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THE STRUCTURE OF SECTORS OF ZEROS OF ENTIRE FLOWS

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ABSTRACT. Dynamical systems or flows $\dot{z} = f(z)$, where f is entire on $\mathbb{C}$, are considered. The nature of the separatrices, the structure of sectors and the boundaries of sectors of the flow at the zeros of $f(z)$ are determined.

1. Introduction

This paper continues the study, commenced in [1, 2, 3, 9] and continued in [4], to explore the local and global properties of complex functions, especially those useful in number theory, using topological methods based on dynamical systems. This requires an investigation of the flow

$$\dot{z} = \frac{dz}{dt} = f(z), z \in \Omega$$

where f is a complex valued function of a complex variable, t is a real parameter and Ω a non-empty open subset of $\mathbb{C}$.

When $f(z)$ is required to be holomorphic there are strong implications for the topology of the flow which results. Some of these were detailed in [3]. For example there are no saddle points. There are no limit cycles on simply connected subdomains. Here this work is taken further.

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Potential applications include a deeper understanding of the zeros of the Riemann zeta function. It is already known [4] that each simple zero on the critical line is a focus. From the results in this paper it follows that each sector at each zero, simple or otherwise, is unbounded.

In Section 2 the term separatrix is defined, as is transit time. In the main Section 3, additional to what was shown in [3], it is proved that the neighborhood of a center is simply connected and that the sum of the transit times for boundary orbits is less than or equal to the period at the center. Each elliptic sector at a higher order zero has a very similar structure, i.e. it is simply connected and the boundary consists of disjoint orbits. The zero itself is the only zero of the flow on the boundary of a sector. Where the zero is a node or focus the picture is somewhat more complicated: zeros on the boundary are possible, but only as part of "separatrix cycles" with at most one or two orbits and a maximum of one zero. In all cases (center, node, focus, elliptic sector) the neighborhood is unbounded. This explains a major feature of the phase portrait of $\zeta(s)$, as may be observed in [4, Figure 2] reproduced here (with permission) as Figure 1.

There is an occasional need to embed flows onto the Riemann sphere, but the structure of the point at infinity is not used. Implicitly, the flow is replaced by

$$\dot{z} = \frac{f(z)}{1 + |z|^2 + |f(z)|^2}$$

which is real analytic on $\mathbb{C}$ and continuous on $\mathbb{C} \cup \{\infty\}$ when f is holomorphic, has a zero at infinity, and the same phase portrait at $\dot{z} = f(z)$.

2. Holomorphic dynamical systems structure

When considering the flow $\dot{z} = f(z)$, where f is holomorphic on an open subset of $\mathbb{C}$, special properties of f restrict the type of flow. Each zero is isolated. If the zero is simple and the eigenvalues of the characteristic polynomial of the linearization are real, then, by the Cauchy-Riemann equations, the eigenvalues must be equal. So saddle points do not exist.

Simple zeros can be categorized as **centers** surrounded by closed integral paths, **nodes** where all integral paths near the points either

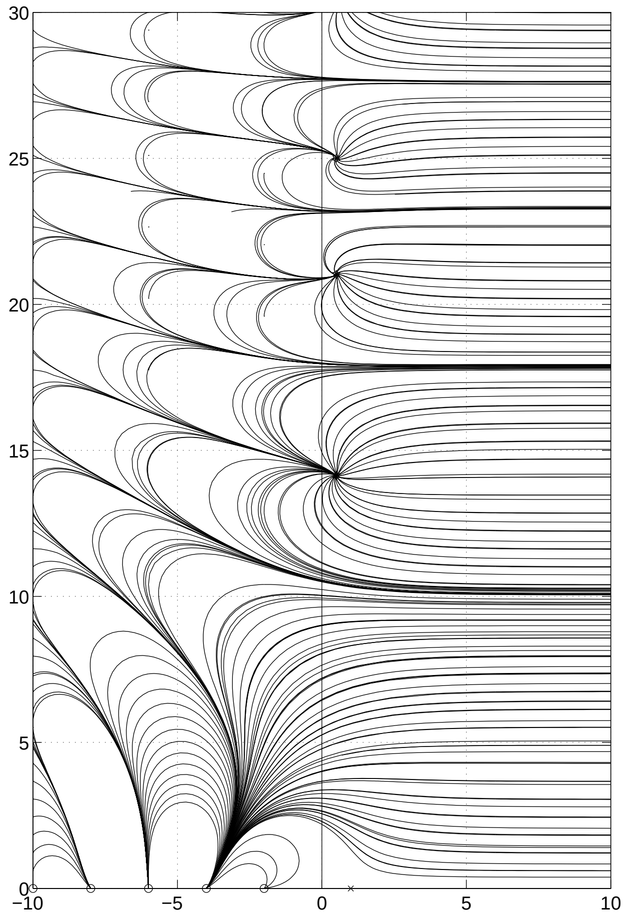

FIGURE 1. Lower section of the flow $\dot{z} = \zeta(z)$.

end or originate at the point, **saddles** where exactly four integral paths meet at the point, two beginning and two ending, and **foci** where the integral paths in the neighborhood of the zero never reach the point but spiral endlessly about it.

The local form at a simple zero $z = \rho$ is $\dot{z} = f'(\rho)(z - \rho)$. The type of zero is related to the type of the coefficient $f'(\rho)$: if pure imaginary the zero is a center, if real a node and if both real and complex parts are non-zero then the zero is a focus. If the zero is not simple then it must have some finite order $n \geq 2$ and the local

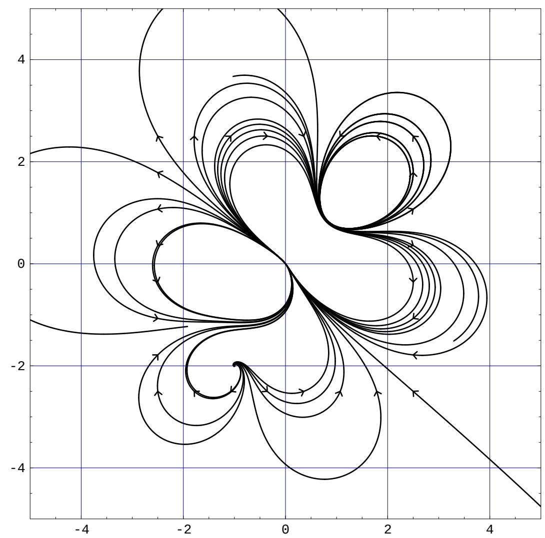

FIGURE 2. Example polynomial flow: $\dot{z} = p(z)$

approximation is $\dot{z} = f^{(n)}(\rho)(z-\rho)^n/n!$. Because of this the flow in the neighborhood of $z = \rho$ has a finite number (indeed $2n-2$) of **elliptic sectors**, where within each sector the flow begins and ends at the zero. The factor $f^{(n)}(\rho)/n!$ determines the local orientation of the flow, but not in this case its type.

An example is given in Figure 2. This is the polynomial flow

$$\dot{z} = p(z) = (2-3i)z^2(z-i-1)(z+2i+1)$$

which has three zeros, a doublet with two sectors at $z = 0$, a center at $z = 1 + i$ and a focus at $z = -1 - 2i$.

Using the Riemann mapping theorem and Schwarz lemma it can be shown [3, Theorem 3.2] that there are no limit cycles on simply connected domains. (The author conjectures that limit cycles never exist on any sort of domain for these flows.)

More details and proofs of these results may be found in [3]. In [4, Section 1.1] a glossary of dynamical systems terminology is given.

3. Definition of Separatrix

In the examples of meromorphic flows considered in [3, 4] the most interesting features have been the zeros and the separatrices. The former are specified very completely in the literature but not the latter. For example if there is a definition of separatrix in [6] or [7] it is hard to find. The definitions which do occur (see below) are adequate for polynomial or rational function flows, but not for more general holomorphic flows. For example [10, page 290] defines a separatrix to be either a critical point, limit cycle or trajectory on the boundary of a hyperbolic sector at a critical point. This does not take account of separatrices with endpoints "at infinity" or at a pole. The definition in [8, page 223] is better: "a path which is either a limit cycle or a path terminating or beginning on the projective plane with a side of a hyperbolic sector". However, in the examples of special interest, e.g. the exponential, hyperbolic, gamma, Riemann zeta and Riemann xi functions, behaviour at infinity is far from regular, but each of these functions exhibits strong separatrix phenomena.

The definition below avoids the use of points at infinity and covers the separatrixes which have appeared in the examples. Limit cycles do not occur so are not part of the definition. Zeros have their own classification, so have been left out also.

Definition 3.1. We say the orbit γ is a **positive separatrix** if for some $z \in \gamma$ the maximum interval of existence of the path commencing at z and proceeding in positive time is finite. We say the orbit γ is a **negative separatrix** if for some $z \in \gamma$ the maximum interval of existence of the path commencing at z and proceeding in negative time is finite. The orbit γ is a **separatrix** if it is a positive or negative separatrix.

This definition works for functions like $e^z, \zeta(z), \xi(z)$, polynomials, rational functions and the like. For the exponential flow (Figure 3), the separatrixes are the lines through $y = n\pi i, n \in \mathbb{Z}$ running parallel to the x-axis. For ze^z there is a node at 0 and the separatrix through $y = 0$ is lost, with the remaining separatrixes of e^z being distorted. For $f(z) = 1/z$ the x and y axes, make up four separatrixes, consistent with the given definition.

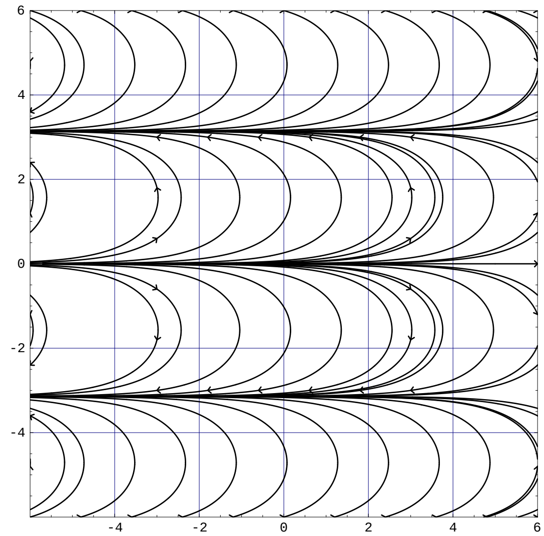

FIGURE 3. Exponential function flow: $\dot{z} = e^z$

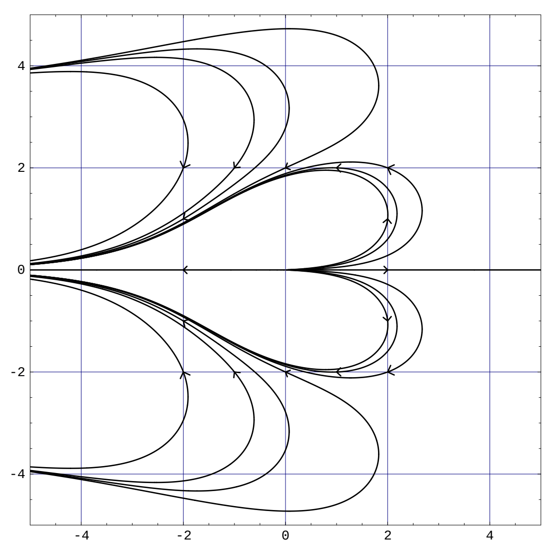

FIGURE 4. Flow with one zero: $\dot{z} = ze^z$

With this definition of separatrix, the union of all separatrixes and zeros remains closed, as it does under the definition of Marcus [9].

Definition 3.2 (transit time)**.** Let $\dot{z} = f(z)$ be a meromorphic flow and γ an orbit. If $a, b \in \gamma$ we define the **transit time** from a to b, denoted $\tau(a, b)$, to be the value of the integral

$$\tau(a, b) = \int_a^b \frac{dz}{f(z)}$$

where the integral is evaluated along the path γ. Note that any continuous deformation of this path will give the same value of the integral provided it does not cross a zero of $f(z)$. Note also that if $a = \gamma(t_1)$ and $b = \gamma(t_2)$ then $\tau(a, b) = t_2 - t_1$.

If γ is an orbit we define the **transit time** of γ, denoted $\tau(\gamma)$, to be the length of the maximum interval of existence for the flow commencing at any $z \in \gamma$, if this is bounded above and below, otherwise let $\tau(\gamma) = \infty$.

Transit time is simply the time it takes to go from one point to another on an orbit. It is not defined for points which are on different orbits.

Definition 3.3. Let α and β be two orbits in the same elliptic sector at a zero (periodic orbits about the same zero). We say β is **outside** α if every path from a point in the interior of $\alpha \cup \{z_o\}$ (respectively α) to a point on β cuts α.

Lemma 3.4. *Let $\dot{z} = f(z)$ be an entire flow. Let C be the graph of an orbit on the boundary of a center or elliptic sector at z_o. Let $x, y \in C$ and let $\epsilon > 0$ be given. Then there is an orbit γ about z_o such that for all orbits β about z_o, with β outside γ, $\beta \cap B_\epsilon(x) \neq \emptyset$ and $\beta \cap B_\epsilon(y) \neq \emptyset$.*

Proof. There exists an orbit γ_1 which meets $B_\epsilon(x)$ and an orbit γ_2 which meets $B_\epsilon(y)$. If γ is the orbit of the pair γ_1, γ_2 which is outside the other, then γ meets both $B_\epsilon(x)$ and $B_\epsilon(y)$: if $\gamma = \gamma_1$ and z_o is a center and γ did not meet $B_\epsilon(y)$ then the two open sets consisting of the interior region of γ and the exterior region of γ would disconnect the connected set $C \cup B_\epsilon(y) \cup \gamma_2$. The case where C is on the boundary of an elliptic sector is similar. $\square$

4. Main Results

In this section the neigbourhoods of a center (begun in [3, Theorem 3.3]), elliptic sector at a zero, and focus or node of an entire flow $\dot{z} = f(z)$ are described in separate theorems.

Theorem 4.1 (neighbourhood of a center)**.** *Let z_o be a center for the entire flow $\dot{z} = f(z)$ with open neighbourhood P and boundary components $(C_\lambda, \lambda \in \Lambda)$. Then P is simply connected and each C_λ is a separatrix. The sum of the transit times of the C_λ is bounded by the (common) period of the orbits which circulate about z_o.*

Proof. 1. P is simply connected: If not there exists a closed path Γ in P such that the interior of Γ has a non-empty intersection with the complement of P. Call this intersection A. Then A is closed and compact and has a boundary consisting of orbits and zeros of the flow. (These are finite in number since they are contained in a compact subset of $\mathbb{C}$.) Since there are no limit cycles there must be at least one zero on the boundary. But this is impossible, since the zero would have a hyperbolic sector (from the orbits of P).

2. Let T be the period, $T = 2\pi i/f'(z_o)$ [3, Theorem 2.3]. Then $\sum_{\lambda\in\Lambda} \tau(C_\lambda) \leq T$:

Claim A: For all $\epsilon > 0$ and $x, y \in C_\lambda$, there exists a $\delta > 0$ such that for all orbits γ with

$$x' \in \gamma \cup B_\delta(x) \neq \emptyset, y' \in \gamma \cup B_\delta(y) \neq \emptyset$$

such that $|\tau(x', y') - \tau(x, y)| < \epsilon$: Chose $\delta > 0$ such that, for some constant $M > 0$, $M \leq |f(z)|$ on $B_\delta(x)$ and on $B_\delta(y)$. Then

$$\begin{aligned} |\tau(x', y') - \tau(x, y)| &= \left| \int_{x'}^{y'} \frac{dz}{f(z)} - \int_x^y \frac{dz}{f(z)} \right| \\ &= \left| \int_x^{x'} \frac{dz}{f(z)} + \int_{y'}^y \frac{dz}{f(z)} \right| \\ &< \frac{2\delta}{M} < \epsilon. \end{aligned}$$

Claim B: If $x \in C_\lambda$ and $\epsilon > 0$ are given and $\alpha \cap B_\epsilon(x) \neq \emptyset$ and β is outside α, then $\beta \cap B_\epsilon(x) \neq \emptyset$: the proof of this is similar to that of Lemma 2.4.

By [3, Theorem 3.3] the index set is countable (or finite) so we identify it with $\mathbb{N}$.

Now, for each $i \in \mathbb{N}$ let $x_i, y_i \in C_i$ be arbitrary distinct points with the positive direction of flow being from x_i to y_i. Choose $\epsilon_1 > 0$ such that $B_{\epsilon_1} \cap B_{\epsilon_2} = \emptyset$ and, inductively, $\epsilon_i > 0$ such that the entire set of $B_{\epsilon_i}(x_i)$ and $B_{\epsilon_i}(y_i)$ are disjoint.

By Claim A, for each i we can find a $\delta_i > 0$ such that any orbit γ_i with

$$x_i' \in B_{\epsilon_i}(x_i) \cap \gamma_i, y_i' \in B_{\epsilon_i}(y_i) \cap \gamma_i$$

satisfies

$$|\tau(x_i', y_i') - \tau(x_i, y_i)| < \frac{\epsilon}{2^i}.$$

Then for each $N \in \mathbb{N}$ one of the orbits $(\gamma_i : 1 \leq i \leq N)$ is outside all of the others in the set, and, by Claim B and Claim A, satisfies

$$\sum_{i=1}^{N} \tau(x_i', y_i') \leq T.$$

But this implies

$$\sum_{i=1}^{N} \tau(x_i, y_i) \leq T + \epsilon.$$

Since the points x_i and y_i, and N are arbitrary, this implies

$$\sum_{i=1}^{\infty} \tau(C_i) \leq T + \epsilon,$$

and the result follows.

3. Each C_λ is a separatrix: this is imediate since each $\tau(C_\lambda)$ is bounded. $\square$

Figure 5 represents the flow of a function with 7 zeros, all of them simple:

$$\dot{z} = z(z^2 + 1)(z^2 - 1)(z^2 - (i + 1)^2).$$

There are centers at 0 and $i + 1$. The boundary of the neighbourhood of 0 consists of two disjoint separatrixes, whereas that of $i+1$ has one. Figure 2 of [3] is an example with 4 disjoint separatrixes on the boundary of a center. Figure 1 of [3] and Figure 1 above, are examples of elliptic sectors, the next type to be considered.

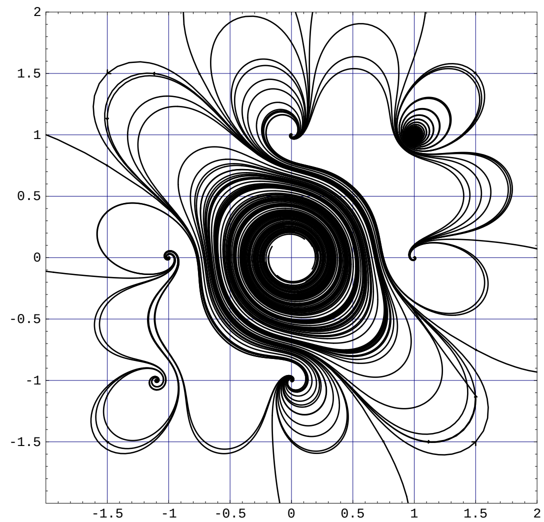

FIGURE 5. Flow with two centers.

Theorem 4.2 (structure of an elliptic sector)**.** *Let $\dot{z} = f(z)$ be an entire flow with a zero at z_o having order $n \geq 2$. Let P be the set consisting of the union of all of the orbits of the flow in a given sector at z_o which satisfy $L_\alpha(\gamma) = L_\omega(\gamma) = z_o$. Then P is a simply connected open subset of $\mathbb{C}$ and ∂P consists of an at most countable union of a set of closed separatrices $\{\gamma(x_\lambda, t) : \lambda \in \Lambda, t \in D_\lambda\}$, D_λ being the maximum interval of existence of the flow through x_λ, and where each $\gamma(x_\lambda, t)$ has an unbounded graph, together with two unbounded separatrices, u, v which satisfy $L_\omega(u) = z_o = L_\alpha(v)$.*

Proof. 1. P is open: If $z \in P$ then any orbit through a point sufficiently close to z, by continuous dependence on initial conditions, comes arbitrarily close to z_o in both positive and negative time. Therefore it must tend to z_o since z_o is elliptic. Hence P is open.

2. P is connected: If G, H is a disconnecting partition of P then necessarily $z_o \in G$ or $z_o \in H$ and not both. But this is a contradiction since points in either set must come arbitrarily close to z_o since each must contain complete orbits.

3. P is simply connected: If not there exists a closed path Γ in P such that the interior of Γ has a non-empty intersection with the complement of P. Call this intersection A. Then A is closed and compact and has a boundary consisting of orbits and zeros of the flow. Since there are no limit cycles there must be at least one zero on the boundary. But this is impossible, since the zero would have a hyperbolic sector (from the orbits of P).

4. $\partial P = B$ is closed. If $B = \emptyset$ the proof is complete. Otherwise proceed as follows:

5. Necessarily B consists of disjoint orbits and zeros. Any zero, other than z_o, would constitute a critical point with a hyperbolic sector, so cannot occur. Chose one point x_λ for each orbit. Let $D_\lambda = (\alpha, \beta)$ be the maximal interval of existence of $\gamma(x_\lambda, t)$. Let

$$B_\lambda = \{\gamma(x_\lambda, t) \mid t \in D_\lambda\}$$

Then

$$B = \{z_o\} \cup_{\lambda \in \Lambda} B_\lambda$$

the union being disjoint.

6. Consider $t \to \beta-$. The argument for $t \to \alpha+$ is similar. If the image of $[0, \beta)$ is bounded in $\mathbb{C}$ then necessarily $\beta = \infty$ and, since the flow has no limit cycle, $\omega(\gamma) = x_1$ which would be a critical point with a hyperbolic sector, impossible for a holomorphic flow. Therefore B_λ is unbounded and

$$B = \sqcup_{\lambda \in \Lambda} \{\gamma(x_\lambda, t) \mid t \in D_\lambda\} = \sqcup_{\lambda \in \Lambda} B_\lambda.$$

7. Each B_λ is closed: If not there is an $x \in \omega(B_\lambda)$ or $x \in \alpha(B_\lambda)$. Since the flow has no limit cycle, x must be a critical point, so must be center, focus, node or point with only elliptic sectors. Since B is closed, $x \in B$, so it must have at least one hyperbolic sector, which is false.

8. Each B_λ is a separatrix: by choosing $r > 0$ sufficiently small and integrating about an arc of a circle center z_o radius r it follows that, for any two points a, b on an orbit γ at z_o and on a circle center z_o radius r:

$$|\tau(a, b)| \ll_{r,f} \left| \int_a^b \frac{dz}{z^n} \right| \ll_{r,f} \frac{1}{r^{n-1}}.$$

Call this upper bound M_r. Then if $x, y \in B_\lambda$ are any two points and $n \in \mathbb{N}$ is given, by the Lemma 2.4, there is an orbit γ_n at z_o such that $B_{\frac{1}{n}}(x) \cap \gamma \neq \emptyset$ and $B_{\frac{1}{n}}(y) \cap \gamma \neq \emptyset$. If x_n is in this first intersection and y_n in the second, then $|\tau(x_n, y_n)| \leq M_r$. Taking the limit as $n \to \infty$ shows that $\tau(x, y)| \leq M_r$. Since this is true for any pair $x, y \in B_\lambda$, we have $\tau(B_\lambda) \leq M_r$, so B_λ is a positive and negative separatrix.

9. The orbits u, v define the sector. Their behaviour as $t \to \pm\infty$ can be deduced in a similar manner to those of other orbits on the boundary of P.

10. $|\Lambda| \leq \aleph_o$: On the Riemann sphere the B_λ enclose open disjoint regions, which therefore must be at most countable in number. □

Theorem 4.3 (structure of a node or focus basin)**.** *Let $\dot{z} = f(z)$ be an entire flow with a simple zero of at z_o which is a node or a focus. Let P be the set of all points in $\mathbb{C}$ with orbits which tend to z_o in positive time if it is a sink (or in negative time if it is a source). Assume, without loss in generality, that z_o is a sink. Then $P \cup \{z_o\}$ is a simply connected open subset of $\mathbb{C}$ and ∂P consists of an at most countable union of closed connected subsets each being of one of three types: (1) zeros z_1 each with an attached orbit γ_1 such that $L_\alpha(\gamma_1) = z_1$ and $L_\omega(\gamma_1) = \infty$, (2) zeros z_2 each with an attached pair of distinct orbits u, v with $L_\alpha(u) = L_\alpha(v) = z_2$ and $L_\omega(u) = L_\omega(v) = \infty$, and (3) orbits of the form γ_λ where each γ_λ is a positive and negative separatrix.*

Proof. 1. P is open: this follows from the continuous dependence of the flow on initial conditions.

2. P is connected: because $P \cup \{z_o\}$ is path connected and contains a neighbourhood $B_\epsilon(z_o)$, P is (path) connected.

3. If $B = \partial P$ then B is closed. The point $z_o \in B$.

4. B consists of the disjoint union of closed connected subsets, each being the union of zeros and orbits: if

$$B = \sqcup_{\lambda \in \Lambda} B_\lambda$$

with each B_λ connected, then $\overline{B_\lambda} \subset B$ is also connected. If $z \in \overline{B_\lambda} \setminus B_\lambda$ then, by the Poincaré-Bendixson theorem and the absence of limit cycles [3, Theorem 3.2], $f(z) = 0$ and $z \in B_\lambda$.

Then each B_λ can be expressed as the union of a chain of distinct orbits and zeros which is either finite or takes the form:

$$B_\lambda = \sqcup_{i \in \mathbb{Z}} \{z_i\} \cup \gamma_i$$

with $L_\alpha(g_i) = z_i, L_\omega(g_i) = z_{i+1}$ or $L_\alpha(g_i) = z_{i+1}, L_\omega(g_i) = z_i$ for each $i \in \mathbb{Z}$.

5. Each component of B cannot contain more than one zero of $f(z)$: if the component B_λ contains two (or more) distinct zeros then it must contain a section with three orbits and two zeros, say $(\alpha, z_1, \beta, z_2, \gamma)$. Neither z_1 nor z_2 can be a sink since they are on the boundary of a sink. If z_1 is a node (source) then the flow is hyperbolic in one sector at z_2 which is impossible for entire flows. The same applies to all of the other possible configurations.

6. If follows from 5. that each B_λ must be one of the types (1), (2) or (3) given in the statement of the theorem. In case (1) and (2) since $L_\alpha(g) = z_1 \in \mathbb{C}$, and there is only one zero, we must have $L_\omega(g) = \infty$, so the orbits are unbounded. In case (3) the orbits must be unbounded in both time directions.

To show that each B_λ is a separatrix consider type (1). The proofs for the other types are similar. Let z_1 be the associated zero and let $r > 0$ be sufficiently small that there are no other zeros in $B_r(z_o)$ or $B_r(z_1)$ and that these sets are disjoint. Let z_2 be a point on B_λ not in the closure of $B_r(z_1)$ and let $\epsilon > 0$ be given. There is an orbit γ with $L_\alpha(\gamma) = z_1,\ L_\omega(\gamma) = z_o$ and $\gamma \cap B_\epsilon(z_2) \neq \emptyset$, which cuts the boundaries of the circular neighborhoods of z_o and z_1. The rest of the proof is similar to that given for Theorem 3.1 (2) - it consists in showing that the time on any orbit starting at a point on the boundary of $B_r(z_o)$ and lying outside γ (and hence tending to B_λ) is bounded by a fixed bound, not dependent on z_2.

7. $P \cup \{z_o\}$ is simply connected: if not there is a bounded region with boundary meeting B in a closed connected subset. This consists of a set of type (1),(2) or (3) so must be unbounded, a contradiction.

8. $|\Lambda| \leq \aleph_o$: by embedding $\mathbb{C}$ in the Riemann sphere, the number of boundary components of types (2) and (3) are seen to be at most countable. Each component of type (1) is associated with a zero of $f(z)$, and there are an at most countable number of these, so they form an at most countable set also. $\square$

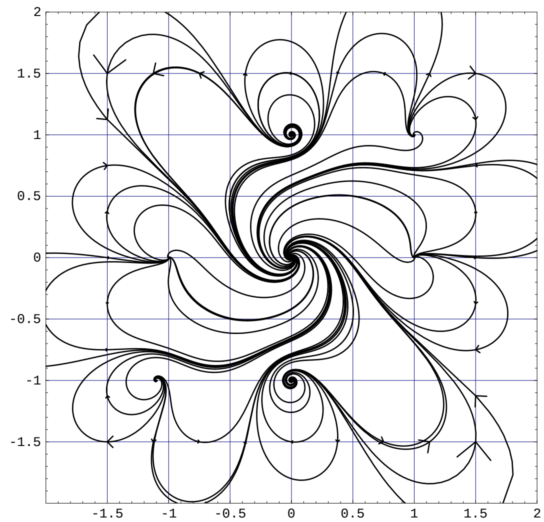

FIGURE 6. Neighbourhood of a focus.

Figure 6 is a neighbourhood of a zero at 0 with 6 zeros on the boundary, all of type (1):

$$f(z) = (1+i)z(z^2+1)(z^2-1)(z^2-(i+1)^2).$$

The separatrixes which tend to 0 (there are 6 of these also), are all included in the interior of the neighbourhood.

Remark 4.4. (1) The reader might suspect that for each of the results given in this paper, $|\Lambda| < \infty$. It would be well to keep in mind entire functions like:

$$f(z) = \alpha z^n \prod_{j\in\mathbb{N}} \Big(\prod_{q\in F_j} (1 - \frac{z}{j} e^{2\pi i q}) \Big) exp(g_j(z))$$

where, for each j, the $g_j(z)$ are functions chosen to make the product converge, where the F_j are the positive Farey rationals other than zero, where the integer n has $n \geq 1$, and where α is a complex number of unit modulus. For example when $\alpha = i, n = 1$, the flow has a center at 0 with $|\Lambda| = \aleph_o$.

(2) When a separatrix occurs on the boundary of a sector at a zero then, in the main, that separatrix is associated also with another zero or zeros. Attempts to prove this in general, using

devices such as the time advance map for the flow, the Riemann mapping theorem, Schwarz reflection, and properties of conformal maps [11, 12], have not succeeded. The example $\dot{z} = z\exp(z)$ (Figure 4) shows that a boundary separatrix need not be associated with any other finite zero.

(3) It is expected that a deeper understanding of separatrixes might be found by considering the singularities of the flow with complex time s:

$$\frac{\partial\gamma(z,s)}{\partial s} = f(\gamma(z,s)), \gamma(z,0) = z$$

where $\gamma : \mathbb{C}^2 \to \mathbb{C}$ is a locally holomorphic function of two complex variables, and where $f(z)$ is an entire function of one complex variable. An example of a modern reference to this very classical topic would be the work of the Costin's including [5].

(4) It is conjectured that for entire flows, the vector field is complete and time advance map holomorphic on a dense open subset of $\mathbb{C}$. This would follow if it could be shown that, for each $n \in \mathbb{N}$, the sets:

$$F^n = \{z : (\alpha_z, \beta_z) \subset (-\infty, n]\}, F_n = \{z : (\alpha_z, \beta_z) \subset [-n, \infty)\}$$

have empty interior.

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