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CERTAIN COVERING-MAPS AND k-NETWORKS, AND RELATED MATTERS

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ABSTRACT. In this paper, we introduce a new general type of covering-maps. Then we unify many characterizations for certain (quotient) images of metric spaces, and obtain new ones by means of these maps.

Finding characterizations of nice images of metric spaces is one of the most important problems in general topology. Various kinds of characterizations have been obtained by means of certain k-networks. For a survey in this field, see [33], for example.

In this paper, we shall introduce a new general type of covering-maps, σ-(P)-maps associated with certain covering-properties (P), in terms of σ-maps defined by [11]. Then we unify many characterizations and obtain new ones by means of these maps.

All spaces are regular and T_1, and all maps are continuous and onto.

For a cover $\mathcal{P}$ of a space X, let (P) be a (certain) covering-property of $\mathcal{P}$. Let us say that $\mathcal{P}$ has property σ-(P) if $\mathcal{P}$ can be expressed as $\cup\{\mathcal{P}_i : i \in N\}$, where each $\mathcal{P}_i$ is a cover of X having the property (P) such that $\mathcal{P}_i \subset \mathcal{P}_{i+1}$, and $\mathcal{P}_i$ is closed under finite intersections. (We may assume that $X \in \mathcal{P}_i$, if necessary). When $\mathcal{P} = \mathcal{P}_i = \mathcal{P}_{i+1}$ for all $i \in N$, we shall say that $\mathcal{P}$ has property (P) (instead of σ-(P)).

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We recall that a cover $\mathcal{P}$ of X is point-countable (compact-countable; star-countable, respectively), if every point (compact set; element of $\mathcal{P}$, respectively) meets at most countably elements of $\mathcal{P}$. A point-finite (compact-finite; or star-finite) cover is similarly defined. Clearly, every σ-point-countable (σ-compact-countable; σ-star-countable, respectively) cover is precisely point-countable (compact-countable; star-countable, respectively) cover.

For a cover $\mathcal{P}$ of a space X, we assume that $(*)$: X has a cover each of whose element meets at most countably many elements of $\mathcal{P}$. Then, as (P), we shall consider locally finite locally countable, star-countable, and point-countable covering-property, etc.

Definition 1. Let (P) be a covering-property. We shall say that a map $f : X \to Y$ is a σ-(P)-map ((P)-map, respectively) if, for some base $\mathcal{B} = \{B_\alpha : \alpha \in \wedge\}$ for X, the family $f(\mathcal{B}) = \{f(B_\alpha) : \alpha \in \wedge\}$ has property σ-(P) ((P), respectively). Obviously, $f(\mathcal{B})$ is a network for Y. In this paper, we will use notations, such as σ-(locally finite)-map instead of σ-locally finite-map.

Recall that a map $f : X \to Y$ is an s-map (compact; Lindelöf-map, respectively) if for any $y \in Y$, $f^{-1}(y)$ is separable (compact; Lindelöf, respectively).

Remark 1. In Definition 1, we assume that the family $f(\mathcal{B}) = \{f(B_\alpha) : \alpha \in \wedge\}$ is to be interpreted in the strict "indexed" sense; hence, the sets $f(B_\alpha)$ are not required to be different. Thus, by the assumption $(*)$, the base $\mathcal{B}$ must be at least point-countable, and f be an s-map. When $f(\mathcal{B})$ is σ-locally finite, then X is a metrizable space with the σ-locally finite base $\mathcal{B}$; Y is a σ-space with the σ-locally finite network $f(\mathcal{B})$; and $f^{-1}(L)$ is Lindelöf for every Lindelöf subset L of Y. When $f(\mathcal{B})$ is locally countable or star-countable, then X is a locally separable, metrizable space with the locally countable base $\mathcal{B}$.

For a map $f : X \to Y$, the following hold in view of the above.

(a) If f is a σ-(locally finite)-map, then X is metrizable.

(b) If f is a (locally countable)-map or a (star-countable)-map, then X is locally separable, metrizable.

(c) (i) f is a (countable)-map iff X is separable metric.

(ii) f is a (locally-finite)-map iff X and Y are discrete.

Thus, we do not consider a trivial case of (locally finite)-maps in view of the above.

S. Lin [11] introduced the concept of σ-maps. Namely, a map is a σ-map if it is a σ-(locally finite)-map. Related to σ-maps, we will review certain maps which are useful in the theory of networks. K. Nagami [22] introduced a σ-map $f : X \to Y$ in the following sense: For every σ-locally finite open cover $\mathcal{G}$ of X, $f(\mathcal{G})$ has a refinement $\mathcal{F}$ such that $\mathcal{F}$ is a σ-locally finite closed cover of Y. Let us call a map f a weak σ-map if it satisfies the definition above, but without the requirement that the members of $\mathcal{F}$ be closed. Related to σ-maps of [22], E. Michael [20] (or [21]) defined a σ-locally finite map $f : X \to Y$ as follows: Every σ-locally finite (not necessarily open) cover of X has a refinement $\mathcal{P}$ such that $f(\mathcal{P})$ is a σ-locally finite cover of Y.

For a map, σ-map $\to$ σ-locally finite map $\to$ weak σ-map (cf. [21]). But each converse need not hold; see Remark 2 below.

Definition 2. For a cover $\mathcal{P}$ of a space X, we recall the following definitions. These are generalizations of bases. For a survey around k-networks, see [33], for example.

$\mathcal{P}$ is a k-network for X if for each compact subset K of X and its open nbd V, $K \subset \cup\, \mathcal{P}' \subset V$ for some finite $\mathcal{P}' \subset \mathcal{P}$. (When K is a single point, such a cover $\mathcal{P}$ is called a network.) As is well-known, a space X is called an $\aleph$-space ($\aleph_0$-space, respectively) if X has a σ-locally finite k-network (countable k-network, respectively).

$\mathcal{P}$ is a cs-network for X if for each $x \in X$, its open nbd V and a sequence $\{x_n\}$ converging to x, there exists $P \in \mathcal{P}$ such that $\{x_n : n \geq m\} \cup \{x\} \subset P \subset V$ for some $m \in N$.

$\mathcal{P}$ is a cs^*-network for X if for each $x \in X$, its open nbd V and a sequence $\{x_n\}$ converging to x, there exist $P \in \mathcal{P}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k} : k \in N\} \cup \{x\} \subset P \subset V$.

$\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ is a weak base if (a): for each $x \in X$, $\mathcal{P}_x$ is a network of x in X, (b): for each $x \in X$, if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$; (c): $G \subset X$ is open in X if for each $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$. A space X is g-first countable [26] (or [1]) if each $\mathcal{P}_x$ is countable. A space X is g-metrizable [26] if X has a σ-locally finite weak base.

$\mathcal{P} = \cup\{P_x : x \in X\}$ satisfying the above conditions (a) and (b) is an sn-network [13] (or [18]) (= universal cs-network of [14]) if for each $x \in X$, any $P \in \mathcal{P}_x$ is a sequential nbd of x (i.e., any sequence converging to x is eventually contained in P).

Recall that a space is sequential if every sequentially open set G (i.e., every sequence converging to a point in G is eventually in G) is open. For a space X and its cover $\mathcal{C}$, X is determined by $\mathcal{C}$ [7] (= X has the weak topology with respect to $\mathcal{C}$), if $G \subset X$ is open in X if $G \cap C$ is open in C for each $C \in \mathcal{C}$. Then a space is sequential iff it is determined by a cover of convergent sequences (containing their limit points).

For a cover $\mathcal{P}$ of X, base $\to$ weak base $\to$ sn-network $\to$ cs-network $\to$ cs^*-network. But, sn-network $\Leftrightarrow$ weak base if X is sequential. Also, cs^*-network $\to$ k-network if X is sequential and $\mathcal{P}$ is point-countable [31].

Remark 2. (1) A map $f : X \to Y$ is a weak σ-map if the following (a), (b), or (c) holds:

(a) f is a closed map such that X is a σ-space.

(b) f is an open map such that Y is subparacompact.

(c) f is a quotient Lindelöf map (or quotient s-map) such that Y is Fréchet and subparacompact.

(In fact, for case (a), every open cover $\mathcal{G}$ of X has a refinement $\mathcal{P}$ which is a σ-locally finite closed network for X. But $f(\mathcal{P})$ is a σ-closure preserving closed network for Y. Thus, $f(\mathcal{P})$ has a refinement which is a σ-discrete closed network $\mathcal{F}$ in view of the proof of Theorem in [27]. Then, $\mathcal{F}$ is a σ-locally finite refinement of $f(\mathcal{G})$. Hence, f is a weak σ-map. For case (b), the result is obvious. For case (c), let $\mathcal{C}$ be a σ-locally finite open cover X. Clearly, X is determined by $\mathcal{C}$. But f is a quotient Lindelöf (or quotient s-map) map. Thus, Y is determined by a point-countable cover $f(\mathcal{C})$. Since Y is Fréchet, each point of Y has a nbd which is contained in a countable union of elements of $f(\mathcal{C})$ by [7, Lemma 2.6]. Thus, since Y is subparacompact, $f(\mathcal{C})$ has a σ-locally finite refinement.)

(2) Let $f : X \to Y$ be a map. If the following (a) or (b) holds, then f is σ-locally finite [20] or [21].

(a) f is a closed Lindelöf map, and X or Y is subparacompact.

(b) $f(\mathcal{P})$ is σ-locally finite for some network $\mathcal{P}$ in X. (Thus, X and Y must be σ-spaces.)

Conversely, if $f : X \to Y$ is σ-locally finite, then for any closed and ω_1-compact subset L of Y (i.e., every uncountable subset of L

has an accumulation point), $f^{-1}(L)$ is ω_1-compact; in particular, $f^{-1}(L)$ is Lindelöf when X is subparacompact.

(In fact, let $f^{-1}(L)$ be not ω_1-compact for some ω_1-compact closed subset L of Y. Then, there exists an uncountable closed discrete subset D of $f^{-1}(L)$. Thus, X has a σ-discrete cover $\mathcal{C} = \{X - D, \{d\} : d \in D\}$ which is disjoint. Then $\mathcal{C}$ has a refinement $\mathcal{P}$ such that $f(\mathcal{P})$ is a σ-locally finite cover of Y. But for uncountably many elements of $\mathcal{P}$, $f(\mathcal{P})$ meets the ω_1-compact set L, a contradiction.)

(3) Let $f : X \to Y$ be a map. If X is a σ-space, then (a) $\Leftrightarrow$ (b) $\to$ (c) holds. When f is closed, (a), (b), and (c) are equivalent, and (a) and (c) are equivalent under X being, more generally, subparacompact.

(a) f is a σ-locally finite map.

(b) $f(\mathcal{P})$ is σ-locally finite for some network $\mathcal{P}$ in X.

(c) f is a Lindelöf map.

(In fact, (3) holds in view of (2), but for (a) $\Leftrightarrow$ (b), use Proposition 2.2 in [21]).

(3) shows that every σ-locally finite image of a σ-space is a σ-space. However, every weak σ-image (actually, open s-image) of a metric space need not be a σ-space (by the well-known Michael's Line, which is the open s-image (hence, weak σ-image) of a metric space, but not a σ-space).

For closed maps, we have (4) below. In (a) or (b), we cannot weaken σ-map or σ-locally finite map to weak σ-map in view of (1) and (2).

(4) For a closed map $f : X \to Y$ with X metric, (a), (b), and (c) below are equivalent, and (c) $\to$ (d) $\Leftrightarrow$ (e) holds.

(a) f is a σ-map.

(b) f is a σ-locally finite map.

(c) f is an s-map.

(d) Every boundary $\partial f^{-1}(y)$ is separable.

(e) X is an $\aleph$-space.

(Indeed, the implication (a)$\to$ (b) $\to$ (c) is already shown. For (c) $\to$ (a), note that every σ-locally finite base for X has a refinement $\mathcal{B}$ such that $\mathcal{B}$ is a base for X and $f(\mathcal{B})$ is σ-locally finite in Y, because f is a closed s-map with Y is paracompact. (d) $\to$ (e) holds by [5, Theorem 1], and so does (e) $\to$ (d) by [29, Proposition 1].)

We recall that every σ-space X (having a σ-locally finite closed network $\mathcal{P}$ which is closed under finite intersections) is the one-to-one, σ-image of a metric space (which is a copy of X topologized by taking $\mathcal{P}$ as a base). More precisely, as a characterization for σ-spaces by means of maps, we have the following (5). The equivalence between (a) and (b) (between (a) and (c), respectively; among (a), (d), and (e)) is due to [11] ([20] or [21], respectively; [22]).

(5) The following are equivalent for a space X. In (b) and (c), the map can be chosen to be one-to-one. In (d), (e), and (f), the condition of the weak σ-map is essential; see (3).

(a) X is a σ-space.
(b) X is the image of a metric space under a σ-map.
(c) X is the image of a metric space under a σ-locally finite map.
(d) X is the image of a metric space under a one-to-one, weak σ-map.
(e) X is the image of a metric space under a compact, weak σ-map.

Proposition 1. For a map $f : X \to Y$, (1), (2), and (3) below hold.

(1) Let X be a metric space. Then the following are equivalent:
(a) f is a (point-countable)-map.
(b) f is an s-map.
(c) $f(\mathcal{B})$ is point-countable for any point-countable base $\mathcal{B}$ in X.

(2) Let X be a locally separable metric space. Then the following are equivalent:

(a) f is a (locally countable)-map ((star-countable)-map, respectively).
(b) Each point $y \in Y$ has a nbd V_y with $f^{-1}(V_y)$ (each point $x \in X$ has a nbd W_x with $f^{-1}(f(W_x))$, respectively) separable in X.
(c) $f(\mathcal{B})$ is locally countable (star-countable, respectively) for any locally countable (star-countable, respectively) base $\mathcal{B}$ in X.
(d) $f(\mathcal{B})$ is locally countable (star-countable, respectively) for any star-countable base $\mathcal{B}$ in X.

(3) Let X be a locally separable metric space. Then the implications (a)$\rightarrow$ (b) $\rightarrow$ (c); (d) $\rightarrow$ (c); and (e) $\rightarrow$ (f) $\rightarrow$ (b) and (c) hold. When f is quotient, (a) through (g) are equivalent.

(a) f is a (locally countable)-map.
(b) $f^{-1}(L)$ is Lindelöf for any Lindelöf subset L of Y.
(c) f is a (star-countable)-map.
(d) f is a σ-(star-finite)-map.
(e) f is a σ-map.
(f) f is a σ-locally finite map.
(g) $f^{-1}(S)$ is separable for any separable subset S of Y.

Proof: (1) is shown in Remark 1. (2) is routinely shown (cf. [34], Proposition 1.1), but note that any star-countable base for X is also locally countable. We show (3) holds. (a) $\rightarrow$ (b) $\rightarrow$ (c), and (d) $\rightarrow$ (c) are routine. (e) $\rightarrow$ (f) is already shown. (f) $\rightarrow$ (b) holds by Remark 2(2). For (f) $\rightarrow$ (c), let f be a σ-locally finite map, and let $\mathcal{B}$ be a σ-locally finite base for X consisting of hereditarily Lindelöf subsets. Then, $\mathcal{B}$ has a refinement $\mathcal{F}$ such that $f(\mathcal{F})$ is σ-locally finite. For each $B \in \mathcal{B}$, $f(B)$ meets only countably many $f(F_n) \in f(\mathcal{F})$ with $F_n \in \mathcal{F}$, for $f(B)$ is Lindelöf, while each Lindelöf subset F_n meets only countably many elements of $\mathcal{B}$. Hence, each $f(B)$ meets only countably many elements of $f(\mathcal{B})$. Then, $f(\mathcal{B})$ is a star-countable cover of Y. Thus, f is a (star-countable)-map. For the latter part, let (c) hold. Since f is quotient, Y is determined by a star-countable cover $\mathcal{C} = f(\mathcal{B})$ for some base $\mathcal{B}$ in X. Thus, by [9, Lemma 1.1], Y is the topological sum of spaces Y_α, where each Y_α is a countable union of elements of $\mathcal{C}$. Thus, the cover $\mathcal{C}$ is locally countable, σ-locally finite, and σ-star-finite in Y. Thus (c) implies (a), (e), and (g). (g) $\rightarrow$ (c) is routine. □

Remark 3. In view of the equivalence between (a) and (d) in (2), (locally countable)-maps ((star-countable)-maps, respectively) are precisely locally countable maps (star-countable maps, respectively) discussed in [34].

Corollary 1. For a quotient map $f : X \rightarrow Y$ such that X is a locally separable metric space, the following are equivalent:

(a) f is a (locally countable)-map.
(b) f is a (star-countable)-map.
(c) f is a σ-(star-finite)-map.
(d) f is a σ-map.

(e) f is a σ-locally finite map.
(f) $f^{-1}(L)$ is Lindelöf for every Lindelöf subset L of Y.
(g) $f^{-1}(S)$ is separable for every separable subset S of Y.

Definition 3. For a map $f : X \to Y$, let us recall the following definitions:

f is sequence-covering [25], if each convergent sequence in Y is the image of some convergent sequence in X.

f is sequence-covering in the terminology of [7], if each convergent sequence L in Y is the image of some compact subset of X. In this paper, let us call this kind of map a pseudo-sequence-covering map, as in [8]. (When the L are compact sets, such a map f is called compact-covering.)

f is subsequence-covering [17] if, for each $y \in Y$, and each sequence L in Y converging to y, there exists a convergent sequence K in X such that $f(K)$ is a subsequence of L.

f is 1-sequence-covering [13], if for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that for each sequence K converging to y, there exists a sequence L converging to x such that $f(L) = K$. For 1-sequence-covering maps, see [18], for example.

Lemma 1. Let $f : X \to Y$ be a σ-(P)-map, then the following hold:

(1) If f is quotient, then Y has a k-network having property σ-(P).

(2) If f is subsequence-covering (sequence-covering; 1-sequence covering, respectively), then Y has a cs^*-network (cs-network; cn-network, respectively) having property σ-(P).

Proof: For (1), let $f(\mathcal{B})$ have property σ-(P) for some base $\mathcal{B}$ in X. Let $K \subset U$ with K compact and U open in Y. Since $f|f^{-1}(U)$ is quotient, U is determined by a point-countable cover $\mathcal{U} = \{f(B) : B \in \mathcal{B}, f(B) \subset U\}$. Thus, $K \subset \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{U}$ by Proposition 2.1 in [7]. This shows that $f(\mathcal{B})$ is a k-network. (2) is routine. □

Every σ-image of a metric space is a σ-space, but need not be an $\aleph$-space in view of Remark 2(5). But we have the following by Lemma 1 and Corollary 1.

Corollary 2. (1) Every quotient σ-image of a metric space is an $\aleph$-space.

(2) Every quotient σ-locally finite image of a locally separable metric space is an $\aleph$-space.

Lemma 2. A first countable space X is metrizable if X has a σ-compact-finite k-network [19], or X has a star-countable k-network [9].

Remark 4. (1) Every quotient σ-locally finite image of a metric space need not be an $\aleph$-space. This shows that the local separability of the domain is essential in (2) in Corollary 2. (In fact, there exists a non-$\aleph$-space Y which is the image of a metric space X under a (1-sequence-covering) quotient σ-locally finite map f (by the open finite-to-one map $f : X \to Y$ in [28, Example 3.2], where Y is not metrizable, hence, not an $\aleph$-space by Lemma 2, and Y has a σ-discrete cover of singletons).)

(2) Every quotient, finite-to-one, weak σ-image of a locally compact, metric space need not be an $\aleph$-space and need not hold each of (e) through (f) in Corollary 1, even if the range is a paracompact σ-space. (This can be seen by the example in [15, Remark 14(2)].) Hence, we can not replace "σ-locally finite" by "weak σ" in Corollary 1 and Corollary 2(2).

A nice characterization for quotient s-images of metric spaces was obtained in [7]. Since then, lots of characterizations for certain images of metric spaces have been obtained by many topologists; see [33]. To unify these characterizations, we prove a general theorem. First, we need the following lemma due to [23, Lemma 1.17].

Lemma 3. Let $\mathcal{P}$ be a point-countable cs^*-network for a space X. Let $K = \{x_m : m \in \omega\}$ be a sequence converging to x_0. If U is an open nbd of K in X, then there exists a subfamily $\mathcal{A}$ of $\mathcal{P}$ satisfying the following properties:

(a) $\mathcal{A}$ is finite;

(b) $K \subset \cup\mathcal{A} \subset U$, and K meets any element of $\mathcal{A}$;

(c) for each $A \in \mathcal{A}$, if A contains a subsequence of K, then $x_0 \in A$.

General Theorem: The following are equivalent for a space X. We can replace "subsequence-sequence-covering" by "pseudo-sequence-covering" in (b).

(a) X has a cs^*-network (cs-network; sn-network, respectively) having property σ-(P).

(b) X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively) σ-(P)-image of a metric space.

Proof: (b) $\to$ (a) holds by Lemma 1.

(a) $\to$ (b): Let $\mathcal{P} = \cup\{\mathcal{P}_i : i \in N\}$ a σ-(P) cover of X, where each $\mathcal{P}_i = \{P_\alpha : \alpha \in A_i\}$ is a cover of X having the propery (P) such that $\mathcal{P}_i \subset \mathcal{P}_{i+1}$, and $\mathcal{P}_i$ is closed under finite intersections. Here, we can assume that $X \in \mathcal{P}_i$. (For a case where (P) is star-finite, for example, this assumption will not be essential to the result, but it will be formally used here.)

For each $i \in N$, endow A_i with discrete topology, then A_i is a metric space. Let M be the set of all $\alpha = (\alpha_i) \in \prod_{i\in N} A_i$ such that $\{P_{\alpha_i} : i \in N\} \subset \mathcal{P}$ and $\{P_{\alpha_i} : i \in N\}$ forms a network at some point $x(\alpha) \in X$. Endow M with the subspace topology induced from the usual product topology of the family $\{A_i : i \in N\}$ of metric spaces, then M is a metric space. Since X is Hausdorff, $x(\alpha)$ is unique in X for each $\alpha \in M$. We define $f : M \to X$ by $f(\alpha) = x(\alpha)$ for each $\alpha \in M$. Because $\mathcal{P}$ is a network for X, then f is surjective. For each $\alpha = (\alpha_i) \in M$, $f(\alpha) = x(\alpha)$. Suppose V is an open nbd of $x(\alpha)$ in X. There exists $n \in N$ such that $x(\alpha) \in P_{\alpha_n} \subset V$. Set $W = \{c \in M :$ the n-th coordinate of c is $\alpha_n\}$; then, W is an open nbd of α in M, and $f(W) \subset P_{\alpha_n} \subset V$. Hence, f is continuous. We will show that f is a σ-(P)-map.

For each $n \in N$ and $\alpha_n \in A_n$, put

$V(\alpha_1, \cdots, \alpha_n) = \{\beta \in M\text{: for each } i \le n,\ \beta_i = \alpha_i\}$,

$\mathcal{B}_n = \{V(\alpha_1, \cdots, \alpha_n) : \alpha_i \in A_i (i \le n)\}$.

Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in N\}$, then $\mathcal{B}$ is a base for M. The following claim holds.

Claim: $f(V(\alpha_1, \cdots, \alpha_n)) = \bigcap_{i\le n} P_{\alpha_i}$ for each $n \in N$ and $\alpha_n \in A_n$.

For each $n \in N$, $\alpha_n \in A_n$ and $i \le n$, $f(V(\alpha_1, \cdots, \alpha_n)) \subset P_{\alpha_i}$, then $f(V(\alpha_1, \cdots, \alpha_n)) \subset \bigcap_{i\le n} P_{\alpha_i}$. On the other hand. For each $x \in \bigcap_{i\le n} P_{\alpha_i}$, there is $\beta = (\beta_j) \in M$ such that $f(\beta) = x$. For each $j \in N$, $P_{\beta_j} \in \mathcal{P}_j \subset \mathcal{P}_{j+n}$, then there is $\alpha_{j+n} \in A_{j+n}$ such that $P_{\alpha_{j+n}} = P_{\beta_j}$. Set $\alpha = (\alpha_j)$, then $\alpha \in V(\alpha_1, \cdots, \alpha_n)$ and $f(\alpha) = x$. Thus, $\bigcap_{i\le n} P_{\alpha_i} \subset f(V(\alpha_1, \cdots, \alpha_n))$. Hence, $f(V(\alpha_1, \cdots, \alpha_n)) = \bigcap_{i\le n} P_{\alpha_i}$.

Also, for a subset A of X meeting only elements $P_{\alpha_{k,n}}$ $(k \in A(n))$ of $\mathcal{P}_n$, if A meets $f(V(\alpha_1, \cdots, \alpha_n))$, then $\{\alpha_1, \cdots, \alpha_n\} \subset \{\alpha_{k,n} : k \in A(n)\}$ by the Claim. Thus, f is a σ-(P)-map (in the sense of Remark 1) by the Claim.

Next, we show that the map f is sequence-covering, 1-sequence-covering, or subsequence-covering, respectively.

(i) When the cover $\mathcal{P}$ is a cs-network, f is sequence-covering.

For each sequence $\{x_n\}$ converging to x_0, we can assume that all $x_n's$ are distinct, and that $x_n \neq x_0$ for each $n \in N$. Set $K = \{x_m : m \in \omega\}$. Suppose V is an open nbd of K in X. Consider a subfamily $\mathcal{A}$ of $\mathcal{P}_i$ and the following properties:

(a) $\mathcal{A}$ is finite;

(b) for each $P \in \mathcal{A}$, $\emptyset \neq P \cap K \subset P \subset V$;

(c) if $z \in K$, there exists a unique $P_z \in \mathcal{A}$ such that $z \in \mathcal{P}_z$;

(d) if $x_0 \in P \in \mathcal{A}$, then $K \setminus P$ is finite.

(Here, the family $\mathcal{A} \subset \mathcal{P}_i$ should be defined as $\mathcal{A} = \{X\}$ if there exist no elements of $\mathcal{P}_i$ satisfying property (b).)

Since $\mathcal{P}$ is a σ-(P)-cs-network for X, then the above construction can be realized, and we can assume that $\{\mathcal{A} \subset \mathcal{P}_i : \mathcal{A}$ satisfies conditions (a) through (d) with $V = X\} = \{\mathcal{P}_{ij} : j \in N\}$.

For each $n \in N$, put

$$\mathcal{P}'_n = \left\{ P = \bigcap_{i,j \leq n} P_{ij} : P_{ij} \in \mathcal{P}_{ij}, P \cap K \neq \emptyset \right\}.$$

Then, $\mathcal{P}'_n \subset \mathcal{P}_n$ and $\mathcal{P}'_n$ also satisfies (a) through (d) with $V = X$.

For each $i \in N$, $m \in \omega$ and $x_m \in K$, there is $\alpha_{im} \in A_i$ such that $x_m \in P_{\alpha_{im}} \in \mathcal{P}'_i$. Let $\beta_m = (\alpha_{im}) \in \prod_{i \in N} A_i$. It is easy to prove that $\{P_{\alpha_{im}} : i \in N\}$ is a network of x_m in X. Then there is a $\beta_m \in M$ such that $f(\beta_m) = x_m$ for each $m \in \omega$. For each $i \in N$, there is $n(i) \in N$ such that $\alpha_{in} = \alpha_{io}$ when $n \geq n(i)$. Hence, the sequence $\{\alpha_{in}\}$ converges to α_{io} in A_i. Thus, the sequence $\{\beta_n\}$ converges to β_0 in M. This show that f is a sequence-covering map.

(ii) When the cover $\mathcal{P}$ is an sn-network, f is 1-sequence-covering in the same way as in (i), but change (d) to (d'): $P \in \mathcal{P}_{x_0} \cap \mathcal{P}_i$.

(iii) When the cover $\mathcal{P}$ is a $cs*$-network, f is subsequence-covering in a way similar to (i), but using Lemma 3.

Finally, for the latter part of the theorem, we show that f is actually pseudo-sequence-covering for case (iii).

This can be done as in (i), but change (c) to (c'): $\mathcal{A}$ covers K, and (d) to (d'): for each $A \in \mathcal{A}$, if A contains a subsequence of K, then $x_0 \in A$. (In fact, since $\mathcal{P}$ is a point-countable cs^*-network for X, there exists a finite $\mathcal{A} \subset \mathcal{P}_i$ for some $i \in N$ satisfying the conditions (a), (b), (c'), and (d') by means of Lemma 3. Let $\mathcal{P}'_n = \{P_\alpha : \alpha \in B_n\}$, where B_n is a finite subset of A_n. Put $L = \{\beta = (\alpha_i) \in \Pi\{B_i : i \in N\}, P_{\alpha_{i+1}} \subset P_{\alpha_i}\}$. Then, in view of the proof of the main theorem in [12], L is a compact subset of $\Pi\{B_i : i \in N\}$, and $L \subset M$ such that $f(L) \subset K$, and $K \subset f(L)$. Thus, L is a compact subset of M and $K = f(L)$. Then, f is pseudo-sequence-covering (hence, f is also subsequence-covering).) That completes the proof. □

As examples of the General Theorem, we will give corollaries 3 through 10 below.

We recall that a space is an $\aleph$-space iff it has a σ-locally finite cs-network (see [4] or [3]). Then we have the following corollary. (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) also holds in view of [11]. The obvious analogue of the corollary remains valid for the σ-locally countable covering-property.

Corollary 3. (1) The following are equivalent for a space X:

(a) X is an $\aleph$-space.

(b) X is the sequence-covering σ-image of a metric space.

(c) X is the subsequence-covering σ-image of a metric space.

(d) X is the pseudo-sequence-covering σ-image of a metric space.

(2) A space X has a σ-locally finite sn-network iff X is the 1-sequence-covering σ-image of a metric space.

In the following corollary, the equivalence (a) $\Leftrightarrow$ (c) is also shown in [12] (also [16] and [13], respectively). (b) $\Leftrightarrow$ (c) holds by Proposition 1.

Corollary 4. The following are equivalent for a space X:

(a) X has a point-countable cs^*-network (cs-network; sn-network, respectively).

(b) X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), (point-countable)-image of a metric space.

(c) X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), s-image of a metric space.

Lemma 4. (1) Let $f : X \to Y$ be a map such that X is sequential. Then f is quotient iff f is subsequence-covering and Y is sequential [17], [31].

(2) Every g-first countable space (in particular, space having a point-countable weak base) is sequential (see [1] or [26]).

Corollary 5. (1) A space X is a sequential space with a point-countable cs^*-network iff X is the quotient s-image of a metric space (see [31] or [12]).

(2) A space X is a sequential space with a point-countable cs-network iff X is the sequence-covering, quotient s-image of a metric space [16].

(3) A space X has a point-countable weak base iff X is the 1-sequence-covering, quotient s-image of a metric space [13].

We recall that a map $f : X \to Y$ is a cs-map [24] if for any compact subset K of Y, $f^{-1}(K)$ is separable. Obviously, when X is metric, every cs-map is precisely a (compact-countable)-map. As an analogue of Corollary 4 the following holds. (a) $\Leftrightarrow$ (c) is also shown in [24] ([10], respectively). For an analogue of Corollary 5, see Corollary 10(3).

Corollary 6. The following are equivalent for a space X:

(a) X has a compact-countable cs^*-network (cs-network; sn-network, respectively).

(b) X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), (compact-countable)-image of a metric space.

(c) X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), cs-image of a metric space.

We shall call a map $f : X \to Y$ σ-point-finite (σ-compact-finite, respectively) if there exists a base $\mathcal{B} = \cup\{\mathcal{B}_i : i \in N\}$ for X such that, for any $y \in Y$ (any compact set $K \subset Y$, respectively), and any $i \in N$, $f^{-1}(y)$ ($f^{-1}(K)$, respectively) meets only finitely many elements of $\mathcal{B}_i$. (For f being a σ-compact-finite map, X must be metrizable by Lemma 2, because X has the σ-compact-finite base $\mathcal{B}$.) Clearly, every σ-point-finite-map (σ-compact-finite-map, respectively) is precisely a σ-(point-finite)-map (σ-(compact-finite)-map, respectively).

Corollary 7. The following are equivalent for a space X. We can replace "point-finite" by "compact-finite" in (a), (b), and (c) together.

(a) X has a σ-point-finite cs^*-network (cs-network; sn-network, respectively).

(b) X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), σ-(point-finite)-image of a metric space.

(c) X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), σ-point-finite-image of a metric space.

Remark 5. In (c), we can not replace "σ-point-finite- image (or σ-compact-finite-image)" by "compact-image." First, recall that every quotient compact-image of a metric space is symmetrizable [1, p. 125], every Fréchet symmetrizable space is first countable, and every closed set is a G_δ-set. For these, see [6, p. 482], for example.

Now, let X be a space having a σ-point-finite and σ-locally countable base, but a closed set which is not a G_δ-set [2, Example 3.3]. Then the first countable space X is not any subsequence-covering compact-image of a metric space (by the above and Lemma 4(1)). Also, let Y be the the sequential fan S_ω (i.e., the quotient space obtained from the topological sum of countably many convergent sequences by identifying all the limit points). Then, Y has the obvious countable (hence, σ-compact-finite) cs-network. But, similarly, the non-first countable space Y is also not any subsequence-covering compact-image of a metric space.

In the following corollary, (1) is (well) known. In (2), the result for the locally countable cs-network is also shown in [34].

Corollary 8. The following hold for a space X. We can replace "subsequence-covering" by "pseudo-sequence-covering" in the following.

(1) X has a countable cs^*-network (cs-network; sn-network, respectively) iff X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively) image of a separable metric space.

(2) X has a locally countable cs^*-network (cs-network; sn-network, respectively) iff X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), (locally-countable)-image of a locally separable metric space.

(3) X has a star-countable cs^*-network (cs-network; sn-network, respectively) iff X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), (star-countable)-image of a locally separable metric space.

(4) X has a σ-star-finite cs^*-network (cs-network; sn-network, respectively) iff X is the subsequence-covering (sequence-covering; 1-sequence-covering, respectively), σ-(star-finite)-image of a locally separable metric space.

We shall call a map $f : X \to Y$ a strong s-map if for any separable subset S of Y, $f^{-1}(S)$ is separable.

Corollary 9. (1) The following are equivalent for a space X:

(a) X is a sequential space with a locally countable cs^*-network.
(b) X is a sequential space with a star-countable cs^*-network.
(c) X is a sequential space with a σ-star-finite cs^*-network.
(d) X is the quotient (locally-countable)-image of a locally separable metric space.
(e) X is the quotient (star-countable)-image of a locally separable metric space.
(f) X is the quotient σ-(star-finite)-image of a locally separable metric space.
(g) X is the quotient σ-image of a locally separable metric space.
(h) X is the quotient σ-locally finite image of a locally separable metric space.
(i) X is the quotient strong s-image of a locally separable metric space.

(2) We can replace "cs^*-network" with "cs-network" ("k-network," respectively) in (a), (b), or (c) ((a), respectively). Also, we can add the prefix "sequence-covering" before "quotient" in any of (d) through (i).

(3) If we replace "cs^*-network" by "weak base" in (a), (b), and (c) simultaneously, then (1) remains valid by adding the prefix "1-sequence-covering" before "quotient" in (d) through (i) simultaneously; here, we may omit "sequential" in (a), (b), and (c).

Proof: (1) and (3) hold by corollaries 1 and 8, and Lemma 4. For (2), it suffices to show that (e) (instead of (a), (b), or (c)) implies that X has a locally countable and σ-star-finite cs-network. But by Lemma 1(1), X is the topological sum of $\aleph_0$-subspaces in view of the proof of Proposition 1(3). Hence, X has the obvious locally countable and σ-star-finite cs-network. For the parenthetic part, note that, for a locally countable cover $\mathcal{P}$ of a sequential space, $\mathcal{P}$ is a k-network iff it is (assumed to be) a cs^*-network, because the "only if" part holds since the elements are assumed to be closed, and the "if" part holds in view of the above. □

Remark 6. (1) We cannot replace "cs^*-network" by "k-network" in (b) or (c) (as demonstrated by the space S_{ω_1}, whose definition is similar to that of S_ω, which has the obvious σ-star-finite and σ-compact-finite k-network, but has no locally countable *networks* since it is not locally Lindelöf).

(2) Every sequential space with a star-countable cs^*-network has a compact-countable cs-network. But every first countable space with a compact-countable cs-network need not have a locally countable network, a star-countable cs^*-(or k-)network, nor a σ-compact-finite cs^*-(or k-)network (by the space X in Remark 5 in view of Lemma 2).

The following holds by corollaries 3, 6, and 7, and Lemma 4.

Corollary 10. (1) A space X is a k-and-$\aleph$-space iff X is the sequence-covering quotient σ-image of a metric space. Here, we can omit "sequence-covering."

(2) A space X is g-metrizable iff X is the 1-sequence-covering quotient σ-image of a metric space.

(3) A space X is a sequential space with a compact-countable cs^*-network iff X is the quotient cs-image of a metric space [24]. We can replace "cs^*-network" by "cs-network" ("weak base," respectively), but add the prefix "sequence-covering" ("1-sequence-covering," respectively)" before "quotient"; here we need not add "sequential" for the parenthetic part [10].

(4) A space X is a sequential space with a σ-point-finite (σ-compact-finite, respectively) cs^*-network iff X is the quotient σ-point-finite-image (σ-compact-finite-image, respectively) of a metric space. The latter part of (3) also remains valid.

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