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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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A NOTE ON AWAY-ALMOST CONTINUOUS FUNCTIONS

PIOTR SZUCA

Abstract. We prove that an AAC_0 function is AAC iff its graph is bilaterally $\mathfrak{c}$-dense in itself.

1. Introduction

Let $\mathbb{I} = [0,1]$. Let $B_x = \{y \in \mathbb{R} \mid \langle x, y\rangle \in B\}$ for every set $B \subset \mathbb{I} \times \mathbb{R}$. A closed set $B \subset \mathbb{I} \times \mathbb{R}$ is blocking if $B_x \neq \mathbb{R}$ for every $x \in \mathbb{I}$ and $g \cap B \neq \emptyset$ for every continuous function $g\colon \mathbb{I} \to \mathbb{R}$.

Given a family $\mathcal{F}$ of real functions, we denote by $\overline{\mathcal{F}}$ the uniform closure of $\mathcal{F}$, i.e., the family of all uniform limits of sequences of functions from $\mathcal{F}$.

We will consider the following properties of functions from $\mathbb{I}$ to $\mathbb{R}$.

- A function f is bilaterally $\mathfrak{c}$-dense in itself ($f \in \mathfrak{D}_{\mathfrak{c}}$) if for every $x \in \mathbb{I}$ and every open neighborhood U of $\langle x, f(x)\rangle$ the set $([0,x) \times \mathbb{R}) \cap U \cap f$ has cardinality $\mathfrak{c}$ for every $x > 0$ and the set $((x,1] \times \mathbb{R}) \cap U \cap f$ has cardinality $\mathfrak{c}$ for every $x < 1$, respectively.
- A function f is Darboux ($f \in \mathrm{D}$) if it maps connected sets onto connected sets.
- $f \in \mathcal{U}_0$ if for each interval $J \subset \mathbb{I}$ the set $f(J)$ is dense in the interval $(\inf_{x\in J} f(x), \sup_{x\in J} f(x))$.

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- $f \in \mathcal{U}$ if for each interval $J \subset \mathbb{I}$ and each $A \subset \mathbb{I}$ of cardinality less than $\mathfrak{c}$, the set $f(J \setminus A)$ is dense in the interval $(\inf_{x\in J} f(x), \sup_{x\in J} f(x))$.
- A function f is almost continuous ($f \in \mathrm{AC}$) if each open neighborhood of f contains the graph of a continuous function $g\colon \mathbb{I} \to \mathbb{R}$. (Recall that $f \in \mathrm{AC}$ iff $f \cap B \neq \emptyset$ for every blocking set B [4].)
- A function f is weakly away-almost continuous ($f \in \mathrm{AAC}_0$) if for every $\varepsilon > 0$ and blocking set B there exists $x \in \mathbb{I}$ such that $(f(x) - \varepsilon, f(x) + \varepsilon) \cap B_x \neq \emptyset$.
- A function $f\colon \mathbb{I} \to \mathbb{R}$ is away-almost continuous ($f \in \mathrm{AAC}$) if for every blocking set B either $f \cap B \neq \emptyset$ or for every $\varepsilon > 0$ the set $\{x \in \mathbb{I} \mid (f(x) - \varepsilon, f(x) + \varepsilon) \cap B_x \neq \emptyset\}$ has cardinality $\mathfrak{c}$.

Recall that $\overline{\mathrm{D}} = \mathcal{U} = \mathcal{U}_0 \cap \mathfrak{D}_{\mathfrak{c}}$ [1]. The similar problem of characterization of the class $\overline{\mathrm{AC}}$ remains still open and it seems to be one of the most interesting problems concerning Darboux-like functions. (See [2, Question 9.14].) In 1974, K. Kellum proved that $\mathrm{AC} \neq \overline{\mathrm{AC}}$ [3]. A few years ago, in 1999, during the Miniconference in Real Analysis at Auburn University, he conjectured that $\overline{\mathrm{AC}} = \mathrm{AAC}_0 \cap \mathcal{U}$. It is easy to see that $\overline{\mathrm{AC}} \subset \mathrm{AAC}_0 \cap \mathcal{U}$ and $\mathrm{AAC}_0 \subset \mathcal{U}_0$, so $\mathrm{AAC}_0 \cap \mathcal{U} = \mathrm{AAC}_0 \cap \mathfrak{D}_{\mathfrak{c}}$. In [5], we define the property AAC and show that $\overline{\mathrm{AC}} \subset \mathrm{AAC} \subset \mathrm{AAC}_0 \cap \mathcal{U}$ and ask whether $\mathrm{AAC} = \mathrm{AAC}_0 \cap \mathcal{U}$ (Problem 1). In this note we answer this question in the positive. This implies the equality $\mathrm{AAC} = \mathrm{AAC}_0 \cap \mathfrak{D}_{\mathfrak{c}}$, similarly to $\mathcal{U} = \mathcal{U}_0 \cap \mathfrak{D}_{\mathfrak{c}}$.

2. The result

We start with the following easy remark.

Remark 2.1. A function $f \in \mathrm{AAC}_0$ iff for every $\varepsilon > 0$ and every open set $G \subset \mathbb{I} \times \mathbb{R}$ such that $\bigcup_{x\in\mathbb{I}} (\{x\} \times (f(x) - \varepsilon, f(x) + \varepsilon)) \subset G$ there exists a continuous function $g\colon \mathbb{I} \to \mathbb{R}$ such that $g \subset G$.

We will use also operators $\mathcal{E}(\cdot)$, $\mathcal{N}(\cdot)$, $\mathrm{A}_{\mathrm{EE}}(\cdot)$, $\mathrm{A}_{\mathrm{NN}}(\cdot)$, and $\mathrm{A}_{\mathrm{NE}}(\cdot)$ introduced in [5].

Definition 2.2. For a blocking set $B \subset \mathbb{I} \times \mathbb{R}$ let

- $\mathcal{E}(B) = \{\langle a, b\rangle \in \mathbb{I}\times\mathbb{R} \mid (\exists h\colon [0, a] \to \mathbb{R})\ (h(a) = b \,\&\, h\cap B = \emptyset \,\&\, h \text{ is continuous})\}$;

- $\mathcal{N}(B) = (\mathbb{I} \times \mathbb{R}) \backslash (B \cup \mathcal{E}(B))$.

It is easy to see that $\mathbb{I} \times \mathbb{R}$ is the union of pairwise disjoint sets B, $\mathcal{E}(B)$, and $\mathcal{N}(B)$.

Definition 2.3. Define also:

- $A_{EE}(B) = \{\langle a, b\rangle \in \mathbb{I} \times \mathbb{R} \mid$ there exists an open set G such that $\langle a, b\rangle \in G$ & $G \subset \mathcal{E}(B)\}$;
- $A_{NN}(B) = \{\langle a, b\rangle \in \mathbb{I} \times \mathbb{R} \mid$ there exists an open set G such that $\langle a, b\rangle \in G$ & $G \subset \mathcal{N}(B)\}$;
- $A_{NE}(B) = \{\langle a, b\rangle \in \mathbb{I} \times \mathbb{R} \mid$ there exists an open set G such that $\langle a, b\rangle \in G$ & $([0, a] \times \mathbb{R}) \cap G \subset \mathcal{N}(B)$ & $((a, 1] \times \mathbb{R}) \cap G \subset \mathcal{E}(B)\}$.

Remark 2.4 [5]. Let $B \subset \mathbb{I} \times \mathbb{R}$ be a blocking set. Then

(1) $\{0\} \times \mathbb{R} \subset B \cup \mathcal{E}(B)$.
(2) $\{1\} \times \mathbb{R} \subset B \cup \mathcal{N}(B)$.
(3) $\mathcal{E}(B)$ is open.
(4) $\mathcal{N}(B)$ is left-open (i. e., for every $\langle x, y\rangle \in \mathcal{N}(B)$ there exists an open neighborhood U of $\langle x, y\rangle$ such that $([0, x] \times \mathbb{R}) \cap U \subset \mathcal{N}(B)$).
(5) If $\langle x, y_1\rangle \in \mathcal{E}(B)$ and $\langle x, y_2\rangle \in \mathcal{N}(B)$, then there exists $y \in (y_1, y_2)$ such that $\langle x, y\rangle \in B$.
(6) $\mathbb{I} \times \mathbb{R}$ is the union of pairwise disjoint sets B, $A_{EE}(B)$, $A_{NN}(B)$, and $A_{NE}(B)$.

Lemma 2.5. *Assume that $f \in AAC_0$ and f is bilaterally dense in itself. Then for every blocking set B at least one of the following two conditions holds*:

(1) $f \cap B \neq \emptyset$,
(2) *for every $\varepsilon > 0$ there exist $a \in \mathbb{I}$ and open set $U \ni \langle a, f(a)\rangle$ such that $|f(b) - B_b| < \varepsilon$ for every $\langle b, f(b)\rangle \in U$.*

Proof: Suppose f is bilaterally dense in itself, f is weakly away-almost continuous and neither (1) nor (2) hold. Then there exist a blocking set $B \subset \mathbb{I} \times \mathbb{R}$ and $\varepsilon > 0$, such that:

(1) $B \cap f = \emptyset$ and

(2) for every $a \in \mathbb{I}$ and open set $U \ni \langle a, f(a)\rangle$ there exists $\langle b, f(b)\rangle \in U$ such that $|f(b) - B_b| \geq \varepsilon$.

Since $B \cap f = \emptyset$, $f \subset \mathrm{A}_{\mathrm{EE}}(B) \cup \mathrm{A}_{\mathrm{NN}}(B) \cup \mathrm{A}_{\mathrm{NE}}(B)$ (see Remark 2.4 (6)). For every $\langle x, f(x)\rangle$ we will construct a rectangular open neighborhood $S_{\langle x,f(x)\rangle} = (x - \tau_x, x + \tau_x) \times (f(x) - \varepsilon, f(x) + \varepsilon)$ such that:

- $S_{\langle x,f(x)\rangle} \cap f \subset \mathcal{E}(B)$ if $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{EE}}(B)$;
- $S_{\langle x,f(x)\rangle} \cap f \subset \mathcal{N}(B)$ if $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{NN}}(B)$;
- $S_{\langle x,f(x)\rangle} \cap f \cap ([0, x] \times \mathbb{R}) \subset \mathcal{N}(B)$ and $S_{\langle x,f(x)\rangle} \cap f \cap ((x, 1] \times \mathbb{R}) \subset \mathcal{E}(B)$ if $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{NE}}(B)$.

To show that such a neighborhood exists take $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{NE}}(B)$. (The cases $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{EE}}(B)$ and $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{NN}}(B)$ are analogous.)

There exists an open rectangle $(x-\tau_x, x+\tau_x)\times(f(x)-\tau_x, f(x)+\tau_x)$ such that $(x-\tau_x, x]\times(f(x)-\tau_x, f(x)+\tau_x) \subset \mathcal{N}(B)$ and $(x, x+\tau_x)\times(f(x)-\tau_x, f(x)+\tau_x) \subset \mathcal{E}(B)$. Set $S_{\langle x,f(x)\rangle} = (x-\tau_x, x+\tau_x)\times(f(x)-\varepsilon, f(x)+\varepsilon)$. Suppose there exists $x_1 \in (x-\tau_x, x]$ such that $\langle x_1, f(x_1)\rangle \in \mathcal{E}(B) \cap S_{\langle x,f(x)\rangle}$ or there exists $x_2 \in (x, x + \tau_x)$ such that $\langle x_2, f(x_2)\rangle \in \mathcal{N}(B) \cap S_{\langle x,f(x)\rangle}$.

In the first case, since f is left side dense at $\langle x_1, f(x_1)\rangle$ and $\mathcal{E}(B)$ is open, the set

$$P_1 = \left\{p \in (x - \tau_x, x) \mid \langle p, f(p)\rangle \in \mathcal{E}(B) \cap S_{\langle x,f(x)\rangle}\right\}$$

is non-empty. Now, if we take any $\langle a, f(a)\rangle \in P_1 \times \mathbb{R}$ and its open neighborhood $U \subset \mathcal{E}(B) \cap S_{\langle x,f(x)\rangle}$, for every $\langle b, f(b)\rangle \in U$ we have $\langle b, f(b)\rangle \in \mathcal{E}(B)$, and $\langle b, f(x)\rangle \in \mathcal{N}(B)$, and $|f(x) - f(b)| < \varepsilon$. According to Remark 2.4 (5), there exists y_b such that $\langle b, y_b\rangle \in B$ and $|f(b) - y_b| < \varepsilon$, contrary to Lemma 2.5 (2).

The second case is analogous if we take into consideration left side neighborhood of $\langle x_2, f(x_2)\rangle$ and the set

$$P_2 = \left\{p \in (x, x + \tau_x) \mid \langle p, f(p)\rangle \in \mathcal{N}(B) \cap S_{\langle x,f(x)\rangle}\right\}.$$

For every $x \in \mathbb{I}$, let $R_{\langle x,f(x)\rangle} = (x_l, x_r) \times (f(x) - \frac{\varepsilon}{3}, f(x) + \frac{\varepsilon}{3}) \subset S_{\langle x,f(x)\rangle}$ be an open rectangular neighborhood of $\langle x, f(x)\rangle$ such that:

- $(x_l, x_r) \subset (x - \frac{\tau_x}{3}, x + \frac{\tau_x}{3})$,
- $f(x_l) \in (f(x) - \frac{\varepsilon}{3}, f(x) + \frac{\varepsilon}{3})$ for every $x > 0$,
- $f(x_r) \in (f(x) - \frac{\varepsilon}{3}, f(x) + \frac{\varepsilon}{3})$ for every $x < 1$.

Note that for $x > 0$, the rectangle $R_{\langle x,f(x)\rangle}$ does not contain points with abscissa 0. Respectively, the $R_{\langle x,f(x)\rangle}$ does not contain points with abscissa 1 for $x < 1$. Moreover, for every $x < 1$, the distance between $R_{\langle x,f(x)\rangle}$ and $\{1\} \times \mathbb{R}$ is positive.

Note also that if $R_{\langle a,b\rangle} \cap R_{\langle c,d\rangle} \neq \emptyset$, then $R_{\langle a,b\rangle} \subset S_{\langle c,d\rangle}$ if $\tau_a \leq \tau_c$ or $R_{\langle c,d\rangle} \subset S_{\langle a,b\rangle}$ if $\tau_c \leq \tau_a$, respectively.

For every $x \in \mathbb{I}$, the set $R_{\langle x,f(x)\rangle}$ fulfills the following conditions:

($\circ_1$) if $\langle r, f(r)\rangle \in \mathcal{E}(B) \cap S_{\langle x,f(x)}$ for a $r \leq x$, then $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{EE}}(B)$, and for every $t > x_l$ there exists $z < t$ such that $\langle z, f(z)\rangle \in R_{\langle x,f(x)\rangle} \cap \mathcal{E}(B)$;

($\circ_2$) if $\langle r, f(r)\rangle \in \mathcal{E}(B) \cap S_{\langle x,f(x)\rangle}$ for an $r > x$, then $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{EE}}(B) \cup \mathrm{A}_{\mathrm{NE}}(B)$, and for every $t > x$ there exists $z \in (x, t)$ such that $\langle z, f(z)\rangle \in R_{\langle x,f(x)\rangle} \cap \mathcal{E}(B)$;

($\bullet_1$) if $R_{\langle x,f(x)\rangle} \cap ((r, x) \times \mathbb{R}) \cap f \subset \mathcal{E}(B)$ for a $r \in (x_l, x)$, then $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{EE}}(B)$, and there exists $z < r$ such that $\langle z, f(z)\rangle \in R_{\langle x,f(x)\rangle} \cap \mathcal{E}(B)$;

($\bullet_2$) if $R_{\langle x,f(x)\rangle} \cap ((r, x_r) \times \mathbb{R}) \cap f \subset \mathcal{E}(B)$ for a $r \in (x, x_r)$, then $\langle x, f(x)\rangle \in \mathrm{A}_{\mathrm{EE}}(B) \cup \mathrm{A}_{\mathrm{NE}}(B)$, and there exists $z < r$ such that $\langle z, f(z)\rangle \in R_{\langle x,f(x)\rangle} \cap \mathcal{E}(B)$.

Let $H = \bigcup_{x\in\mathbb{I}} R_{\langle x,f(x)\rangle}$. H is an open set and $f \subset H$, so by Remark 2.1, there exists a continuous function $g\colon \mathbb{I} \to \mathbb{R}$ such that $g \subset H$. Let $\mathcal{R}$ be a finite subfamily of $\{R_{\langle x,f(x)\rangle} \mid x \in \mathbb{I}\}$ such that $g \subset \bigcup \mathcal{R}$.

Since only the $R_{\langle 0,f(0)\rangle}$ contains points with abscissa 0 and only the $R_{\langle 1,f(1)\rangle}$ contains points with abscissa 1, $R_{\langle 0,f(0)\rangle} \in \mathcal{R}$ and $R_{\langle 1,f(1)\rangle} \in \mathcal{R}$. Moreover, since $\mathcal{R}$ is finite and the distance between $R_{\langle x,f(x)\rangle}$ and $\{1\} \times \mathbb{R}$ is positive for every $x < 1$,

$$\sup\Big\{x \in \mathbb{I} \mid \langle x, y\rangle \in \bigcup\big(\mathcal{R} \setminus \{R_{\langle 1,f(1)\rangle}\}\big)\Big\} < 1. \qquad (\star)$$

Let $C = \{x \in \mathbb{I} \mid (\exists R \in \mathcal{R})\, (\langle x, g(x)\rangle \in R \,\&\, (\exists x_1 \leq x)\, \langle x_1, f(x_1)\rangle \in \mathcal{E}(B) \cap R)\}$ and let $s = \sup C$. Since $\langle 0, f(0)\rangle \in \mathcal{E}(B)$ and $\langle 0, g(0)\rangle \in R_{\langle 0,f(0)\rangle}$, there exist $x > 0$ and $x_1 \in (0, x)$ such that $\langle x, g(x)\rangle \in R_{\langle 0,f(0)\rangle}$, $\langle x_1, f(x_1)\rangle \in \mathcal{E}(B) \cap R_{\langle 0,f(0)\rangle}$, so $s \geq x > 0$. Analogously, the condition $(\star)$ implies $s < 1$.

Since $\mathcal{R}$ is finite and g is continuous, there exist $R_{\langle p,f(p)\rangle} \in \mathcal{R}$ and $p_1 \le s$ such that $\langle s, g(s)\rangle \in \overline{R_{\langle p,f(p)\rangle}}$ and $\langle p_1, f(p_1)\rangle \in \mathcal{E}(B) \cap R_{\langle p,f(p)\rangle}$.

Let $R_{\langle q,f(q)\rangle} \in \mathcal{R}$ be an open rectangle such that $\langle s, g(s)\rangle \in R_{\langle q,f(q)\rangle}$. Since $\langle s, g(s)\rangle \in \overline{R_{\langle p,f(p)\rangle}} \cap R_{\langle q,f(q)\rangle}$, $R_{\langle p,f(p)\rangle} \cap R_{\langle q,f(q)\rangle} \neq \emptyset$.

We have two cases:

(1) $R_{\langle p,f(p)\rangle} \subset S_{\langle q,f(q)\rangle}$, if $\tau_p \le \tau_q$;
(2) $R_{\langle q,f(q)\rangle} \subset S_{\langle p,f(p)\rangle}$, otherwise.

Case 1. Then $\langle p_1, f(p_1)\rangle \in \mathcal{E}(B) \cap S_{\langle q,f(q)\rangle}$. There exists $q_1 \le s$ such that $\langle q_1, f(q_1)\rangle \in R_{\langle q,f(q)\rangle} \cap \mathcal{E}(B)$. Indeed, if $p_1 \le q$ then $\langle p_1, f(p_1)\rangle \in ([0,q] \times \mathbb{R}) \cap \mathcal{E}(B) \cap S_{\langle q,f(q)\rangle}$, so $\langle q, f(q)\rangle \in \mathrm{A}_{\mathrm{EE}}(B)$ and there exists $q_1 \le s$ such that $\langle q_1, f(q_1)\rangle \in \mathcal{E}(B) \cap R_{\langle q,f(q)\rangle}$ (see the condition $(\circ_1)$). If $p_1 > q$, then $\langle p_1, f(p_1)\rangle \in ((q,1] \times \mathbb{R}) \cap \mathcal{E}(B) \cap S_{\langle q,f(q)\rangle}$ and $s > q$, so $\langle q, f(q)\rangle \in \mathrm{A}_{\mathrm{EE}}(B) \cup \mathrm{A}_{\mathrm{NE}}(B)$ and there exists $q_1 \le s$ such that $\langle q_1, f(q_1)\rangle \in \mathcal{E}(B) \cap R_{\langle q,f(q)\rangle}$ (see the condition $(\circ_2)$).

Now, since g is continuous and $s < 1$, there exists $s_1 > s$ such that $\langle s_1, g(s_1)\rangle \in R_{\langle q,f(q)\rangle}$, so $s_1 \in C$, a contradiction.

Case 2. There exists $q_1 > s$ such that $\langle q_1, f(q_1)\rangle \in \mathcal{N}(B) \cap R_{\langle q,f(q)\rangle}$. Indeed, suppose by contradiction that no $q_1 > s$ fulfills the claim. Since g is continuous and $s < 1$, there exists $s_1 > s$ such that $\langle s_1, g(s_1)\rangle \in R_{\langle q,f(q)\rangle}$ and $s_1 \neq q$. By supposition, $((s_1,1] \times \mathbb{R}) \cap R_{\langle q,f(q)\rangle} \cap \mathcal{N}(B) \cap f = \emptyset$, so $((s_1,1] \times \mathbb{R}) \cap R_{\langle q,f(q)\rangle} \cap f \subset \mathcal{E}(B)$. If $s_1 < q$ then there exists $z \le s_1$ such that $\langle z, f(z)\rangle \in \mathcal{E}(B) \cap R_{\langle q,f(q)\rangle}$ (see the condition $(\bullet_1)$). If $s_1 > q$ then there exists $z \le s_1$ such that $\langle z, f(z)\rangle \in \mathcal{E}(B) \cap R_{\langle q,f(q)\rangle}$ (see the condition $(\bullet_2)$). In any case, $s_1 \in C$. Since this is a contradiction, there exists $q_1 > s$ such that $\langle q_1, f(q_1)\rangle \in \mathcal{N}(B) \cap R_{\langle q,f(q)\rangle}$.

Since $\langle q_1, f(q_1)\rangle \in \mathcal{N}(B) \cap R_{\langle q,f(q)\rangle}$, $\langle q_1, f(q_1)\rangle \in \mathcal{N}(B) \cap S_{\langle p,f(p)\rangle}$. But $\langle p_1, f(p_1)\rangle \in \mathcal{E}(B) \cap S_{\langle p,f(p)\rangle}$ and $p_1 < q_1$ which is impossible because $\langle p, f(p)\rangle \in \mathrm{A}_{\mathrm{EE}}(B) \cup \mathrm{A}_{\mathrm{NN}}(B) \cup \mathrm{A}_{\mathrm{NE}}(B)$. □

Theorem 2.6. *The following equalities hold.*

$$\mathrm{AAC} = \mathrm{AAC}_0 \cap \mathcal{U} = \mathrm{AAC}_0 \cap \mathfrak{D}_{\mathfrak{c}}$$

Proof: The inclusion "$\mathrm{AAC} \subset \mathrm{AAC}_0 \cap \mathcal{U}$" is proved in [5]. Since $\mathcal{U} \subset \mathfrak{D}_{\mathfrak{c}}$ [1], we have $\mathrm{AAC}_0 \cap \mathcal{U} \subset \mathrm{AAC}_0 \cap \mathfrak{D}_{\mathfrak{c}}$. To prove the inclusion "$\mathrm{AAC}_0 \cap \mathfrak{D}_{\mathfrak{c}} \subset \mathrm{AAC}$," take a bilaterally $\mathfrak{c}$-dense in itself weakly-away almost continuous function f and fix a blocking set B. By Lemma 2, either $f \cap B \neq \emptyset$, or for every $\varepsilon > 0$ there exist $a \in \mathbb{I}$ and an open set $U \ni \langle a, f(a)\rangle$ such that $U \cap f \subset \{\langle x, f(x)\rangle \mid (f(x)-\varepsilon, f(x)+\varepsilon) \cap B_x \neq \emptyset\}$. In the second case, since $f \in \mathfrak{D}_{\mathfrak{c}}$, the set $U \cap f$ has cardinality continuum, and therefore, the set $\{x \in \mathbb{I} \mid (f(x)-\varepsilon, f(x)+\varepsilon) \cap B_x \neq \emptyset\}$ also has cardinality continuum. Thus, in both cases, $f \in \mathrm{AAC}$. □

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Department of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland

E-mail address: pszuca@radix.com.pl