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ON COMPACT IMAGES OF LOCALLY SEPARABLE METRIC SPACES

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Abstract. In this paper, we give a characterization of sequentially-quotient compact images of locally separable metric spaces to prove that a space X is a sequentially-quotient compact image of a locally separable metric space if and only if X is a pseudo-sequence-covering compact image of a locally separable metric space. As an application of the above result, we obtain that a space X is a quotient compact image of a locally separable metric space if and only if X is a pseudo-sequence-covering quotient compact image of a locally separable metric space, which answers a question posed by Y. Ikeda.

1. Introduction

To determine what spaces are the images of "nice" spaces under "nice" mappings is one of the central questions of general topology (see, e.g., [1]). In the past, some noteworthy results on sequence-covering images of metric spaces have been obtained (see [4], [9], [10], [11], [12], [13], [16], [17]). Notice that pseudo-sequence-covering mapping and sequentially-quotient mapping are two important generalizations of sequence-covering mapping. In his book, S. Lin [8] asked: If the domains are (locally separable) metric spaces, are

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pseudo-sequence-covering compact mappings equivalent to sequentially-quotient compact mappings? This question is still open, which arouses our interest in the relations between pseudo-sequence-covering compact images and sequentially-quotient compact images for these metric domains. In [16], P. Yan proved that pseudo-sequence-covering compact images are equivalent to sequentially-quotient compact images for metric domains. Later, Lin and Yan [10] proved that pseudo-sequence-covering compact images are equivalent to sequentially-quotient compact images for separable metric domains. Thus, it is natural to raise the following question, which is helpful in solving [8, Question 3.4.8].

Question 1.1. Are pseudo-sequence-covering compact images equivalent to sequentially-quotient compact images for locally separable metric domains?

Finding the internal characterizations of certain images of metric spaces is also of considerable interest in general topology. Recently, many topologists were engaged in research of internal characterizations of compact images of metric spaces and separable metric spaces, and some noteworthy results have been obtained (see [2], [4], [6], [10], [11], [13], [14], [15], [16]). This leads us to be interested in compact images of locally separable metric spaces ([8]), that is, we are interested in the following question.

Question 1.2. How are compact images of locally separable metric spaces characterized?

Related to the above question, Lin, C. Liu, and M. Dai [9] gave an answer on quotient compact images of locally separable metric spaces, and so far there are no other results.

In this paper, we give an internal characterization of sequentially-quotient compact images of locally separable metric spaces, and make use of the characterization to prove that a space X is a sequentially-quotient compact image of a locally separable metric space if and only if X is a pseudo-sequence-covering compact image of a locally separable metric space. As an application of the above result, we obtain that a space X is a quotient compact image of a locally separable metric space if and only if X is a pseudo-sequence-covering quotient compact image of a locally separable metric space,which answers a question posed by Y. Ikeda in [5].

Throughout this paper, all spaces are T_2 and all mappings are continuous and onto. N and ω denote the set of all natural numbers and first infinite ordinal, respectively. Let A be a subset of a space X, $x \in X$, $\mathcal{U}$ be a family of subsets of X, and f be a mapping. We write $(\mathcal{U})_x = \{U \in \mathcal{U} : x \in U\}$, $st(x,\mathcal{U}) = \cup\{U \in \mathcal{U} : x \in U\}$, $st(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \phi\}$, $\mathcal{U} \cap A = \{U \cap A : U \in \mathcal{U}\}$, $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$. The sequence $\{x_n : n \in N\}$, the sequence $\{P_n : n \in N\}$ of subsets, and the sequence $\{\mathcal{P}_n : n \in N\}$ of families of subsets are abbreviated $\{x_n\}$, $\{P_n\}$, and $\{\mathcal{P}_n\}$, respectively. Notice that definitions of some mappings are different in different references. Definitions of mappings in this paper are quoted from [10], [8], [13], [16]. For terms which are not defined here, please refer to [3].

Definition 1.3 ([8])**.** Let X be a space, $x \in P \subset X$. P is said to be a sequential neighborhood of x, if every sequence $\{x_n\}$ converging to x is eventually in P; i.e., there is $k \in N$ such that $x_n \in P$ for $n > k$.

Remark 1.4. (1) P is a sequential neighborhood of x iff $x \in P$ and every sequence $\{x_n\}$ converging to x is cofinally in P; i.e., for each $k \in N$, there is $n > k$ such that $x_n \in P$.

(2) The intersection of finitely many sequential neighborhoods of x is a sequential neighborhood of x.

Definition 1.5 ([11])**.** Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X.

(1) $\{\mathcal{P}_n\}$ is a point-star network of X, if $\{st(x,\mathcal{P}_n)\}$ is a network at x in X for each $x \in X$;

(2) $\{\mathcal{P}_n\}$ is a refinement of X, if $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in N$;

(3) $\{\mathcal{P}_n\}$ is point-finite, if $\mathcal{P}_n$ is point-finite for each $n \in N$.

We suppose every convergent sequence in the following definitions contains its limit point.

Definition 1.6 ([2], [9], [16])**.** Let $f : X \longrightarrow Y$ be a mapping.

(1) f is compact, if $f^{-1}(y)$ is compact in X for each $y \in Y$;

(2) f is sequence-covering (pseudo-sequence-covering)[1], if for every convergent sequence S in Y, there is a convergent sequence L in X (compact subset K in X) such that $f(L) = S$ ($f(K) = S$).

[1] "pseudo-sequence-covering" was called "sequence-covering" by E. Michael.

(3) f is sequentially-quotient (subsequence-covering), if for every convergent sequence S in Y, there is a convergent sequence L in X (compact subset K in X) such that $f(L)$ $(f(K))$ is an infinite subsequence of S.

Remark 1.7. The following implications are obvious by Definition 1.6 and [8] and cannot be reversed.

(1) sequence-covering mapping$\Longrightarrow$pseudo-sequence-covering mapping and sequentially-quotient mapping;

(2) pseudo-sequence-covering mapping$\Longrightarrow$subsequence-covering mapping;

(3) sequentially-quotient mapping$\Longrightarrow$subsequence-covering mapping.

2. Main Results

Lin [7, Proposition 2.1.7] proved that pseudo-sequence-covering mappings on spaces in which points are $G'_\delta s$ are sequentially-quotient mappings. In fact, pseudo-sequence-covering mappings in this result can be relaxed to subsequence-covering mappings.

Proposition 2.1. *Let $f : X \longrightarrow Y$ be a subsequence-covering mapping, where points in X are G_δ. Then, f is sequentially-quotient.*

Proof: Let S be a sequence converging to y in Y. f is subsequence-covering, so there is a compact subset K in X such that $f(K) = S'$ is a subsequence of S. Put $S' = \{y\} \cup \{y_n : n \in N\}$, where $\{y_n\}$ converging y. Pick $x_n \in f^{-1}(y_n) \cap K$. Then, $\{x_n\} \subset K$. Notice that K is a compact subspace in which points are $G'_\delta s$. Thus, K is first countable, hence sequentially compact, so there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to $x \in f^{-1}(y)$. This proves that f is sequentially-quotient. □

Theorem 2.2. *For a space X, the following are equivalent.*

(1) X is a pseudo-sequence-covering compact image of a locally separable metric space;

(2) X is a subsequence-covering compact image of a locally separable metric space;

(3) X is a sequentially-quotient compact image of a locally separable metric space;

(4) X has a point-finite cover $\{X_\alpha : \alpha \in \Lambda\}$, where every X_α has a sequence $\{\mathcal{P}_{\alpha,n}\}$ of countable and point-finite refinements, and the following conditions (a) and (b) are satisfied.

(a) $\{\mathcal{P}_n\}$ is a point-star network of X, where $\mathcal{P}_n = \cup\{\mathcal{P}_{\alpha,n} : \alpha \in \Lambda\}$;

(b) for each $x \in X$, there is finite subset Λ' of Λ such that $st(x, \mathcal{P}(n, \Lambda'))$ is a sequential neighborhood of x for each $n \in \omega$, where $\mathcal{P}(n, \Lambda') = \cup\{\mathcal{P}_{\alpha,n} : \alpha \in \Lambda'\}$.

Proof: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) by Remark 1.7 and Proposition 2.1. We prove only (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

(3) $\Longrightarrow$ (4). Let $f : M \longrightarrow X$ be sequentially-quotient, and M be a locally separable metric space. By [3, 4.4.F], $M = \oplus_{\alpha\in\Lambda} M_\alpha$, where every M_α is separable. As M is metric, using [3, 5.4.E], there is a sequence $\{\mathcal{B}_n\}$ of locally finite open covers of M such that every $\mathcal{B}_{n+1}$ is a refinement of $\mathcal{B}_n$, and for every compact subset $K \subset M$ and any open set $U \supset K$, there is $n \in \omega$ such that $st(K, \mathcal{B}_n) \subset U$. For each $n \in \omega$, there is a countable subfamily $\mathcal{B}_{\alpha,n}$ of $\mathcal{B}_n$, which covers M_α. We can assume $\mathcal{B}_{\alpha,n+1}$ refines $\mathcal{B}_{\alpha,n}$ for each $n \in N$. For each $\alpha \in \Lambda$, $n \in \omega$, put

$$X_\alpha = f(M_\alpha); \mathcal{P}_{\alpha,n} = f(\mathcal{B}_{\alpha,n})\}; \mathcal{P}_n = \cup\{\mathcal{P}_{\alpha,n} : \alpha \in \Lambda\}.$$

As f is compact, it is easy to check that $\{X_\alpha : \alpha \in \Lambda\}$ is a point-finite cover of X, and $\{\mathcal{P}_{\alpha,n}\}$ is a sequence of countable and point-finite refinements of X_α for each $\alpha \in \Lambda$.

(a) $\{\mathcal{P}_n\}$ is a point-star network of X:

Let $x \in U$ and U be open in X. Then $f^{-1}(x) \subset f^{-1}(U)$. $f^{-1}(x)$ is compact in M, so there is $k \in \omega$ such that $st(f^{-1}(x), \mathcal{B}_k) \subset f^{-1}(U)$. It is easy to check $x \in st(x, \mathcal{P}_k) \subset f(st(f^{-1}(x), \mathcal{B}_k)) \subset ff^{-1}(U) = U$. So $\{st(x, \mathcal{P}_n)\}$ is a net- work at x. This proves that $\{\mathcal{P}_n\}$ is a point-star network of X.

For $x \in X$ and $n \in \omega$, put $\Lambda' = \{\alpha \in \Lambda : x \in X_\alpha\}$, and $\mathcal{P}(n, \Lambda') = \cup\{\mathcal{P}_{\alpha,n} : \alpha \in \Lambda'\}$.

(b) $st(x, \mathcal{P}(n, \Lambda'))$ is a sequential neighborhood of x for each $n \in \omega$:

Let S be a sequence in X converging to x. f is sequentially-quotient, so there is a sequence L in M converging to t such that $f(L)$ is a subsequence of S. There is $\alpha' \in \Lambda'$ such that $t \in M_{\alpha'}$. M is a topological sum of $\{M_\alpha : \alpha \in \Lambda\}$, so $M_{\alpha'}$ is open in M; hence, L is eventually in $M_{\alpha'}$. $\mathcal{B}_{\alpha',n}$ is an open cover of $M_{\alpha'}$; there is $B \in \mathcal{B}_{\alpha',n}$ such that $t \in B$, so L is eventually in $B \cap M_{\alpha'}$. Thus, $f(L)$ is eventually in $f(B) \in \mathcal{P}_{\alpha',n} \subset \mathcal{P}(n, \Lambda')$; hence, S is cofinally in $f(B) \subset st(x, \mathcal{P}(n, \Lambda'))$. So $st(x, \mathcal{P}(n, \Lambda'))$ is a sequential neighborhood of x by Remark 1.4(1).

(4) $\Longrightarrow$ (1). We can assume $\mathcal{P}_{\alpha,0} = \{X_\alpha\}$ for each $\alpha \in \Lambda$. For $\alpha \in \Lambda$ and $n \in \omega$, we write $\mathcal{P}_{\alpha,n} = \{P_\beta : \beta \in A_{\alpha,n}\}$, where $A_{\alpha,n}$ is countable. Put $A_n = \cup\{A_{\alpha,n} : \alpha \in \Lambda\}$, then $\mathcal{P}_n = \cup\{\mathcal{P}_{\alpha,n} : \alpha \in \Lambda\} = \{P_\beta : \beta \in A_n\}$, and $\{\mathcal{P}_n : n \in \omega\}$ is a sequence of point-finite covers of X. We can assume $\{A_{\alpha,n} : \alpha \in \Lambda\}$ and $\{A_n : n \in \omega\}$ are all mutually disjoint. Let every A_n be endowed with the discrete topology. Put

$M = \{b = (\beta_n) \in \Pi_{n\in\omega} A_n$: there is $\alpha \in \Lambda$ such that $P_{\beta_n} \in \mathcal{P}_{\alpha,n}$ for each $n \in \omega$, and $\{P_{\beta_n}\}$ is a network at some point x_b in $X\}$.

Then M, which is a subspace of Tychonoff product space $\Pi\{A_n : n \in N\}$, is a metric space. It is easy to check that $f : M \longrightarrow X$ defined by $f(b) = x_b$ is a mapping.

Claim 1. M is locally separable.

Let $b = (\beta_n) \in M$. Put $M_b = \{c = (\gamma_n) \in M : \gamma_0 = \beta_0\}$. Then $b \in M_b$ and M_b is open. Let $\alpha \in \Lambda$ such that $P_{\beta_0} \in \mathcal{P}_{\alpha,0} = \{X_\alpha\}$.

(i) $M_b \subset \Pi_{n\in\omega} A_{\alpha,n}$: Let $c = (\gamma_n) \in M_b$. Then $P_{\gamma_0} = P_{\beta_0} = X_\alpha \in \mathcal{P}_{\alpha,0}$. For each $n \in \omega$, $P_{\gamma_n} \in \mathcal{P}_{\alpha,n} = \{P_\beta : \beta \in A_{\alpha,n}\}$, hence $\gamma_n \in A_{\alpha,n}$, so $c \in \Pi_{n\in\omega} A_{\alpha,n}$.

(ii) M_b is separable: Notice that every $A_{\alpha,n}$ is a countable and discrete space; $\Pi_{n\in\omega} A_{\alpha,n}$ is separable; so, M_b is separable.

By (i),(ii) above, M is locally separable.

Claim 2. f is compact.

Let $x \in X$ and let $\Lambda' = \{\alpha \in \Lambda : x \in X_\alpha\}$. Put $B_{\alpha,n} = \{\beta \in A_{\alpha,n} : x \in P_\beta\}$ for each $\alpha \in \Lambda$. Then $B_{\alpha,n}$ is finite; hence, $K = \cup_{\alpha\in\Lambda'}(\Pi_{n\in\omega} B_{\alpha,n})$ is a compact subset of M. We prove only $f^{-1}(x) = K$.

(i) $K \subset f^{-1}(x)$: Let $b = (\beta_n) \in K$. Then there is $\alpha \in \Lambda'$ such that $b = (\beta_n) \in \Pi_{n\in\omega} B_{\alpha,n}$. So $\beta_n \in B_{\alpha,n}$ for each $n \in \omega$, i.e., $P_{\beta_n} \in \mathcal{P}_{\alpha,n}$, and $x \in P_{\beta_n}$. Hence, $\{P_{\beta_n}\}$ is a network at x; thus, $f(b) = x$, i.e., $b \in f^{-1}(x)$.

(ii) $f^{-1}(x) \subset K$: Let $b = (\beta_n) \in f^{-1}(x)$. Then $f(b) = x$; there is $\alpha \in \Lambda'$ such that $P_{\beta_n} \in \mathcal{P}_{\alpha,n}$ for each $n \in N$, and $\{P_{\beta_n}\}$ is a network at x. So $\beta_n \in A_{\alpha,n}$, and $x \in P_{\beta_n}$, i.e., $\beta_n \in B_{\alpha,n}$; thus, $b = (\beta_n) \in \Pi_{n\in\omega} B_{\alpha,n} \subset K$.

By (i),(ii) above, $f^{-1}(x) = K$.

Claim 3. f is pseudo-sequence-covering.

Let $x \in X$, $\{x_n\}$ be a sequence in X converging to x. Put $S = \{x_n : n \in N\} \cup \{x\}$. There is a finite $\Lambda' \subset \Lambda$ such that for each $n \in \omega$, $st(x, \mathcal{P}(n, \Lambda'))$ is a sequential neighborhood of x. We can assume $\mathcal{P}(n, \Lambda')$ covers S. For each $n \in \omega$, put

$$\mathcal{P}^{''}(n, \Lambda') = \{P_\beta : \beta \in A_{\alpha,n}, \alpha \in \Lambda', (S - st(x, \mathcal{P}(n, \Lambda'))) \cap P_\beta \neq \phi\},$$

$$\mathcal{P}^{'}(n, \Lambda') = (\mathcal{P}(n, \Lambda'))_x \cup \mathcal{P}^{''}(n, \Lambda').$$

Then $\mathcal{P}^{'}(n, \Lambda')$ covering S is a finite subfamily of $\mathcal{P}(n, \Lambda')$. It is easy to check $st(x, \mathcal{P}^{'}(n, \Lambda')) = st(x, \mathcal{P}(n, \Lambda'))$.

Put $\mathcal{P}^{'}(n, \Lambda') = \cup\{\mathcal{P}^{'}_{\alpha,n} : \alpha \in \Lambda'\}$, where each $\mathcal{P}^{'}_{\alpha,n} \subset \mathcal{P}_{\alpha,n}$. Put $\mathcal{P}^{'}_{\alpha,n} = \{P_\beta : \beta \in A^{'}_{\alpha,n}\}$, then $A^{'}_{\alpha,n}$ is a finite subset of $A_{\alpha,n}$.

(i) For each $\beta \in A^{'}_{\alpha,n}$, where $\alpha \in \Lambda'$, $n \in \omega$, we construct S_β as follows:

Put $S_\beta = S \cap P_\beta$ if $x \in P_\beta$ and $S_\beta = (S - st(x, \mathcal{P}(n, \Lambda'))) \cap P_\beta$ if $x \notin P_\beta$. Then S_β is compact in X.

(ii) $S = \cup\{S_\beta : \beta \in A^{'}_{\alpha,n}, \alpha \in \Lambda\}$ for each $n \in \omega$:

Let $y \in S$. If there is $\alpha \in \Lambda$ and $\beta \in A^{'}_{\alpha,n}$ such that $\{x, y\} \subset P_\beta$, then $y \in S \cap P_\beta = S_\beta$; If any $\alpha \in \Lambda$ and any $\beta \in A^{'}_{\alpha,n}$, $\{x, y\} \not\subset P_\beta$, since $\mathcal{P}^{'}(n, \Lambda')$ covers S. Pick $\alpha \in \Lambda$ and $\beta \in A^{'}_{\alpha,n}$ such that $y \in P_\beta$, then $y \notin st(x, \mathcal{P}(n, \Lambda'))$. Thus, $y \in (S - st(x, \mathcal{P}(n, \Lambda'))) \cap P_\beta = S_\beta$.

Put $K = \cup_{\alpha \in \Lambda'}(\Pi_{n\in\omega} A^{'}_{\alpha,n})$, then K is compact. Put $L = \{b = (\beta_n) \in K : \cap_{n\in\omega} S_{\beta_n} \neq \phi\}$.

(iii) L is a closed subset of K; hence, L is compact:

Let $a = (\alpha_n) \in K - L$. Then there is $\alpha' \in \Lambda'$ such that $a \in \Pi_{n\in\omega} A^{'}_{\alpha',n}$, and $\cap_{n\in\omega} S_{\alpha_n} = \phi$. So there is $n_0 \in \omega$ such that $\cap_{n\leq n_0} S_{\alpha_n} = \phi$. Put $W = \{b = (\beta_n) \in K : \beta_n = \alpha_n \text{ for } n \leq n_0\}$. Then W is an open neighborhood of x in K. We claim $W \cap L = \phi$; thus, L is closed subset of K. If not, let $c = (\gamma_n) \in W \cap L$. Then $\cap_{n\in\omega} S_{\gamma_n} \subset \cap_{n\leq n_0} S_{\gamma_n} = \cap_{n\leq n_0} S_{\alpha_n} = \phi$ by $c \in W$, and $\cap_{n\in\omega} S_{\gamma_n} \neq \phi$ by $c \in L$. This is a contradiction.

(iv) $L \subset M$ and $f(L) \subset S$:

Let $b = (\beta_n) \in L$. Then $\cap_{n\in\omega} S_{\beta_n} \neq \phi$, and there is $\alpha \in \Lambda'$ such that $\beta_n \in A^{'}_{\alpha,n} \subset A_{\alpha,n}$ for any $n \in \omega$. Pick $y \in \cap_{n\in\omega} S_{\beta_n} \subset$

$\cap_{n\in\omega}P_{\beta_n}$, then $y \in P_{\beta_n} \in \mathcal{P}_{\alpha,n}$ for any $n \in \omega$. So $\{P_{\beta_n}\}$ is a network at y; thus, $b \in M$ and $f(b) = y \in S_{\beta_n} \subset S$.

(v) $S \subset f(L)$:

For limit point x of S, pick $\alpha \in \Lambda'$ such that $x \in X_\alpha$. Then for any $n \in \omega$, there is $\alpha_n \in A'_{\alpha,n}$ such that $x \in P_{\alpha_n}$. So $x \in S \cap P_{\alpha_n} = S_{\alpha_n}$; hence, $\cap_{n\in\omega}S_{\alpha_n} \neq \phi$. It is easy to see that $a = (\alpha_n) \in L$ and $f(a) = x$. For $y \in S$ and $y \neq x$, by (a), there is $n \in \omega$ such that $y \notin st(x, \mathcal{P}(n, \Lambda'))$. Put $m = \min\{n : y \notin st(x, \mathcal{P}(n, \Lambda'))\}$. Then $y \in st(x, \mathcal{P}(m-1, \Lambda'))$, so there is $\alpha \in \Lambda'$ and $\beta_{m-1} \in A'_{\alpha,m-1}$ such that $\{x, y\} \subset P_{\beta_{m-1}}$. Thus, $y \in S \cap P_{\beta_{m-1}} = S_{\beta_{m-1}}$. For $n < m-1$, as $\mathcal{P}_{\alpha,m-1}$ refines $\mathcal{P}_{\alpha,n}$, there is $\beta_n \in A_{\alpha,n}$ such that $P_{\beta_{m-1}} \subset P_{\beta_n}$. So $\{x, y\} \subset P_{\beta_n}$; thus, $\beta_n \in A'_{\alpha,n}$ and $y \in S \cap P_{\beta_n} = S_{\beta_n}$. For $n \geq m$, as $\mathcal{P}_{\alpha,n}$ refines $\mathcal{P}_{\alpha,m-1}$, $y \notin st(x, \mathcal{P}(n, \Lambda'))$, i.e., $y \in S - st(x, \mathcal{P}(n, \Lambda'))$. Notice that $y \in X_\alpha$, pick $\beta_n \in A_{\alpha,n}$ such that $y \in P_{\beta_n}$. Obviously, $\beta_n \in A'_{\alpha,n}$ and $y \in (S - st(x, \mathcal{P}(n, \Lambda'))) \cap P_{\beta_n} = S_{\beta_n}$. It is easy to see that $b = (\beta_n) \in L$ and $f(b) = y$. So $S \subset f(L)$.

By (i)-(v) above, f is pseudo-sequence-covering.

By the above, X is a pseudo-sequence-covering compact image of a locally separable metric space. □

Corollary 2.3. *For a space X, the following are equivalent.*

(1) X is a pseudo-sequence-covering, quotient compact image of a locally separable metric space;

(2) X is a quotient compact image of a locally separable metric space.

Proof: (1) $\Longrightarrow$ (2) is obvious; we prove only (2) $\Longrightarrow$ (1).

By [7, Proposition 2.1.16(5)], quotient mappings on sequential spaces are sequentially-quotient, so X is a sequentially-quotient compact image of a locally separable metric space; hence, X is a pseudo-sequence-covering compact image of a locally separable metric space from Theorem 2.2. Quotient mappings preserve sequential spaces, so X is sequential. By [7, Proposition 2.1.16(2)], X is a pseudo-sequence-covering, quotient compact image of a locally separable metric space. □

Remark 2.4. In [5], Ikeda asked if every quotient compact image of a locally separable metric space must be the pseudo-sequence-covering, quotient compact images of a locally separable metric space. Corollary 2.3 answers this question affirmatively.

Remark 2.5. By [17, Theorem 1], Theorem 2.2, and [10, Theorem 4.6], sequentially-quotient compact images of metric spaces (locally separable metric spaces, separable metric spaces) are pseudo-sequence-covering compact images of metric spaces (locally separable metric spaces, separable metric spaces). But we don't even know whether sequentially-quotient compact mappings on separable metric spaces are pseudo-sequence-covering. So we raised the following question. By the above, we conjecture the answer is positive.

Question 2.6. Let $f : X \longrightarrow Y$ be a sequentially-quotient compact mapping from a separable metric spaces X onto a space Y. Is f pseudo-sequence-covering?

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References

1. P. S. Alexandroff, *On some results concerning topological spaces and their continuous mappings.* 1962 General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961) pp. 41–54. NY: Academic Press.
2. J. R. Boone and F. Siwiec, *Sequentially quotient mappings*, Czechoslovak Math. J. **26**(101) (1976), no. 2, 174–182.
3. R. Engelking, *General Topology.* Warszawa: Polish Scientific Publishers, 1977.
4. G. Gruenhage, E. Michael, and Y. Tanaka, *Spaces determined by point-countable cover*, Pacific J. Math. **113** (1984), 303–332.
5. Y. Ikeda, *σ-strong networks and quotient compact images of metric spaces*, Questions Answers Gen. Topology **17** (1999), 269-279.
6. Y. Ikeda and Y. Tanaka, *Spaces having star-countable k-networks*, Topology Proc. **18** (1993), 107–132.
7. S. Lin, *Generalized Metric Spaces and Mappings.* (Chinese). Beijing: Chinese Science Press, 1995.
8. S. Lin, *Point-Countable Covers and Sequence-Covering Mappings.* (Chinese). Beijing: Chinese Science Press, 2002.
9. S. Lin., C. Liu, and M. Dai, *Images on locally separable metric spaces*, Acta Math. Sinica (N. S.) **13** (1997), 1–8.
10. S. Lin and P. Yan, *Sequence-covering maps of metric spaces*, Topology Appl. **109** (2001), 301–314.
11. S. Lin and P. Yan, *On sequence-covering compact mappings*, (Chinese), Acta Math. Sinica **44** (2001), 175–182.
12. E. Michael and K. Nagami, *Compact-covering images of metric spaces*, Proc. Amer. Math. Soc. **37** (1973), 260–266.

13. F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology and Appl. **1** (1971), 143–154.
14. Y. Tanaka, *Point-countable covers and k-networks*, Topology Proc. **12** (1987), 327–349.
15. Y. Tanaka, *Theory of k-networks, II*, Questions Answers Gen. Topology **19** (2001), 27–46.
16. Y. Tanaka, C. Liu, and Y. Ikeda, *Around quotient compact images of metric spaces*. To appear in Topology Appl.
17. P. Yan, *Compact images of metric spaces*, (Chinese), J. Math. Study **30** (1997), no. 2, 185–187.

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