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A PRODUCT THEOREM IN COHOMOLOGICAL DIMENSION FOR CYCLIC GROUPS AND METRIZABLE SPACES

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Abstract. Our main result is a product theorem in cohomological dimension for arbitrary metrizable spaces and groups that are direct sums of cyclic groups.

Theorem. Let $F = \bigoplus_{i \in R} F_i$ where for each $i \in R$, either $F_i = \mathbb{Z}$ or there exist $p_i \in \mathbb{P}$ and $k_i \in \mathbb{P}$ such that $F_i = \mathbb{Z}/p_i^{k_i}$. Then, $\dim_F X \times Y \leq \dim_F X + \dim_F Y$.

1. Introduction

For covering dimension, dim, it is a classical result [3, 4.1.21 and 4.1.3], that $\dim X \times Y \leq \dim X + \dim Y$ for metrizable spaces X and Y. This logarithmic formula does not hold in general for cohomological dimension, $\dim_F$, even when X and Y are compact and metrizable. (See Theorem 7 of [5].) So it is interesting to determine whether the logarithmic law,

$$\dim_F X \times Y \leq \dim_F X + \dim_F Y$$

holds for some groups F where X and Y are arbitrary metrizable spaces.

In this work we shall prove the following product theorem in cohomological dimension for metrizable spaces. Let $\mathbb{P}$ designate the set of all primes.

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Theorem 1.1 (Product Theorem)**.** *Let $F = \bigoplus_{i \in R} F_i$ where for each $i \in R$, either $F_i = \mathbb{Z}$ or there exist $p_i \in \mathbb{P}$ and $k_i \in \mathbb{N}$ such that $F_i = \mathbb{Z}/p_i^{k_i}$. Suppose that $n \geq 0$, $m \geq 0$, X, Y are metrizable spaces,* $\dim_F X \leq n$, *and* $\dim_F Y \leq m$. *Then* $\dim_F X \times Y \leq n+m$.

2. Proof of Theorem

Recall that for a given abelian group F and metrizable space X, one says that X is F-acyclic if all the reduced Čech cohomology groups $\widetilde{\check{H}}^k(X;F)$ are trivial and that a map is called F-acyclic if each of its fibers is F-acyclic. We begin with Theorem 1.2 of [4].

Theorem 2.1. *Let X be a metrizable space, $n \in \mathbb{N}$, and $p \in \mathbb{P}$. If* $\dim_{\mathbb{Z}/p} X \leq n$, *then there exists a metrizable space Z with* $\dim Z \leq n$ *and a proper surjective $\mathbb{Z}/p$-acyclic map of Z onto X.*

Since the product of two cell-like sets is cell-like and hence acyclic with respect to any (abelian) coefficient group, the next result follows from the main theorem of [6].

Theorem 2.2. *Let X be a metrizable space, $n \in \mathbb{N}$, and $p \in \mathbb{P}$. If* $\dim_{\mathbb{Z}} X \leq n$, *then there exists a metrizable space Z with* $\dim Z \leq n$ *and a proper surjective $\mathbb{Z}$-acyclic map of Z onto X.*

Lemma 2.3. *Let A and B be metrizable continua.*

(a) *If A and B are $\mathbb{Z}$-acyclic, then so is $A \times B$.*
(b) *Let $p \in \mathbb{P}$. If A and B are $\mathbb{Z}/p$-acyclic, then so is $A \times B$.*

Proof: From [7, Example 5.2.3, p. 220], one sees that the torsion product $\mathbb{Z} * \mathbb{Z} = 0$. That $\mathbb{Z}/p * \mathbb{Z}/p = 0$ can be found also in [7] (on page 221 near the bottom of the page). One then applies Exercise 6.E.5 [7, p. 360] to complete the proof. □

Using theorems 2.1 and 2.2, Lemma 2.3, and the Vietoris-Begle mapping theorem, we get the next result.

Corollary 2.4. *Let X and Y be metrizable spaces, $p \in \mathbb{P}$, and $F \in \{\mathbb{Z}, \mathbb{Z}/p\}$. Then* $\dim_F X \times Y \leq \dim_F X + \dim_F Y$.

Let us mention that this result for $F = \mathbb{Z}$ originally was observed by R. Millspaugh and the first author. It also has come to the attention of others, e.g., [2, p. 1649].

For the next lemma, see the bottom of page 32 of [5].

Lemma 2.5. *Let X be a metrizable space, $p \in \mathbb{P}$, and $k \in \mathbb{N}$. Then* $\dim_{\mathbb{Z}/p^k} X = \dim_{\mathbb{Z}/p} X$.

We recall Corollary 2.7 of [1].

Lemma 2.6. *Suppose X is a metrizable space and $F = \bigoplus_{i \in R} F_i$ is the direct sum of abelian groups F_i. Then*

$$\dim_F X = \sup\{\dim_{F_i} X \mid i \in R\}.$$

Using the short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \longrightarrow \mathbb{Z}/p^k \longrightarrow 0$, one may readily obtain a proof of the following.

Lemma 2.7. *For each $p \in \mathbb{P}$, $n \in \mathbb{N}$, and metrizable space X,* $\dim_{\mathbb{Z}/p} X \leq \dim_{\mathbb{Z}} X$.

Proof of Theorem 1.1: If $\mathbb{Z}$ is one of the summands, then apply lemmas 2.6 and 2.7 and Corollary 2.4 with $F = \mathbb{Z}$. Otherwise, apply lemmas 2.6 and 2.5, and then Corollary 2.4 with $F = \mathbb{Z}/p$ for an appropriate value of p. $\square$

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