B' we have that $B \cap A' = \emptyset$ and $A \cup B = X$, whence $A' \subset X \setminus B \subset A$. Therefore, $A' \subset \mathrm{int}(A)$. □

Theorem 4.14. *Let X be a continuum such that $\dim\big(C(p, X)\big) < 3$ for each $p \in X$ and the set $\{p \in X : C(p, X)$ has cut points$\}$ is at most countable.*

Let $Y \in C(X) \setminus \{X\}$ and let $A', B' \in C(Y)$ be such that Y is irreducible between A' and B'. If $A_1, A_2 \in C(Y)$ satisfy that $A' \subset A_1 \cap A_2$, then either $A_1 \subset A_2$ or $A_2 \subset A_1$.

Proof: Assume $A' \subsetneq A_i$ for each $i \in \{1, 2\}$, and let $a \in A'$. As a consequence of Lemma 4.3 and [8, §48, II, Theorem 5], $\mathrm{cl}\big(\mathrm{int}_Y(A_1)\big)$, $\mathrm{cl}\big(\mathrm{int}_Y(A_2)\big) \in \mathbb{D}_{(Y,a)}$. Thus, by [8, §48, III, Theorem 2], we may assume that $\mathrm{cl}\big(\mathrm{int}_Y(A_1)\big) \subset \mathrm{cl}\big(\mathrm{int}_Y(A_2)\big)$. Let $K = \mathrm{cl}\big(\mathrm{int}_Y(A_1)\big)$. If A_1 and A_2 are not comparable, then $K \subsetneq A_1$ and $K \subsetneq A_2$.

Choose $w \in K \setminus A'$ (Lemma 4.13) and let $b \in B'$. Since Y is irreducible between A' and B', there exists $B \in C(Y) \setminus \{Y\}$ such that $w, b \in B$. Hence, $A' \subset Y \setminus B$. Applying now Theorem 4.9, we get that $A_i \cap B \in C(Y)$, for $i \in \{1, 2\}$. Using again the irreducibility of Y, we have that $A_i \cup B = Y$ and $K \cup B = Y$. In particular, $Y \setminus A_i \subset B$.

Now, observe that $A' \subset K \setminus \Big((A_1 \cap B) \cup (A_2 \cap B)\Big)$ and that $w \in (A_1 \cap B) \cap (A_2 \cap B) \cap K$.

On the other hand, one can prove that $(A_i \cap B) \setminus \big((A_j \cap B) \cup K\big) = A_i \setminus A_j \neq \emptyset$, for $i \neq j$. As a consequence of the statements above, we get that $A_1 \cap B$, $A_2 \cap B$, and K form a weak triod. This contradicts Theorem 4.7. □

Lemma 4.15. *Let X be a continuum such that $\dim(C(p, X)) < 3$ for each $p \in X$ and let $Y \in C(X)$. Suppose that Y is irreducible between A' and B', where $A', B' \in C(Y)$.*

Let $w \in Y \setminus (A' \cup B')$ and let $Z \in C(w, X)$ be such that $Z \setminus Y \neq \emptyset$. Denote by Z_0 the component of $Z \cap Y$ that contains w. Then either $A' \subset Z_0$ or $B' \subset Z_0$.

Proof: Claim 1. $A' \cap Z \neq \emptyset$ or $B' \cap Z \neq \emptyset$.

Choose $a \in A'$ and $A \in C(Y) \setminus \{Y\}$ such that $a, w \in A$. Then $A \cap B' = \emptyset$. Similarly, let $b \in B'$ and let $B \in C(Y) \setminus \{Y\}$ be such that $b, w \in B$ and $B \cap A' = \emptyset$. Hence, $A \cup B = Y$; thus, $Z \setminus (A \cup B) \neq \emptyset$.

If we suppose that $Z \cap A' = \emptyset$ and $Z \cap B' = \emptyset$, we get that $a \in A' \cap A \subset A \setminus (Z \cup B)$. Similarly, $b \in B \setminus (Z \cup A)$. According to this, we can conclude that A, B, and Z form a weak triod. However, this contradicts Theorem 4.7. The claim is proved.

Claim 2. $A' \subset Z_0$ or $B' \subset Z_0$.

Let α be an order arc from Z_0 to Z. By Lemma 4.1, we know that $Z \cap Y$ has at most two components, so there exists $\delta > 0$ which satisfies $\alpha(\delta) \cap Y = Z_0$. Applying Claim 1 to the subcontinuum $\alpha(\delta)$, we may suppose that $A' \cap \alpha(\delta) \neq \emptyset$. Then clearly $\emptyset \neq A' \cap \alpha(\delta) = A' \cap Z_0$. Finally, by Lemma 4.13, and the fact that $w \in Z_0 \setminus A'$, we conclude that $A' \subset Z_0$. □

5. On class $\mathcal{P}$

Definition 5.1. Let $\mathcal{P}$ be the class of continua X such that $C(p, X)$ is an arc or a 2-cell for each $p \in X$, and the set $\{p \in X : C(p, X) \text{ is arc}\}$ is at most countable.

Note, for example, that an arc is such a continuum (Theorem 3.17), whereas the continuum Y of Example 3.22 is not.

Theorem 5.2. *Let $X \in \mathcal{P}$ and let $Y \in C(X)$. Suppose that $A', B' \in C(Y)$ are such that Y is irreducible between A' and B'. Let $w \in Y \setminus (A' \cup B')$. Then $C(w, Y)$ has no cut points.*

Proof: By Lemma 3.5, neither Y nor $\{w\}$ are cut points of $C(w, Y)$. Let $W \in C(w, Y)$ be such that $\{w\} \subsetneq W \subsetneq Y$. We shall analyze two cases, in order to see that W is not a terminal subcontinuum of Y, at w.

Case 1. $A' \subset W$ or $B' \subset W$.

Suppose that $A' \subset W$, then $W \cap B' = \emptyset$. Since w is not a point of irreducibility of Y, we can take $b \in B'$ and $B \in C(w, Y) \setminus \{Y\}$ such that $b \in B$. Therefore, $A' \subset W \setminus B$ and $b \in B \cap B' \subset B \setminus W$. Hence, W is not terminal at w in Y.

Case 2. $A' \not\subset W$ and $B' \not\subset W$.

Notice that w is not a point of irreducibility of Y. Take $a \in A'$, $b \in B'$, and $A, B \in C(w, Y) \setminus \{Y\}$ such that $a \in A$ and $b \in B$.

According to this, $b \in B \setminus A$ and $a \in A \setminus B$. Thus, by Lemma 3.9, $C(w, X)$ is not an arc. Since $X \in \mathcal{P}$, $C(w, X)$ must be a 2-cell, and therefore, by Lemma 3.7, W is not a subcontinuum of X terminal at w.

Let $Z \in C(w, X)$ be such that $Z \setminus W \neq \emptyset$ and $W \setminus Z \neq \emptyset$. If $Z \subset Y$, we obtain directly that W is not a subcontinuum of Y terminal at w; thus, we may suppose that $Z \setminus Y \neq \emptyset$. We shall also suppose that $Z \cup Y \subsetneq X$. Let $Z_0 = Z \cap Y$. Then, according to Theorem 4.9, $Z_0 \in C(w, Y)$.

On the other hand, by Lemma 4.15, we may assume that $A' \subset Z_0$. Now, we have that $\emptyset \neq A' \setminus W \subset Z_0 \setminus W$ and that $\emptyset \neq W \setminus Z_0$. Hence, W is not a subcontinuum of Y terminal at w.

As a result of either case, by Lemma 3.7, we get that W is not a cut point of $C(w, Y)$. □

The next theorem follows from Theorem 4.14 and Lemma 3.9.

Theorem 5.3. *Let $X \in \mathcal{P}$ and let Y be a proper and nondegenerate subcontinuum of X. If Y is irreducible between A' and B' for some $A', B' \in C(Y)$, then $C(A', Y)$ and $C(B', Y)$ are arcs.*

The main theorem in this section states that proper and nondegenerate subcontinua of continua in class $\mathcal{P}$ are arcs. We shall proceed to develop auxiliary results to this aim.

Theorem 5.4. *Let $X \in \mathcal{P}$ and let Y be a proper and nondegenerate subcontinuum of X. Suppose that Y is irreducible between A' and B' for some $A', B' \in C(Y)$.*

Let $a \in A'$, and let $\alpha : I \to C(A', Y)$ be an order arc from A' to Y. Then the set $T = \{t \in I : \alpha(t) \in \mathbb{D}_{(Y,a)}\}$ is dense in I.

Proof: Suppose that T is not dense in I, and take $r \in (0, 1)$ and $\varepsilon \in (0, r)$ in such a way that $0 < r - \varepsilon < r + \varepsilon < 1$ and $(r - \varepsilon, r + \varepsilon) \cap T = \emptyset$.

Define $s = \inf\{t \in [r + \varepsilon, 1] : t \in T\}$, $Z = [0, r - \varepsilon] \cap T$. Further, let $t_0 = \sup Z$, if $Z \neq \emptyset$; otherwise, define $t_0 = 0$. As a consequence of Theorem 5.3 and [9, Theorem 3.1], there exists a point $y \in \alpha(s) \setminus \bigcup\{\alpha(t) : 0 < t < s\}$. Let $b \in B'$. We shall proceed with the proof in a series of steps.

`Step 1.` $t_0 > 0$, $t_0 \in T$ and $\mathrm{cl}\big(\mathrm{int}_Y(\alpha(s))\big) = \mathrm{cl}\big(\mathrm{int}_Y(\alpha(t_0))\big) = \alpha(t_0)$.

As a consequence of Lemma 4.13, Lemma 4.3, and [8, §48, II, Theorem 5], $\mathrm{cl}\big(\mathrm{int}_Y(\alpha(s))\big) \in \mathbb{D}_{(Y,a)}$. Now, applying Theorem 5.3 and Lemma 3.9, we obtain that $\mathrm{cl}\big(\mathrm{int}_Y(\alpha(s))\big) = \alpha(s_0)$ for some $s_0 \in [0,s]$. According to this and to Lemma 4.13, we deduce that $A' \subsetneq \alpha(s_0) \subset \alpha(s)$. Therefore, $s_0 > 0$, and from the construction, it follows that $0 < s_0 \leq t_0$.

Take now an increasing sequence of elements in T which converges to t_0. According to [8, p. 196], $\alpha(t_0) \in \mathbb{D}_{(Y,a)}$; thus, $\mathrm{cl}\big(\mathrm{int}_Y(\alpha(s))\big) = \alpha(s_0) \subset \alpha(t_0) = \mathrm{cl}\big(\mathrm{int}_Y(\alpha(t_0))\big)$.

On the other hand, since $t_0 \leq s$, then $\mathrm{cl}\big(\mathrm{int}_Y\big(\alpha(t_0)\big)\big) \subset \mathrm{cl}\big(\mathrm{int}_Y\big(\alpha(s)\big)\big)$. This step is finished.

`Step 2.` $s \notin T$. In particular, $s < 1$.

By construction we have that

$$\big\{\alpha(t) \in C(A',Y) : \alpha(t_0) \subsetneq \alpha(t) \subsetneq \alpha(s)\big\} \cap \mathbb{D}_{(Y,a)} = \emptyset.$$

If we suppose that $s \in T$, applying Lemma 4.13, Step 1, and [8, §48, VII, Theorem 2] to $\alpha(t_0)$ and $\alpha(s)$, we obtain that $\mathrm{cl}\big(\alpha(s) \setminus \alpha(t_0)\big)$ is an indecomposable subcontinuum of Y. Since $X \in \mathcal{P}$, using Theorem 3.16, we contradict the statement above. Hence, $s \notin T$.

`Step 3.` Let $K' = \mathrm{cl}\big(\alpha(s) \setminus \alpha(t_0)\big)$. Then $K' \in C(y,Y)$ and $\{y\} \subsetneq K' \subsetneq Y$.

Applying Theorem 5.3 and Lemma 3.10 to $\alpha(s)$ and $\alpha(t_0)$, we obtain that $\alpha(s) \setminus \alpha(t_0)$ is connected. Moreover, $K' \in C(y,Y)$ and $K' \setminus \{y\} \neq \emptyset$. Finally, since $s < 1$, it follows that $K' \subset \alpha(s) \subsetneq Y$.

The aim of the following steps is to show that K' is terminal at the point y, in the subcontinuum Y.

`Step 4.` Let K' be defined as in Step 3, and let $K \in C(y,Y)$ be such that $(K \setminus K') \cap \alpha(t_0) \neq \emptyset$. Then $K' \subset K$.

We shall prove that $\alpha(s) \setminus \alpha(t_0) \subset K$.

Suppose there is a point $w \in \Big(\alpha(s) \setminus \alpha(t_0)\Big) \setminus K$. We may then assume that $w \in \alpha(r')$ for some $r' \in (t_0, s)$.

Consider now $\alpha(t_0) \cup K$ and $\alpha(r')$. According to the hypotheses of this step, $\alpha(t_0) \cup K \in C(A',X)$; however, $y \in \Big(\alpha(t_0) \cup K\Big) \setminus \alpha(r')$ and $w \in \alpha(r') \setminus \Big(\alpha(t_0) \cup K\Big)$. Thus, $\alpha(t_0) \cup K$ and $\alpha(r')$ are two noncomparable elements of $C(A',Y)$. According to this and to Lemma 3.9, we obtain a contradiction with Theorem 5.3. This step is finished.

`Step 5.` Let K' be defined as in Step 3, and let $K \in C(y, Y)$ be such that *i)* $(K \setminus K') \cap (Y \setminus \alpha(t_0)) \neq \emptyset$ and *ii)* $\alpha(s) \nsubseteq \alpha(t_0) \cup K$. Then $\alpha(t_0) \cap K = \emptyset$.

Suppose that $\alpha(t_0) \cap K \neq \emptyset$, then clearly $\alpha(t_0) \cup K \in C(A', X)$. Now, according to our hypotheses, it is not difficult to prove that $\emptyset \neq (K \setminus K') \cap (Y \setminus \alpha(t_0)) \subset (\alpha(t_0) \cup K) \setminus \alpha(s)$. Therefore, $\alpha(s)$ and $\alpha(t_0) \cup K$ are two noncomparable elements of $C(A', Y)$. Applying now Lemma 3.9, we have that $C(A', Y)$ is not an arc, which yields a contradiction with Theorem 5.3. The step is complete.

`Step 6.` Let K' be defined as in Step 3, and let $K \in C(y, Y)$ be such that $(K \setminus K') \cap (Y \setminus \alpha(t_0)) \neq \emptyset$. Then $\alpha(s) \subset \alpha(t_0) \cup K$.

By construction, we have that $\alpha(s) \cup K \in C(A', Y)$. As a consequence of Theorem 5.3 and Lemma 3.9, we obtain that $\alpha(s) \cup K = \alpha(s_1)$ for some $s_1 \in [s, 1]$.

On the other hand, according to the hypotheses, one can prove that $\emptyset \neq (K \setminus K') \cap (Y \setminus \alpha(t_0)) \subset K \setminus \alpha(s)$. Thus, $s_1 > s$. Let $s_2 \in [s, s_1) \cap T$.

Suppose that $\alpha(s) \nsubseteq \alpha(t_0) \cup K$, then, as a consequence of Step 5, there exists $\delta > 0$ such that $\alpha(t_0 + \delta) \cap K = \emptyset$. Since $y \in \alpha(s) \cap K$, we have that $t_0 + \delta < s$. Let $z \in \alpha(t_0 + \delta) \setminus \alpha(t_0)$. Applying Step 1, it follows that $z \notin \mathrm{cl}\big(\mathrm{int}_Y(\alpha(s))\big)$, and we already know that $z \notin K$.

Now, since $s_2 < s_1$, it is easy to see that $\mathrm{cl}\big(\alpha(s_2) \setminus \alpha(s)\big) \subset K$. Thus, we have that $z \notin \mathrm{cl}\big(\mathrm{int}_Y\big(\alpha(s)\big)\big) \cup \mathrm{cl}\big(\alpha(s_2) \setminus \alpha(s)\big)$. Hence, $z \in X \setminus \mathrm{cl}\ \big(\mathrm{int}_Y\big(\alpha(s_2)\big)\big) = X \setminus \alpha(s_2) \subset X \setminus \alpha(t_0 + \delta)$, which is a contradiction. The step is complete.

`Step 7.` If K' is defined as in Step 3, then $A' \cap K' = \emptyset$ and $B' \cap K' = \emptyset$. In particular, $\{a, b\} \cap K' = \emptyset$ and $y \notin A' \cup B'$.

Since $s < 1$, we have $K' = \mathrm{cl}\big(\alpha(s) \setminus \alpha(t_0)\big) \subset Y \setminus B'$. In particular, $b \notin K'$. On the other hand, $\mathrm{int}_Y\big(\alpha(t_0)\big) \cap \mathrm{cl}\big(\alpha(s) \setminus \alpha(t_0)\big) = \emptyset$. As a consequence of this, Step 1, and Lemma 4.13, $A' \cap K' = \emptyset$. In particular, $a \notin K'$. Finally, in Step 3, we saw that $K' \in C(y, Y)$. Hence, $y \notin A' \cup B'$.

`Step 8.` If K' is defined as in Step 3, then K' is a subcontinuum of Y terminal at y.

Let $K \in C(y, Y)$ be such that $K \setminus K' \neq \emptyset$. If $(K \setminus K') \cap \alpha(t_0) \neq \emptyset$, by Step 4, we have directly that $K' \subset K$. Suppose now that $(K \setminus K') \cap (Y \setminus \alpha(t_0)) \neq \emptyset$. Then, applying Step 6, it follows that $\alpha(s) \subset \alpha(t_0) \cup K$. Therefore, $K' = \mathrm{cl}\big(\alpha(s) \setminus \alpha(t_0)\big) \subset K$.

Hence, K' is a subcontinuum of Y terminal at y.

Now, using Theorem 5.2 and Step 7, we deduce that $C(y,Y)$ has no cut points. However, Lemma 3.7 and the statements above contradict the terminality of K' at y proved in Step 8. □

Proposition 5.5. *Let $X \in \mathcal{P}$ and let Y be a proper nondegenerate subcontinuum of X. If Y is irreducible between A' and B', for some $A', B' \in C(Y)$, and $a \in A'$, then $C(A', Y) \setminus \{A'\} \subset \mathbb{D}_{(Y,a)}$.*

Proof: As a consequence of Theorem 5.3 and Lemma 3.9, there exists a unique order arc α from A' to Y. Let $D \in C(A', Y) \setminus \{A'\}$. Then $D = \alpha(s)$ for some $s > 0$. Now, by Theorem 5.4 we can take an increasing sequence $\{s_n\}_{n=1}^{\infty} \subset I$ such that $\{\alpha(s_n)\}_{n=1}^{\infty} \subset \mathbb{D}_{(Y,a)}$ and $s_n \to s$. According to [8, p. 196], we get that $D = \alpha(s) \in \mathbb{D}_{(Y,a)}$. Therefore, $C(A', Y) \setminus \{A'\} \subset \mathbb{D}_{(Y,a)}$. □

The proof of the following lemma is straightforward and is left to the reader.

Lemma 5.6. *Let X be a continuum and let L and K be two closed subsets of X. If $w \in \mathrm{bd}(L) \setminus K$, then $w \in \mathrm{bd}(L \cup K)$.*

Proposition 5.7. *Let $X \in \mathcal{P}$ and let Y be a proper and nondegenerate subcontinuum of X. Suppose that Y is irreducible between A' and B' for some $A', B' \in C(Y)$.*

Then $\mathrm{bd}_Y(D)$ is a one-point set for each $D \in C(A', Y) \setminus \{A', Y\}$.

Proof: Let $D \in C(A', Y) \setminus \{A', Y\}$. According to Theorem 5.3 and Lemma 3.9, there exists a unique order arc α from A' to Y; thus, we can choose $t \in (0,1)$ such that $D = \alpha(t)$. Suppose that the boundary of $\alpha(t)$ in Y has more than one point and let $x \in \mathrm{bd}_Y\big(\alpha(t)\big)$.

As a consequence of Lemma 4.13, $x \notin A'$. Since $B' \cap \alpha(t) = \emptyset$, we have that $x \notin B'$. Thus, using Theorem 5.2, it follows that $C(x, Y)$ has no cut points. Thus, using Lemma 3.12 and Lemma 3.7, we obtain that $\mathrm{bd}_Y\big(\alpha(t)\big)$ is not terminal at x. Let $K \in C(x, Y)$ be such that $K \setminus \mathrm{bd}_Y\big(\alpha(t)\big) \neq \emptyset$ and $\mathrm{bd}_Y\big(\alpha(t)\big) \setminus K \neq \emptyset$.

Case 1. $K \setminus \alpha(t) \neq \emptyset$.

Let $a \in A'$ and let $w \in \mathrm{bd}_Y\big(\alpha(t)\big) \setminus K$. Then $w \in K \cup \alpha(t) \in C(A', Y)$ and, nevertheless, $\alpha(t) \cup K$ is not irreducible between a and w. Now, by Proposition 5.5, we know that $\alpha(t) \cup K \in \mathbb{D}_{(Y,a)}$

and according to Lemma 5.6, $w \in \mathrm{bd}_Y\big(\alpha(t) \cup K\big)$. However, this contradicts [8, §48, III, Theorem 1].

Case 2. $K \subset \alpha(t)$.

Let $b \in B'$. In this case, we have that $K \setminus \mathrm{cl}\big(Y \setminus \alpha(t)\big) \neq \emptyset$ and, by Lemma 3.10, we know that $\mathrm{cl}\big(Y \setminus \alpha(t)\big)$ is connected. Moreover, since Y is irreducible and $t \in (0,1)$, $B' \subset Y \setminus \alpha(t)$. Therefore, $K \cup \mathrm{cl}\big(Y \setminus \alpha(t)\big) \in C(B', Y) \setminus \{B'\}$. Applying Proposition 5.5, we get that $K \cup \mathrm{cl}\big(Y \setminus \alpha(t)\big) \in \mathbb{D}_{(Y,b)}$ and that $\alpha(t) \in \mathbb{D}_{(Y,a)}$. Thence, it is not difficult to see that $\mathrm{bd}_Y\big(\alpha(t)\big) = \mathrm{bd}_Y\big(\mathrm{cl}\big(Y \setminus \alpha(t)\big)\big)$. Take a point $w \in \mathrm{bd}_Y\big(\alpha(t)\big) \setminus K = \mathrm{bd}_Y\big(\mathrm{cl}\big(Y \setminus \alpha(t)\big)\big) \setminus K$. Thus, by Lemma 5.6, $w \in \mathrm{bd}_Y\big(\mathrm{cl}\big(Y \setminus \alpha(t)\big) \cup K\big)$. Finally, since $b, w \in \mathrm{cl}\big(Y \setminus \alpha(t)\big) \subsetneq K \cup \mathrm{cl}\big(Y \setminus \alpha(t)\big)$, it is easy to see that $K \cup \mathrm{cl}\big(Y \setminus \alpha(t)\big)$ is not irreducible between b and w. However, this contradicts [8, §48, III, Theorem 1]. □

Lemma 5.8. *Let X be a continuum irreducible between the points a and b. If $A, B \in C(X)$ are such that $a \in A \subsetneq B$ and $B \in \mathbb{D}_{(X,a)}$, then cl$(X \setminus B) \subsetneq$ cl$(X \setminus A)$.*

Proof: Notice that $\mathrm{cl}(X \setminus B) \subset \mathrm{cl}(X \setminus A)$. Moreover, according to [8, §48, III, Theorem 1], we have that $\mathrm{bd}(A) \cap \mathrm{bd}(B) = \emptyset$. However, this yields $\emptyset \neq \mathrm{bd}(A) \subset \mathrm{cl}(X \setminus A) \setminus \mathrm{cl}(X \setminus B)$. □

Proposition 5.9. *Let $X \in \mathcal{P}$ and let Y be a proper and nondegenerate subcontinuum of X. Suppose that Y is irreducible between A' and B' for some $A', B' \in C(Y)$.*

Let $\alpha : I \to C(A', Y)$ be an order arc from A' to Y. Then, for each $x \in Y \setminus (A' \cup B')$, there exists $t \in (0,1)$ such that $x \in \mathrm{bd}_Y\big(\alpha(t)\big)$.

Proof: Let $x \in Y \setminus (A' \cup B')$, $t = \min\{s \in I : x \in \alpha(s)\}$, $a \in A'$, and $b \in B'$; note that $0 < t < 1$. By Lemma 3.10, $\mathrm{cl}\big(Y \setminus \alpha(t)\big) \in C(B', Y)$. Moreover, by Theorem 5.3 and Lemma 3.9, there exists a unique order arc β from B' to Y; hence, $\mathrm{cl}\big(Y \setminus \alpha(t)\big) = \beta(s)$ for some $s \in [0,1)$. Since $x \in \alpha(t)$, it is enough to show that $x \in \mathrm{cl}\big(Y \setminus \alpha(t)\big)$. Suppose that $x \notin \mathrm{cl}\big(Y \setminus \alpha(t)\big) = \beta(s)$.

Let $s' \in (s,1)$ be such that $x \notin \beta(s')$. Again, by Lemma 3.10, we get that $\mathrm{cl}\big(Y \setminus \beta(s')\big) \in C(A', Y)$. Hence, $\mathrm{cl}\big(Y \setminus \beta(s')\big) = \alpha(r)$ for some $r \in I$ and $x \in \mathrm{cl}\big(Y \setminus \beta(s')\big) = \alpha(r)$. Now, according to Proposition 5.5, $\beta(s') \in \mathbb{D}_{(Y,b)}$, and by Lemma 5.8, $\alpha(r) =$

$\mathrm{cl}\big(Y \setminus \beta(s')\big) \subsetneq \mathrm{cl}\big(Y \setminus \beta(s)\big)$. Therefore, $\alpha(r) \subsetneq \mathrm{cl}\Big(Y \setminus \mathrm{cl}\big(Y \setminus \alpha(t)\big)\Big)$ $= \mathrm{cl}\big(\mathrm{int}_Y\big(\alpha(t)\big)\big) \subset \alpha(t)$. Thus, $r < t$, but this is a contradiction. □

Theorem 5.10. *Let $X \in \mathcal{P}$ and let $Y \in C(X) \setminus \{X\}$. If Y is nondegenerate, then Y is irreducible between A' and B' for some $A', B' \in C(Y)$.*

Proof: By theorems 4.9 and 4.7, we know that Y is unicoherent, and that it contains no triods. Therefore, according to [13, Theorem 11.34], Y is irreducible. Moreover, by means of Theorem 3.16, Y is hereditarily decomposable. The conclusion follows from [10, Lemma A]. □

Theorem 5.11. *Let $X \in \mathcal{P}$ and let Y be a proper and nondegenerate subcontinuum of X. If Y is irreducible between the subcontinua A' and B', then $|A'| = 1 = |B'|$.*

Proof: Suppose that $|A'| > 1$ and take $a \in A'$ such that $C(a, X)$ has no cut points. Then, by Lemma 3.7, we can take $K' \in C(a, X)$ such that $K' \setminus A' \neq \emptyset \neq A' \setminus K'$. However, according to Lemma 4.13, we obtain that $K' \setminus Y \neq \emptyset$. Let L be the component of $A' \cap K'$ containing a, and let α be an order arc from L to K'. By Lemma 4.1, we know that $A' \cap K'$ has at most two components, so we can choose $\delta \in (0, 1)$ such that *i)* $\alpha(\delta) \cap Y = L \subset A'$, *ii)* $Y \cup \alpha(\delta) \neq X$, and *iii)* $\alpha(\delta) \setminus Y$ is connected.

Let $K = \alpha(\delta)$. Then it is not difficult to see that $K \in C(a, X)$, $K \setminus A' \neq \emptyset$, $A' \setminus K \neq \emptyset$, and $Y \subsetneq Y \cup K \subsetneq X$. Define $Y' = Y \cup K$. By Theorem 5.10, Y' is irreducible between two subcontinua C' and D' in $C(Y')$. We may assume that $C' \subset K \setminus Y$ and $D' \subset Y \setminus K$.

Suppose that there exists a point $d \in D' \setminus B'$. Then there is a proper subcontinuum H of Y such that $d \in H$ and $H \cap A' \neq \emptyset$. Therefore, $H \cup A' \cup K$ is a proper subcontinuum of Y', which contains both d and C', contradicting the irreducibility of Y. Hence, $D' \subset B'$.

Now let γ be an order arc from C' to Y'. Since $\mathrm{cl}(K \setminus Y)$ and $K \cup A'$ are elements of $C(C', Y')$, by means of Theorem 5.3 and Lemma 3.9, we can take $s_1, s_2 \in I$ such that $\gamma(s_1) = \mathrm{cl}(K \setminus Y)$ and $\gamma(s_2) = K \cup A'$. Therefore, $\gamma(s_1) \subset K \subsetneq \gamma(s_2)$, and thus, $s_1 < s_2$.

Let $z \in \mathrm{bd}_{Y'}\big(\gamma(s_1)\big)$. We shall see next that $z \in \mathrm{bd}_{Y'}\big(\gamma(s_2)\big)$. Take an open subset U of Y' such that $z \in U$. We know that

$K \cap Y = K \cap A'$ and that $Y' = K \cup Y = (K \setminus Y) \cup (K \cap Y) \cup (Y \setminus K)$. On the other hand, we have that $\emptyset \neq U \setminus \gamma(s_1) \subset U \setminus (K \setminus Y)$. Hence, $\emptyset \neq U \cap \big[(K \cap Y) \cup (Y \setminus K)\big] \subset U \cap \big[A' \cup (Y \setminus K)\big] = U \cap Y$. Now, it is well known that $\mathrm{int}_Y(A') = \emptyset$ (see [8, §48, VIII, Theorem 5]). Thus, $U \cap Y \not\subseteq A'$. Therefore, $\emptyset \neq (U \cap Y) \setminus (K \cup A') \subset U \setminus \gamma(s_2)$. Hence, $z \in \mathrm{bd}_{Y'}\big(\gamma(s_2)\big)$.

As we showed above, $\mathrm{bd}_{Y'}\big(\gamma(s_1)\big) \cap \mathrm{bd}_{Y'}\big(\gamma(s_2)\big) \neq \emptyset$. If $c \in C'$, applying Proposition 5.5 to X and Y', we get that $\gamma(s_1), \gamma(s_2) \in \mathbb{D}_{(Y',c)}$, contradicting [8, §48, III, Theorem 2]. Hence, $|A'| = 1$. Similarly, $|B'| = 1$. □

Theorem 5.12. *Let $X \in \mathcal{P}$ and let Y be a proper and nondegenerate subcontinuum of X. Then Y is an arc.*

Proof: As a consequence of Theorem 5.10 and Theorem 5.11, we obtain that Y is irreducible beetween two points a and b. Let α be an order arc from $\{a\}$ to Y and let $x \in Y \setminus \{a, b\}$. According to Proposition 5.9, there exists $t \in (0, 1)$ such that $x \in \mathrm{bd}_Y\big(\alpha(t)\big)$. Then, by Proposition 5.5, $\alpha(t) \in \mathbb{D}_{(Y,a)}$, whence $\mathrm{int}_Y\big(\alpha(t)\big) \neq \emptyset$. Now, by Proposition 5.7, $Y \setminus \{x\} = \mathrm{int}_Y\big(\alpha(t)\big) \cup \mathrm{ext}_Y\big(\alpha(t)\big)$; hence, x is cut point of Y. The conclusion follows from [13, Theorem 6.17]. □

Theorem 5.13. *If $X \in \mathcal{P}$ is decomposable, then X is an arc or a simple closed curve.*

Proof: Let $A, B \in C(X) \setminus \{X\}$ be such that $X = A \cup B$. Then, according to Theorem 5.12 and Theorem 4.7, X is the atriodic union of two arcs. Thus, X is an arc or a simple closed curve. □

Definition 5.14. A continuum X is *circle-similar* if $C(p, X)$ is a 2-cell for each $p \in X$.

As direct consequences of the previous theorem and theorems 3.17 and 3.18, we have the following results.

Theorem 5.15. *A continuum X is an arc if and only if X is a decomposable arc-similar continuum.*

Theorem 5.16. *A continuum X is a simple closed curve if and only if X is a decomposable circle-similar continuum.*

Note that the condition of decomposability in the previous theorem is essential: A solenoid is a circle-similar continuum which is not a simple closed curve.

Example 5.17. Let Y be the continuum presented in Example 3.21, and let $X = Y/\sim$, where the relation $\sim$ identifies the points a and b. Then X is an example of a circle-similar continuum, which is neither a simple closed curve nor a solenoid.

Recall that a point p is an *end point* of a continuum X if for any $A, B \in C(p, X)$, we have that A and B are comparable. The next observation follows from Lemma 3.9.

Observation 5.18. p is an end point of the continuum X if and only if $C(p, X)$ is an arc.

A continuum X has the *property of Kelley* provided that for each $p \in X$, each $A \in C(p, X)$, and each sequence $\{p_n\}_{n=1}^{\infty}$ which converges to p, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ converging to A in such a way that $A_n \in C(p_n, X)$ for every $n \in \mathbb{N}$.

In [7], P. Krupski introduces the class $\mathcal{K}$ as being the class of continua satisfying the following criteria: *i*) X is chainable, *ii*) X has the property of Kelley, *iii*) X has one or two end points, and *iv*) the proper and nondegenerate subcontinua of X are arcs.

Consider the class $\mathcal{K}_1 = \{X \in \mathcal{K} : X \text{ has two end points}\}$. As far as we have seen, all our examples of arc-similar continua belong to the class $\mathcal{K}_1$. Therefore, it is natural to ask whether these two classes coincide. The following examples give a negative answer to this question.

Example 5.19. *An arc-similar continuum that is not chainable.*

J. Doucet constructs in [2] a chainable continuum Y, with four end points $\{a, b, c, d\}$, and such that its proper and nondegenerate subcontinua are arcs. Let X be the space resulting by identifying the points a and d. Thus, X is a nonchainable continuum, with two end points, and such that its proper and nondegenerate subcontinua are arcs. Thence, X is an arc-similar continuum which is not chainable (this will be a consequence of Theorem 5.21).

Example 5.20. In [5, p. 426], there are presented a buckethandle continuum and a modification of it which does not have the property of Kelley. In a similar way, we can modify the continuum of Example 3.21 to obtain an arc-similar continuum not having the property of Kelley.

In what follows, we shall denote the image of a function f by $\mathrm{im}(f)$, and the Hausdorff distance in $C(X)$ by H.

Theorem 5.21. *Let X be an indecomposable continuum such that its proper and nondegenerate subcontinua are arcs. Let $p \in X$. If p is not an end point of X, then $C(p,X)$ is a 2-cell.*

Proof: Since p is not an end point of X, we can take two non-comparable elements, B and C of $C(p,X)$. Let $A = B \cup C$. By the indecomposability of X, it is not difficult to see that A is an arc containing p; moreover, p is not an end point of A. Let a and b be the end points of this arc and give an order $<$ on A, satisfying $a < b$. Clearly, this order is unique.

Now, let $K \in C(p,X) \setminus \{X\}$. Then, by the indecomposability of X, we get that $K \cup A \subsetneq X$, so $K \cup A$ is an arc containing a and b. Give the arc $K \cup A$ the unique order in which $a < b$, and note that this order, restricted to A, coincides with the order we already gave A. Let $l_K = \min K$ and let $r_K = \max K$. Then we may denote K as an interval $[l_K, r_K]$.

Let μ be a Whitney map for $C(p,X)$ and let $f : C(p,X)\setminus\{X\} \to I \times I$ be given by $f(K) = \Big(\mu\big([l_K,p]\big), \mu\big([p,r_K]\big)\Big)$.

Claim 1. f is one-to-one.

Let $K, K' \in C(p,X)\setminus\{X\}$. Then $A\cup K\cup K'$ is an arc containing a and b, so we can give it the order in which $a < b$, which, by the way, coincides with the order defined in $K\cup A$ and $K'\cup A$. Then it is easy to see that $[l_K,p]$ and $[l_{K'},p]$ are comparable. Likewise, $[p,r_K]$ and $[p,r_{K'}]$ are comparable. Hence, if we suppose $f(K) = f(K')$, then we get that $[l_K,p] = [l_{K'},p]$ and $[p,r_K] = [p,r_{K'}]$, which means that f is one-to-one.

Claim 2. f is continuous.

Take a sequence $\{K_n\}_{n=1}^{\infty} \subset C(p,X) \setminus \{X\}$ which converges to $K \in C(p,X) \setminus \{X\}$. One can easily prove that under these conditions the set $W = K \cup \bigcup\{K_n : n \in \mathbb{N}\}$ is a continuum. On the other hand, let $M, M' \in \mathbb{N}$ be such that $K_n \subset \mathrm{cl}\big(N(\frac{1}{M'},K)\big) \subsetneq X$ whenever $n > M$. Since $\mathrm{int}(K_n) = \emptyset$ for every n, clearly $\mathrm{cl}\big(N(\frac{1}{M'},K)\big) \cup \bigcup_{n=1}^{M} K_n \subsetneq X$; therefore, $W \in C(p,X) \setminus \{X\}$, and thus, we can consider the arc $[l_W, r_W]$. Now, since $K_n \to K$, and we are working in the arc W, we necessarily have that $l_{K_n} \to l_K$ and $r_{K_n} \to r_K$. Hence, $f(K_n) \to f(K)$ and f is continuous.

Claim 3. f is open.

By Claim 1 we know that f is a bijection onto $\operatorname{im}(f)$; thus, it is enough to show that f^{-1} is continuous. Let $\{(x_n, y_n)\}_{n=1}^{\infty} \subset \operatorname{im}(f)$ be a sequence converging to $(x, y) \in \operatorname{im}(f)$. Then $(x_n, y_n) = f(K_n)$ and $(x, y) = f(K)$, for some $K, K_n \in C(p, X) \setminus \{X\}$, for each $n \in \mathbb{N}$. Suppose that the sequence$\{K_n\}_{n=1}^{\infty}$ does not converge to K, then there is a subsequence $\{K_{n_i}\}_{i=1}^{\infty}$ which converges to some $J \in C(p, X) \setminus \{K\}$. If $J \neq X$, then, by continuity of f, $\lim f(K_{n_i}) = f(J)$. But by construction, we have that $\lim f(K_n) = f(K)$, so according to Claim 1, we conclude that $J = K$, a contradiction. Therefore, we may assume that $J = X$. Since $\lim \big([l_{K_{n_i}}, p] \cup [p, r_{K_{n_i}}]\big) = \lim K_{n_i} = J = X$, and X is indecomposable, we may suppose that $\lim [l_{K_{n_i}}, p] = X$. But then $\lim x_{n_i} = \lim \mu\big([l_{K_{n_i}}, p]\big) = \mu(X)$. Thus, $\mu\big([l_K, p]\big) = x = \mu(X)$, a contradiction with the monotoneity of μ. Hence, $\lim K_n = K$ and f^{-1} is continuous.

Claim 4. If $(x, y) \in \operatorname{im}(f)$, then $[0, x] \times [0, y] \subset \operatorname{im}(f)$.

Let $(z, w) \in [0, x] \times [0, y]$, and consider $A \in C(p, X) \setminus \{X\}$ such that $f(A) = (x, y)$. Let α and β be order arcs from $\{p\}$ to $[l_A, p]$ and $[p, r_A]$, respectively. Since μ, α, and β are continuous, there exist $s, t \in I$ such that $\mu\big(\alpha(s)\big) = z$ and $\mu\big(\beta(t)\big) = w$. Then it is not difficult to see that $f\big(\alpha(s) \cup \beta(t)\big) = \Big(\mu\big(\alpha(s)\big), \mu\big(\beta(t)\big)\Big) = (z, w)$. The claim is proved.

Claim 5. $C(p, X)$ is a 2-cell.

As a consequence of Claim 4, it can be shown that $\operatorname{im}(f)$ is homeomorphic to one of the following spaces: $I \times I, I \times [0, 1), [0, 1) \times I$ or $[0, 1) \times [0, 1)$ (note that the last three are pairwise homeomorphic). Moreover, since f is a homeomorphism onto its image, and $C(p, X) \setminus \{X\}$ is not compact, we have that $\operatorname{im}(f) \approx I \times [0, 1)$. Furthermore, $C(p, X)$ is homeomorphic to the one-point compactification of $\operatorname{im}(f)$; hence, $C(p, X)$ is a 2-cell. □

Corollary 5.22. *A continuum X is arc-similar if and only if it has exactly two end points and its proper and nondegenerate subcontinua are arcs.*

Proof: The necessity is a direct consequence of Observation 5.18 and Theorem 5.12. Suppose now that X has exactly two end points and that its proper and nondegenerate subcontinua are arcs. If X is decomposable, X is an arc or a simple closed curve. Since the latter case is impossible, applying Theorem 3.17, we get that X is arc-similar. On the other hand, if X is indecomposable with two end points a and b, by Observation 5.18, we know that $C(a, X)$ and $C(b, X)$ are arcs. The conclusion follows from Theorem 5.21. $\square$

Corollary 5.23. *If X belongs to class $\mathcal{K}_1$, then X is arc-similar.*

Corollary 5.24. *A continuum X is circle-similar if and only if it has no end points and its proper and nondegenerate subcontinua are arcs.*

Proof: The necessity follows from Observation 5.18 and Theorem 5.12.

Suppose now that X has no end points and that its proper and nondegenerate subcontinua are arcs. If X is decomposable, then necessarily it is a simple closed curve. Hence, by Theorem 3.18, we get that X is circle-similar. If X is indecomposable, the conclusion follows from Theorem 5.21. $\square$

Definition 5.25. Let $X \in \mathcal{P}$ and let $n \in \mathbb{N} \cup \{0, \omega\}$. We will say that X is of *size n,* provided that the cardinality of the set $\{p \in X : C(p, X) \text{ is an arc }\}$ is n.

Theorem 5.26. *Let $n \in \mathbb{N} \cup \{0, \omega\}$. Then a continuum X belongs to class $\mathcal{P}$ and is of size n if and only if its proper and nondegenerate subcontinua are arcs, and X has exactly n end points.*

Proof: The necessity is a direct consequence of Observation 5.18 and Theorem 5.12. To prove the sufficiency, we shall suppose first that X is decomposable. Then X is an arc or a simple closed curve. If $n = 2$, from Corollary 5.22, we deduce that $X \in \mathcal{P}$ and X is of size 2. On the other hand, if $n = 0$, from Corollary 5.24, we conclude that $X \in \mathcal{P}$ and X is of size 0. Now, if we suppose that X is indecomposable and E is the set of end points of X, then $C(p, X)$ is an arc, for each $p \in E$. Moreover, for each $p \in X \setminus E$, by Theorem 5.21, we know that $C(p, X)$ is a 2-cell. Therefore, $X \in \mathcal{P}$ and X is of size n. $\square$

Finally, in examples 3.21, 5.19, and 5.20, we showed a continuum $X \in \mathcal{P}$ of size 2; the arc is such an example too. On the other hand, S^1, a solenoid, and the continuum presented in Example 5.17 are examples of continua belonging to class $\mathcal{P}$ with size 0. Let

$n \in \mathbb{N} \cup \{\omega\}$. In his paper [2], Doucet constructs indecomposable continua, with n end points, in such a way that its proper and nondegnerate subcontinua are arcs. Thus, according to Theorem 5.26, such continua are of size n and belong to class $\mathcal{P}$.

Questions

1. Let T be a simple triod. Does there exist a continuum X, $X \neq T$, such that $\mathcal{K}(X)$ coincides with $\mathcal{K}(T)$? If so, must X be indecomposable?

2. More generally, if G is a finite graph, does there exist a continuum X such that $\mathcal{K}(X)$ coincides with $\mathcal{K}(G)$? If so, must X be indecomposable?

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