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SUBMETACOMPACTNESS IN COUNTABLE PRODUCTS

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Abstract. In this paper, we prove the following: If $\{X_n : n \in \omega\}$ is a countable collection of C-scattered submetacompact spaces, then the product $\prod_{n\in\omega} X_n$ is submetacompact.

1. Introduction

A space X is said to be *metacompact* if every open cover of X has a point finite open refinement, and X is said to be *submetacompact* if for every open cover $\mathcal{U}$ of X, there is a sequence $(\mathcal{V}_n)_{n\in\omega}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$, there is an $n \in \omega$ with $\mathrm{Ord}(x, \mathcal{V}_n) < \omega$. For $x \in X$ and $n \in \omega$, let $\mathcal{V}_{n_x} = \{V \in \mathcal{V}_n : x \in V\}$ and $\mathrm{Ord}(x, \mathcal{V}_n) = |\mathcal{V}_{n_x}|$. We call this sequence $(\mathcal{V}_n)_{n\in\omega}$ a *θ-sequence* of open refinemenets of $\mathcal{U}$. Clearly, every paracompact space is metacompact and every metacompact space is submetacompact. It is well known that if X is countably compact and submetacompact, then X is compact.

Since the notion of C-scattered spaces was introduced by R. Telgársky [8], C-scattered spaces play the fundamental role in the study of covering properties of products. A space X is said to be *scattered* if every nonempty subset A of X has an isolated point in A, and X is said to be *C-scattered* if for every nonempty closed subset A of X, there is a point $x \in A$ which has a compact neighborhood in A. Scattered spaces and locally compact spaces are

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C-scattered. Telgársky proved that if X is a C-scattered paracompact (Lindelöf) space, then $X \times Y$ is paracompact (Lindelöf) for every paracompact (Lindelöf) space Y.

Telgársky [9] also introduced the notion of $\mathcal{DC}$-like spaces, using topological games. The class of $\mathcal{DC}$-like spaces includes all spaces with a σ-closure-preserving closed cover by compact subsets and all C-scattered paracompact spaces. Telgársky proved that if X is a paracompact (Lindelöf) $\mathcal{DC}$-like space, then $X \times Y$ is paracompact (Lindelöf) for every paracompact (Lindelöf) space Y. Furthermore, G. Gruenhage and Y. Yajima [3] proved that if X is a metacompact (submetacompact) $\mathcal{DC}$-like space, then $X \times Y$ is metacompact (submetacompact) for every metacompact (submetacompact) space Y, and that if X is a C-scattered metacompact (submetacompact) space, then $X \times Y$ is metacompact (submetacompact) for every metacompact (submetacompact) space Y. For covering properties of countable products, the author proved the following.

(A) ([5]) If Y is a perfect paracompact (hereditarily Lindelöf) space and $\{X_n : n \in \omega\}$ is a countable collection of paracompact (Lindelöf) $\mathcal{DC}$-like spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact (Lindelöf).

(B) ([6, 7]) If $\{X_n : n \in \omega\}$ is a countable collection of metacompact (submetacompact) $\mathcal{DC}$-like spaces, then the product $\prod_{n\in\omega} X_n$ is metacompact (submetacompact).

(C) ([6]) If $\{X_n : n \in \omega\}$ is a countable collection of C-scattered metacompact spaces, then the product $\prod_{n\in\omega} X_n$ is metacompact.

The author asked whether the product $\prod_{n\in\omega} X_n$ is submetacompact whenever X_n is a C-scattered submetacompact space for each $n \in \omega$.

Our purpose in this paper is to give an affirmative answer to this question.

All spaces are assumed to be regular T_1 spaces. Let ω denote the set of natural numbers and $|A|$ denote the cardinality of a set A. Undefined terminology can be found in R. Engelking [2].

2. Submetacompactness

Let X be a space. For a closed subset A of X, let

$$A^* = \{x \in A : x \text{ has no compact neighborhood in } A\}.$$

Let $A^{(0)} = A, A^{(\alpha+1)} = (A^{(\alpha)})^*$ and $A^{(\alpha)} = \cap_{\beta<\alpha} A^{(\beta)}$ for a limit ordinal α. Note that every $A^{(\alpha)}$ is a closed subset of X and if A and B are closed subsets of X such that $A \subset B$, then $A^{(\alpha)} \subset B^{(\alpha)}$ for each ordinal α. Furthermore, X is C-scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α. Let X be a C-scattered space and $A \subset X$. Put

$$\lambda(X) = \inf\{\alpha : X^{(\alpha)} = \emptyset\} \text{ and}$$
$$\lambda(A) = \inf\{\alpha : A \cap X^{(\alpha)} = \emptyset\} \leq \lambda(X).$$

It is clear that if A, B are subsets of X such that $A \subset B$, then $\lambda(A) \leq \lambda(B)$. A subset A of X is said to be *topped* if there is an ordinal $\alpha(A)$ such that $A \cap X^{(\alpha(A))}$ is a nonempty compact subset of X and $A \cap X^{(\alpha(A)+1)} = \emptyset$. Thus, $\lambda(A) = \alpha(A) + 1$. For each $x \in X$, there is a unique ordinal α such that $x \in X^{(\alpha)} - X^{(\alpha+1)}$, which is denoted by $\mathrm{rank}(x) = \alpha$. There is a neighborhood base $\mathcal{B}_x$ of x in X, consisting of open subsets of X, such that for each $B \in \mathcal{B}_x$, $\mathrm{cl}B$ is topped in X and $\alpha(\mathrm{cl}B) = \mathrm{rank}(x)$. If A is a topped subset of X and B is a subset of A such that $B \cap (A \cap X^{(\alpha(A))}) = B \cap X^{(\alpha(A))} = \emptyset$, then $\lambda(B) \leq \alpha(A) < \lambda(A) = \alpha(A) + 1$.

The following plays the fundamental role in the study of submetacompactness.

Lemma 2.1 (Gruenhage and Yajima [3]). *There is a filter $\mathcal{F}$ on ω satisfying: For every submetacompact space X and every open cover $\mathcal{U}$ of X, there is a sequence $(\mathcal{V}_n)_{n\in\omega}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$,*

$$\{n \in \omega : Ord(x, \mathcal{V}_n) < \omega\} \in \mathcal{F}.$$

By Lemma 2.1, let $\mathcal{F}^{n+1}$ denote the filter on ω^{n+1} generated by sets of the form

$$\textstyle\prod_{i\leq n} F_i, \text{ where } F_i \in \mathcal{F} \text{ for each } i \leq n.$$

Put

$$\Phi_n = \textstyle\prod_{i\leq n} \omega^{i+1} \text{ for each } n \in \omega \text{ and } \Phi = \cup\{\Phi_n : n \in \omega\}.$$

For $\mu = (\tau_0, \tau_1, \cdots, \tau_n) \in \Phi_n, n \in \omega$ with $n \geq 1$, let $\mu_- = (\tau_0, \tau_1, \cdots, \tau_{n-1}) \in \Phi_{n-1}$. If $\tau \in \Phi_0$, let $\tau_- = \emptyset$. For each $\tau \in \omega^{n+2}$, let $\mu \oplus \tau = (\tau_0, \tau_1, \cdots, \tau_n, \tau) \in \Phi_{n+1}$. Let $\mathcal{U}, \mathcal{V}$ be collections of subsets of a space X. Put $\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

Theorem 2.2. *If $\{X_n : n \in \omega\}$ is a countable collection of C-scattered submetacompact spaces, then the product $\prod_{n\in\omega} X_n$ is submetacompact.*

Proof: : By the method used in the proof of [1, Theorem], we may assume the following:

(1) X is a C-scattered submetacompact space and for each $n \in \omega$, $X_n = X$,
(2) X is topped and there is a point $a \in X$ such that $X^{(\alpha(X))} = \{a\}$.

We shall show that X^ω is submetacompact. Let $\mathcal{B}$ be the base of X^ω, consisting of all basic open subsets of X^ω, i.e., $B = \prod_{n\in\omega} B_n \in \mathcal{B}$ if there is an $n \in \omega$ such that for $i < n, B_i$ is an open subset of X and for $i \geq n, B_i = X$. Let

$$n(B) = \inf\{i : B_j = X \text{ for } j \geq i\}.$$

We call $n(B)$ the *length* of B. Let $\mathcal{O}$ be an open cover of X^ω, which is closed under finite unions and $\mathcal{O}' = \{B \in \mathcal{B} : B \subset O \text{ for some } O \in \mathcal{O}\}$.

Take $B = \prod_{i\in\omega} B_i \in \mathcal{B}$ and let $\mathcal{N}(B) = \{i < n(B) : \mathrm{cl}B_i$ is topped in $X\}$. Take $i < n(B)$ with $i \notin \mathcal{N}(B)$. If $\lambda(\mathrm{cl}B_i)$ is an isolated ordinal, then there is an ordinal γ such that $\lambda(\mathrm{cl}B_i) = \gamma+1$ and $\mathrm{cl}B_i \cap X^{(\gamma)}$ is nonempty and locally compact. For each $x \in \mathrm{cl}B_i \cap X^{(\gamma)}$, there is an open neighborhood B_x of x in X such that $\mathrm{cl}B_x$ is topped in X, $\mathrm{cl}(B_x \cap X^{(\gamma)})$ is compact, and $\alpha\ (\mathrm{cl}B_x) = \mathrm{rank}(x)$. For each $x \in \mathrm{cl}B_i - X^{(\gamma)}$, take an open neighborhood B_x of x in X such that $\mathrm{cl}B_x$ is topped in X, $\mathrm{cl}B_x \cap (\mathrm{cl}B_i \cap X^{(\gamma)}) = \emptyset$, and $\alpha\ (\mathrm{cl}B_x) = \mathrm{rank}(x)$. Then every $\mathrm{cl}B_i \cap \mathrm{cl}B_x$ is topped in X and $\alpha(\mathrm{cl}B_i \cap \mathrm{cl}B_x) = \alpha\ (\mathrm{cl}B_x)$. Next, let $i = n(B)$. Since $X^{(\alpha(X))} = \{a\}$, take a proper open neighborhood B_a of a in X, and for each $x \in X - \{a\}$, take an open neighborhood B_x of x in X such that $a \notin \mathrm{cl}B_x$, $\mathrm{cl}B_x$ is topped in X, and $\alpha(\mathrm{cl}B_x) = \mathrm{rank}(x)$. If $\lambda(\mathrm{cl}B_i)$ is a limit ordinal, then for each $x \in \mathrm{cl}B_i$, there is an open neighborhood B_x of x in X such that $\mathrm{cl}B_x$ is topped in X and $\alpha(\mathrm{cl}B_x) = \mathrm{rank}(x)$.

Since $\mathcal{B}_i(B) = \{B_x : x \in \mathrm{cl}B_i\}$ is an open cover of $\mathrm{cl}B_i$ and X is submetacompact, there is a θ-sequence $(\mathcal{V}^j_{B,i})_{j\in\omega}$ of open (in X) refinements of $\mathcal{B}_i(B), \mathcal{V}^j_{B,i} = \{V_\xi : \xi \in \Xi^j_{B,i}\}, j \in \omega$, such that for each $j \in \omega, \cup\mathcal{V}^j_{B,i} = B_i$ and for each $x \in B_i, \{j \in \omega : Ord(x, \mathcal{V}^j_{B,i}) <$

$\omega\} \in \mathcal{F}$, where $\mathcal{F}$ is the filter on ω described in Lemma 2.1. For each $j \in \omega$ and $\xi \in \Xi^j_{B,i}$, take an $x(\xi) \in \mathrm{cl}B_i$ such that $V_\xi \subset B_{x(\xi)}$. For each $i \in \mathcal{N}(B)$ and $j \in \omega$, let $\Xi^j_{B,i} = \{\xi^j_{B,i}\}$ and $\mathcal{V}^j_{B,i} = \{V_{\xi^j_{B,i}}\} = \{B_i\}$. For each $\eta = (j_0, j_1, \cdots, j_{n(B)}) \in \omega^{n(B)+1}$, put $\Xi_{B,\eta} = \prod_{i \le n(B)} \Xi^{j_i}_{B,i}$. For each $\xi = (\xi(i)) \in \Xi_{B,\eta}$, let $V(\xi) = \prod_{i \le n(B)} V_{\xi(i)} \times X \times \cdots$ and $\mathcal{V}_\eta(B) = \{V(\xi) : \xi \in \Xi_{B,\eta}\}$. Then every $\mathcal{V}_\eta(B)$ is an open cover of B. Take a $\xi = (\xi(i)) \in \Xi_{B,\eta}$ and let $\mathcal{M}(\xi) = \{i \le n(B) : \mathrm{cl}V_{\xi(i)}$ is topped in X$\}$. Then $\mathcal{N}(B) \subset \mathcal{M}(\xi)$. Put $K(\xi) = \prod_{i \in \mathcal{M}(\xi)} (\mathrm{cl}V_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))}) \times \prod_{i \le n(B), i \notin \mathcal{M}(\xi)} V_{\xi(i)} \times \{a\} \times \cdots = \prod_{i \in \omega} K_{\xi,i}$ and $\mathcal{K}(B,\eta) = \{K(\xi) : \xi \in \Xi_{B,\eta}\}$. We consider the following condition for $K(\xi)$.

(*) There is an open set $B' \in \mathcal{O}'$ such that $K(\xi) \subset B'$.

If $K(\xi)$ satisfies (*), define $n(\xi) = \inf\{n(O) : K(\xi) \subset O$ and $O \in O'\}$. Put

$$r(\xi) = \max\{n(B), n(\xi)\}.$$

There is an $O(\xi) = \prod_{i \in \omega} O_{\xi,i} \in \mathcal{O}'$ such that:

(3) $K(\xi) \subset O(\xi)$,

(4) $n(\xi) = n(O(\xi))$.

Take an $H(\xi) = \prod_{i \in \omega} H_{\xi,i} \in \mathcal{O}'$ such that:

(5) (a) $\prod_{i < n(\xi)} H_{\xi,i} \times X \times \cdots \subset O(\xi)$;

(b) for i with $n(\xi) \le i \le n(B)$ or $i \le n(B)$ with $i \notin \mathcal{M}(\xi)$, let $H_{\xi,i} = O_{\xi,i}$;

(c) for $i < n(\xi)$ with $i \in \mathcal{M}(\xi)$, let $H_{\xi,i}$ be an open subset of X such that $K_{\xi,i} = \mathrm{cl}V_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))} \subset H_{\xi,i} \subset \mathrm{cl}H_{\xi,i} \subset O_{\xi,i}$;

(d) for i with $n(B) < i < n(\xi)$, let $H_{\xi,i}$ be an open subset of X such that $K_{\xi,i} = \{a\} \subset H_{\xi,i} \subset \mathrm{cl}H_{\xi,i} \subset O_{\xi,i}$;

(e) if $r(\xi) = n(B)$, let $H_{\xi,i} = X$ for each $i > n(B)$, and if $r(\xi) = n(\xi) > n(B)$ let $H_{\xi,i} = X$ for $i \ge n(\xi)$.

Then we have $K(\xi) \subset H(\xi)$. Let $\mathcal{P}(B) = \{P : P \subset \{0, 1, \cdots, n(B)\}\}$ and $P \in \mathcal{P}(B)$. Define

$$G(\xi) = \prod_{i \in \omega} G_{\xi,i} \text{ and } B(\xi, P) = \prod_{i \in \omega} B_{\xi,P,i}$$

as follows:

(6) (a) Suppose $r(\xi) = n(B)$. For each $i \le n(B)$, let $G_{\xi,i} = V_{\xi(i)} \cap O_{\xi,i}$ and for each $i > n(B)$, let $G_{\xi,i} = X$.
(b) Suppose $r(\xi) = n(\xi) > n(B)$. For each $i \in \omega$, let $G_{\xi,i} = \emptyset$.
(c) In either case, for each $i \le n(B)$, if $i \in P$, let $B_{\xi,P,i} = V_{\xi(i)} -$ cl$H_{\xi,i}$, and if $i \notin P$, let $B_{\xi,P,i} = V_{\xi(i)} \cap O_{\xi,i}$. For each $i > n(B)$, let $B_{\xi,P,i} = X$.

Clearly, if $r(\xi) = n(B)$, then $B(\xi, \emptyset) = G(\xi)$. Notice that for each $i \in \omega, B_{\xi,P,i} \subset B_i$ and if $B(\xi, P) \ne \emptyset$, then $n(B(\xi, P)) = n(B) + 1$. Let $i \le n(B)$. If $i \in P$ and $i \notin \mathcal{M}(\xi)$, then $B_{\xi,P,i} = \emptyset$.

Let

$$\mathcal{B}_{\eta,\xi}(B) = \{B(\xi, P) : P \in \mathcal{P}(B) - \{\emptyset\}\} \text{ if } r(\xi) = n(B),$$

$$\mathcal{B}_{\eta,\xi}(B) = \{B(\xi, P) : P \in \mathcal{P}(B)\} \text{ if } r(\xi) = n(\xi) > n(B),$$

Claim 1. Let $K(\xi)$ satisfy the condition (*), $P \in \mathcal{P}(B)$ and $B(\xi, P) \in \mathcal{B}_{\eta,\xi}(B)$ with $B(\xi, P) \ne \emptyset$. If $r(\xi) = n(B)$, then there is an $i < n(\xi)$ with $i \in P$.

Proof: of Claim 1: Since $K(\xi)$ satisfies the condition (*) and $r(\xi) = n(B)$, by the definition, $P \ne \emptyset$. Take $i \in P$. By (5) (b) and (6) (c), we have $i < n(\xi)$.

Now, assume that $K(\xi)$ does not satisfy the condition (*). Let $G(\xi) = \emptyset$. Take $P \in \mathcal{P}(B)$ and define $B(\xi, P)$ as follows: If $P = \emptyset$, let $B(\xi, P) = V(\xi)$. If $P \ne \emptyset$, let $B(\xi, P) = \emptyset$. Put $\mathcal{B}_{\eta,\xi}(B) = \{B(\xi, P) : P \in \mathcal{P}(B)\} = \{V(\xi)\}$.

Then, in each case, we have $V(\xi) = G(\xi) \cup (\cup \mathcal{B}_{\eta,\xi}(B))$.

Claim 2. Let $i \le n(B)$, $\xi = (\xi(i)) \in \Xi_{B,\eta}$, $K(\xi) = \prod_{t \in \omega} K_{\xi,t}$, $P \in \mathcal{P}(B)$, and $B(\xi, P) = \prod_{t \in \omega} B_{\xi,P,t}$ with $B_{\xi,P,i} \ne \emptyset$.

(a) If $i \in P$, then $K(\xi)$ satisfies (*), $i \in \mathcal{M}(\xi)$ and $\lambda(\text{cl}B_{\xi,P,i}) < \lambda\ (\text{cl}B_i)$.
(b) Let $i \notin P$.
   (i) If $i \in \mathcal{M}(\xi)$, then cl$B_{\xi,P,i}$ is topped in X such that $\lambda(\text{cl}B_{\xi,P,i}) = \lambda(\text{cl}V_{\xi(i)})$ and $K_{\xi,i} = \text{cl}V_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))} = \text{cl}B_{\xi,P,i} \cap X^{(\alpha(clB_{\xi,P,i}))}$. Furthermore, if $i \in \mathcal{N}(B)$, then cl$B_{\xi,P,i}$ is topped in X such that $\lambda(\text{cl}B_{\xi,P,i}) = \lambda\ (\text{cl}B_i)$ and $K_{\xi,i} = \text{cl}B_i \cap X^{(\alpha(clB_i))} = \text{cl}B_{\xi,P,i} \cap X^{(\alpha(clB_{\xi,P,i}))}$.

(ii) If $i \notin \mathcal{M}(\xi)$, then $\lambda(\mathrm{cl}B_{\xi,P,i}) < \lambda(\mathrm{cl}B_i)$.

Proof: of Claim 2: (a) Since $P \neq \emptyset$ and $B_{\xi,P,i} \neq \emptyset$, we have that $K(\xi)$ satisfies (*) and $i \in \mathcal{M}(\xi)$. Assume that $i \notin \mathcal{N}(B)$. Then $\lambda(\mathrm{cl}V_{\xi(i)}) \leq \lambda\ (\mathrm{cl}B_{x(\xi(i))} \cap \mathrm{cl}B_i) \leq \lambda(\mathrm{cl}B_i)$. Since $i \in P, B_{\xi,P,i} = V_{\xi(i)} - \mathrm{cl}H_{\xi,i}$. So, by (5) (c), $\mathrm{cl}V_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))} \subset H_{\xi,i}$. Then $\lambda\ (\mathrm{cl}B_{\xi,P,i}) \leq \alpha(\mathrm{cl}V_{\xi(i)}) < \lambda(\mathrm{cl}V_{\xi(i)})$. It follows that $\lambda(\mathrm{cl}B_{\xi,P,i}) < \lambda(\mathrm{cl}B_i)$. Assume that $i \in \mathcal{N}(B)$. Then $\lambda(\mathrm{cl}B_i) = \alpha(\mathrm{cl}B_i)+1$. Since $i \in P$, as before, we have $\lambda\ (\mathrm{cl}B_{\xi,P,i}) \leq \alpha(\mathrm{cl}B_i) < \lambda(\mathrm{cl}B_i)$.

(b) (i) Assume that $K(\xi)$ satisfies (*). Since $i \notin P, B_{\xi,P,i} = V_{\xi(i)} \cap O_{\xi,i}$. By $i \in \mathcal{M}(\xi)$, $\mathrm{cl}V_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))}$ is nonempty and compact. It follows from $\mathrm{cl}V_{\xi(i)} \cap O_{\xi,i} \subset \mathrm{cl}B_{\xi,P,i}$ that $\mathrm{cl}V_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))} \subset \mathrm{cl}B_{\xi,P,i}$. Hence, $\mathrm{cl}B_{\xi,P,i}$ is topped in X such that $\alpha(\mathrm{cl}B_{\xi,P,i}) = \alpha(\mathrm{cl}V_{\xi(i)})$ and $K_{\xi,i} = \mathrm{cl}V_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))} = \mathrm{cl}B_{\xi,P,i} \cap X^{(\alpha(clB_{\xi,P,i}))}$. Next, assume that $K(\xi)$ does not satisfy (*). Then $P = \emptyset$ and $B_{\xi,P,i} = V_{\xi(i)}$. So, this follows easily. Furthermore, if $i \in \mathcal{N}(\xi)$, since $V_{\xi(i)} = B_i$, we also obtain it.

(b) (ii) Since $i \notin \mathcal{M}(\xi), i \notin \mathcal{N}(B)$. Then $V_{\xi(i)} \subset B_{x(\xi(i))}$. By way of taking $B_{x(\xi(i))}$, $\mathrm{cl}B_{x(\xi(i))} \cap \mathrm{cl}B_i$ is topped in X and $\alpha(\mathrm{cl}B_{x(\xi(i))} \cap \mathrm{cl}B_i) = \alpha(\mathrm{cl}B_{x(\xi(i))}) = \mathrm{rank}(x(\xi(i)))$. So it follows that $\alpha(\mathrm{cl}B_{x(\xi(i))} \cap \mathrm{cl}B_i) < \lambda(\mathrm{cl}B_i)$ and $K(\xi)_i = V_{\xi(i)}$. By $i \notin \mathcal{M}(\xi)$, we have $\mathrm{cl}V_{\xi(i)} \cap (\mathrm{cl}B_i \cap \mathrm{cl}B_{x(\xi(i))} \cap X^{(\alpha(clB_{x(\xi(i))}))}) = \emptyset$. So, $\lambda(\mathrm{cl}V_{\xi(i)}) \leq \alpha(\mathrm{cl}B_{x(\xi(i))})$. Since $i \notin P$, $B_{\xi,P,i} = V_{\xi(i)} \cap O_{\xi,i}$. Hence, we obtain that $\lambda(\mathrm{cl}B_{\xi,P,i}) < \lambda(\mathrm{cl}B_i)$.

For each $\eta \in \omega^{n(B)+1}$, put

$$\mathcal{G}_\eta(B) = \{G_\xi : \xi \in \Xi_{B,\eta}\} \text{ and}$$
$$\mathcal{B}_\eta(B) = \cup\{\mathcal{B}_{\eta,\xi}(B) : \xi \in \Xi_{B,\eta}\}.$$

Then we have

(7) (a) every element of $\mathcal{G}_\eta(B)$ is contained in some member of $\mathcal{O}'$;
(b) $\mathcal{G}_\eta(B) \cup \mathcal{B}_\eta(B)$ is a cover of B;
(c) the length of nonempty element of $\mathcal{B}_\eta(B)$ is $n(B)+1$.

(8) For each $x \in B, \{\eta \in \omega^{n(B)+1} : \mathrm{Ord}(x, \mathcal{V}_\eta) < \omega\} \in \mathcal{F}^{n(B)+1}$.

Take $x = (x_i) \in B$ and for each $i \leq n(B)$, let $F_i = \{j \in \omega : \mathrm{Ord}(x_i, \mathcal{V}^j_{B,i}) < \omega\} \in \mathcal{F}$ and $F = \prod_{i \leq n(B)} F_i \in \mathcal{F}^{n(B)+1}$. For each

$\eta \in F, \mathrm{Ord}(x, \mathcal{V}_\eta(B)) < \omega$. So, $\{\eta \in \omega^{n(B)+1} : \mathrm{Ord}(x, \mathcal{V}_\eta(B)) < \omega\} \in \mathcal{F}^{n(B)+1}$.

(9) For each $x \in B, \{\eta \in \omega^{n(B)+1} : \mathrm{Ord}(x, \mathcal{G}_\eta(B) \cup \mathcal{B}_\eta(B)) < \omega\} \in \mathcal{F}^{n(B)+1}$.

For each $m \in \omega$ and $\tau \in \Phi_m$, let us define $\mathcal{G}_\tau$ and $\mathcal{B}_\tau$ of elements of $\mathcal{B}$. For each $m \in \Phi_0 = \omega$, let $\mathcal{G}_m = \mathcal{G}_m(X^\omega)$ and $\mathcal{B}_m = \mathcal{B}_m(X^\omega)$. Assume that for $m \in \omega$ and $\mu \in \Phi_m$, we have already obtained $\mathcal{G}_\mu$ and $\mathcal{B}_\mu$. Let $\tau \in \Phi_{m+1}$ and $\tau = \mu \oplus \eta$, where $\mu = \tau_- \in \Phi_m$ and $\eta \in \omega^{m+2}$. Let $B \in \mathcal{B}_\mu$. If $B \neq \emptyset$, then we denote $\mathcal{G}_\eta(B)$ and $\mathcal{B}_\eta(B)$ by $\mathcal{G}_\tau(B)$ and $\mathcal{B}_\tau(B)$, respectively. If $B = \emptyset$, let $\mathcal{G}_\tau(B) = \mathcal{B}_\tau(B) = \{\emptyset\}$. Let $\mathcal{G}_\tau = \mathcal{G}_\mu \cup (\cup\{\mathcal{G}_\tau(B) : B \in \mathcal{B}_\mu\})$ and $\mathcal{B}_\tau = \cup\{\mathcal{B}_\tau(B) : B \in \mathcal{B}_\mu\}$. Then every nonempty element of $\mathcal{B}_\tau$ has the length $m+2$.

Claim 3. $\{\mathcal{G}_\tau \cup (\mathcal{B}_\tau \wedge \mathcal{O}') : \tau \in \Phi\}$ is a θ-sequence of open refinements of $\mathcal{O}'$.

Proof: of Claim 3: By (7) (a) and induction, $\mathcal{G}_\tau \cup (\mathcal{B}_\tau \wedge \mathcal{O}')$ is an open refinement of $\mathcal{O}'$. Take an $x = (x_i) \in X^\omega$. By (9), take a $\tau(0) = m(0) \in \omega$ such that $Ord(x, \mathcal{G}_{\tau(0)} \cup \mathcal{B}_{\tau(0)}) < \omega$. Then $\tau(0) \in \Phi_0$. If ${\mathcal{B}_{\tau(0)}}_x = \emptyset$, then we are done. So, assume that $\mathcal{B}_{\tau(0)_x} \neq \emptyset$. By (7) (c), every nonempty element of $\mathcal{B}_{\tau(0)}$ has the length 1. By (9) again, we can take an $\eta(1) \in \omega^2$ such that

$$\eta(1) \in \cap\{\{\eta \in \omega^2 : Ord(x, \mathcal{G}_\eta(B) \cup \mathcal{B}_\eta(B)) < \omega\} : x \in B \in \mathcal{B}_{\tau(0)}\} \in \mathcal{F}^2.$$

Let $\tau(1) = (\eta(0), \eta(1)) \in \Phi_1$. Then we have $Ord(x, \mathcal{G}_{\tau(1)} \cup \mathcal{B}_{\tau(1)}) < \omega$. Assume also that ${\mathcal{B}_{\tau(1)}}_x \neq \emptyset$. Continuing in this manner, we can choose a $\tau(t) = (\eta(0), \eta(1), \cdots, \eta(t)) \in \Phi_t$ such that for each $t \in \omega$,

$$\mathrm{Ord}(x, \mathcal{G}_{\tau(t)} \cup \mathcal{B}_{\tau(t)}) < \omega \text{ and } {\mathcal{B}_{\tau(t)}}_x \neq \emptyset.$$

Since ${\mathcal{B}_{\tau(t)}}_x \neq \emptyset$ and is finite for each $t \in \omega$, it follows from König's lemma (cf. K. Kunen [4]) that there are sequences $\{\xi_t : t \in \omega\}, \{\mathcal{N}(t) : t \in \omega\}, \{\mathcal{M}(t) : t \in \omega\}, \{K(t) : t \in \omega\}, \{P(t) : t \in \omega\}, \{B(t) = B(\xi(t), P(t)) : t \in \omega\}, B(\xi(t), P(t)) = \prod_{i \in \omega} B_{t,i}$ of elements of $\mathcal{B}$ satisfying: for each $t \in \omega$,

(10) (a) $x \in B(t) = \prod_{i \in \omega} B(t)_i \in \mathcal{B}_{\eta(t)}(B(t-1))$ and $n(B(t)) = t+1$, where $B(-1) = X^\omega$;

(b) $\xi_t \in \Xi_{B(t-1), \eta(t)}$;

(c) $\mathcal{N}(t) = \mathcal{N}(B(t-1))$;

(d) $\mathcal{M}(t) = \mathcal{M}(\xi_t)$;
(e) $K(t) = K(\xi_t) = \prod_{i\in\omega} K(t)_i \in \mathcal{K}(B(t-1), \eta(t))$;
(f) $P(t) \in \mathcal{P}(\{0, 1, \cdots, n(B(t-1))\})$;
(g) if $K(t)$ satisfies the condition (*) and $r(\xi_t) = n(B(t-1))$, then there is an $i < n(\xi_t)$ with $i \in P(t)$;
(h) if $i \in P(t)$, then $\lambda(clB(t)_i) < \lambda(clB(t-1)_i)$;
(i) for $i \notin P(t)$,
    (1) if $i \in \mathcal{M}(t)$, then $K(t)_i \subset \mathrm{cl}B(t-1)_i, i \in \mathcal{N}(t+1)$, and furthermore, if $i \in \mathcal{N}(t)$, then $K(t)_i = \mathrm{cl}B(t-1)_i \cap X^{(\alpha(clB(t-1)_i))} = K(t+1)_i = \mathrm{cl}B(t) \cap X^{(\alpha(clB(t)_i))}$, and hence, $\lambda(\mathrm{cl}B(t-1)_i) = \lambda(\mathrm{cl}B(t)_i)$;
    (2) if $i \notin \mathcal{M}(t)$, then $\lambda(\mathrm{cl}B(t)_i) < \lambda(\mathrm{cl}B(t-1)_i)$.

Let $i \in \omega$. By (10)(a), let $\tilde{t} \geq 1$ such that $n(B(\tilde{t})) > i$. By (10)(h), if $i \in P(m)$ for $m \geq \tilde{t}, \lambda(\mathrm{cl}B(m)_i) < \lambda(\mathrm{cl}B(m-1)_i)$. Since there does not exist an infinite decreasing sequence of ordinals, there is a $t_i \in \omega$ with $t_i \geq \tilde{t}$ such that for each $t \geq t_i, i \notin P(t)$. By (10) (i) (2), there is an m_i such that $m_i \geq t_i$ and for each $t \geq m_i, i \in \mathcal{M}(t)$. Then, by (10) (i) (1), for each $t \geq m_i$, $\mathrm{cl}B(t+1)_i$ is topped and $\mathrm{cl}B(t+1)_i \cap X^{(\alpha(clB(t+1)_i))} = K(t+1)_i = K(m_i+1)_i$. Let $K = \prod_{i\in\omega} K(m_i+1)_i$. Then K is a compact subset of X^ω. Since $\mathcal{O}$ is an open cover of X^ω, which is closed under finite unions, there is an $O = \prod_{i\in\omega} O_i \in \mathcal{O}'$ such that $K \subset O$. By (10) (a), take an $s \geq 1$ such that:

(11) (a) $n(O) \leq n(B(s-1))$,
(b) for each $i < n(O), m_i + 1 \leq s$.

For each $i < n(O)$, by (11) (b), $K(s)_i = K(m_i+1)_i \subset O_i$. Then $K(s) \subset O$ and hence, $K(s)$ satisfies the conditon (*). Since $n(\xi_s) \leq n(O), r(\xi_s) = n(B(s-1))$. By (10)(g), there is an $i < n(\xi_s)$ with $i \in P(s)$, which contradicts the way of taking s.

The proof is complete. □

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