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NOTES ON THE INDUCTIVE DIMENSION Ind_0

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Abstract. A variety of conditions is presented under which the sum theorem for Ind_0 holds. The subset theorem for Ind_0 is established for σ-totally paracompact as well as for supernormal spaces. The equality $Ind_0 = ind_0$ is proved for a class of spaces that includes all completely paracompact normal spaces as well as all order totally paracompact, almost Ščepin spaces. A proof is given of the product theorem for Ind_0 if the product is normal and piecewise rectangular.

1. Introduction

The dimension function Ind_0 is defined on all spaces inductively as follows:
(i) $Ind_0 X = -1$ iff $X = \emptyset$.
(ii) For a non-negative integer n, $Ind_0 X \leq n$ iff between any two disjoint closed subsets A, B of X there is a zero partition C with $Ind_0 C \leq n-1$.
(iii) $Ind_0 X = n$ iff $Ind_0 X \leq n$ is true but $Ind_0 X \leq n-1$ is false.
(iv) $Ind_0 X = \infty$ iff $Ind_0 X \leq n$ holds for no integer n.

Observe that $Ind_0 X \leq n$ iff every neighbourhood of each closed set A of X, contains a cozero set G and a zero set F of X with $A \subset G \subset F$ and $Ind_0(F \setminus G) \leq n-1$. The definition of ind_0 is obtained in an analogous manner, setting $ind_0 X \leq n$ iff every neighbourhood of each point x of X, contains a cozero set G and

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a zero set F of X with $x \in G \subset F$ and $ind_0(F \setminus G) \leq n-1$. It is evident that $Ind_0X = 0$ (resp. $ind_0X = 0$) iff $IndX = 0$ (resp. $indX = 0$). Also, $Ind_0X \geq IndX$, $ind_0X \geq indX$, if X is T_1 then $Ind_0X \geq ind_0X$, and, if X is perfectly normal, $Ind_0X = IndX$ and $ind_0X = indX$. It follows from the definition that if X is not normal, then $Ind_0X = \infty$.

The definitions of ind_0 and Ind_0 are attributed by A.V. Ivanov [15], who restricts attention to normal spaces, to V.V. Filippov. The dimension functions ind^* and Ind^* studied in [2] and [3] on all spaces can be trivially seen to agree with ind_0 and Ind_0, respectively, on the class of normal spaces.
These inductive dimension functions have proved instrumental in establishing the equality $ind = Ind$ on some interesting classes of topological spaces (see e.g. [11], [24], [7]).
The purpose of this paper is to extend the known results on Ind_0. In section 2, we look at sum theorems, some of which are needed in the sequel. Subsequent sections deal with subset theorems, the equality $ind_0 = Ind_0$, and the product theorem. The proofs of the results in the last three sections of this paper rely on a technical lemma presented in section 3 that concerns order locally finite collections and sharpens results by other authors.
Apart from the evident closed subset theorem for ind_0 and Ind_0, we recall the *countable sum theorem* for Ind_0: If a normal space X is the countable union of zero subsets with $Ind_0 \leq n$, then $Ind_0X \leq n$ (see [3], [15]). From these one readily deduces the cozero subset theorem: If G is a cozero set of any space X, then $Ind_0G \leq Ind_0X$.

2. Sum Theorems

Definition 1. A collection $\{F_\lambda : \lambda \in \Lambda\}$ of subsets of X will be called z-conservative if $\bigcup\{E_\lambda : \lambda \in \Lambda\}$ is a zero set of X whenever E_λ is a zero subset of F_λ for each λ in Λ.

We note for future use that if $\{F_\lambda : \lambda \in \Lambda\}$ is a z-conservative family of a space X, F_λ is normal and E_λ is a zero set of F_λ for each λ, then $\{E_\lambda : \lambda \in \Lambda\}$ is z-conservative in X. This is a consequence of the fact that a zero set of E_λ is also a zero set of F_λ.

Lemma 1. *Let $\{F_\lambda : \lambda \in \Lambda\}$ be a z-conservative cover of a space X with F_λ normal for each λ. Then X is normal.*

Proof. Let A, B be disjoint closed sets of X. As F_λ is normal for each λ, there is a zero set S_λ of F_λ containing $A \cap F_\lambda$ and disjoint from B. Then $S = \bigcup_\lambda S_\lambda$ is a zero set of X containing A but disjoint from B. Hence there is a zero set T of X containing B but disjoint from S. The result now follows from the fact that the disjoint zero sets S and T are contained in disjoint cozero sets of X. □

A countable union of z-conservative families is called a *σ-z-conservative* family.

Corollary 1. *A space X that can be covered by the interiors of a σ-z-conservative family of normal subspaces is normal.*

Proof. By Lemma 1, X has normal zero subspaces X_n such that $X = \bigcup_{n=1}^{\infty} intX_n$. This assures that X is normal (see e.g. Lemma 1.4.7 of [22]). □

Theorem 1. *Let $\{F_\lambda : \lambda \in \Lambda\}$ be a z-conservative cover of a space X with $Ind_0 F_\lambda \leq n$ for each λ. Then $Ind_0 X \leq n$.*

Proof. We can assume that n is finite, hence every F_λ is normal and, by lemma 1, X is normal. Let us note that for the case when $F_\lambda \cap F_\mu = \emptyset$ for $\lambda \neq \mu$, X is the topological sum of the spaces F_λ and the proof is straightforward (cf. Proposition 11 of [3]).
For the general case, we may assume that Λ is well-ordered. For each λ, the cozero set $F_\lambda \setminus \bigcup_{\mu<\lambda} F_\mu$ of F_λ is the union of countably many zero sets $E_{i,\lambda}$. Now, because $\{F_\lambda\}$ is z-conservative and each F_λ is normal, for each i, $E_i = \bigcup_\lambda E_{i,\lambda}$ is a zero set of X and $\{E_{i,\lambda} : \lambda \in \Lambda\}$ is a z-conservative cover of E_i. Thus, by the first part of the proof, $Ind_0 E_i \leq n$. Finally, as $X = \bigcup_i E_i$, we can now use the countable sum theorem to infer $Ind_0 X \leq n$. □

Corollary 2. *Let $\{F_\lambda : \lambda \in \Lambda\}$ be a σ-z-conservative cover of a normal space X such that $Ind_0 F_\lambda \leq n$ for each λ. Then $Ind_0 X \leq n$.*

Proof. Write $\Lambda = \bigcup_{i\in\mathbb{N}} \Lambda_i$, where $\{F_\lambda : \lambda \in \Lambda_i\}$ is z-conservative in X. Then each $X_i = \bigcup_{\lambda\in\Lambda_i} F_\lambda$ is a zero set of X and, by Theorem 1, $Ind_0 X_i \leq n$. Now, by the countable sum theorem, $Ind_0 X \leq n$. □

Corollary 3. *Let* $\{F_\lambda : \lambda \in \Lambda\}$ *be a* σ*-*z*-conservative family of* X *such that* $Ind_0 F_\lambda \leq n$ *for each* λ *and* $\bigcup_{\lambda\in\Lambda} int F_\lambda = X$*. Then* $Ind_0 X \leq n$*.*

Proof. We may assume n to be finite so that each F_λ is normal and hence, by Corollary 1, X is normal. The result now follows from Corollary 2. □

Lemma 2. *Let* $\{V_\lambda\}$ *be a locally finite cozero cover of* X *and* F *a subset of* X *such that* $F \cap V_\lambda$ *is a zero set of* V_λ *for each* λ*. Then* F *is a zero set of* X*.*

Proof. Note that the cozero set $V_\lambda \setminus F$ of V_λ is cozero in X, and consider a continuous $g_\lambda : X \to [0,1]$ such that $g_\lambda^{-1}(0,1] = V_\lambda \setminus F$. Define $g : X \to \mathbb{R}$ by $g(x) = \sum_\lambda g_\lambda(x)$. By the local finiteness of $\{V_\lambda\}$, g is continuous. Hence $F = g^{-1}(0)$ is a zero set of X. □

The following proposition enables us to recognize some locally finite collections of zero sets as being z-conservative. The result seems to be known, proofs of the equivalence of conditions (i), (iv) and (v) having appeared in both [20] and [17]. We include a proof of Proposition 1 for the convenience of the reader.

Proposition 1. *The following conditions for a locally finite family* $\{F_\lambda\}$ *of zero sets of* X *are equivalent.*
(i) There is a locally finite cozero collection $\{G_\lambda\}$ *of* X *with* $F_\lambda \subset G_\lambda$ *for each* λ*.*
(ii) There is a continuous $f : X \to M$ *into a metric space* M *and a locally finite collection* $\{E_\lambda\}$ *of closed sets of* M *with* $F_\lambda = f^{-1}(E_\lambda)$ *for each* λ*.*
(iii) There is a continuous $f : X \to P$ *into a countably paracompact and collectionwise normal space* P*, and a locally finite collection* $\{E_\lambda\}$ *of closed sets of* P *with* $F_\lambda = f^{-1}(E_\lambda)$ *for each* λ*.*
(iv) The local finiteness of $\{F_\lambda\}$ *is witnessed by a locally finite cozero cover* $\mathcal{V}$ *of* X*.*
(v) The local finiteness of $\{F_\lambda\}$ *is witnessed by a* σ*-locally finite cozero cover* $\mathcal{V}$ *of* X*.*
If one of the above conditions holds and X *is normal, then* $\{F_\lambda\}$ *is a* z*-conservative family of* X*.*

Proof. If (i) holds, let $f_\lambda : X \to I_\lambda$ be a continuous map, where I_λ is a copy of the unit interval $[0,1]$, such that $F_\lambda = f_\lambda^{-1}(1)$ and $X \setminus G_\lambda = f_\lambda^{-1}(0)$. Let M be the subset of $\prod I_\lambda$ consisting of those points with only a finite number of non-zero coordinates with metric $d(x,y) = \sum_\lambda |x_\lambda - y_\lambda|$. Then $\{E_\lambda\}$, where $E_\lambda = \{y \in M : y_\lambda = 1\}$, is a locally finite closed collection of M. Additionally, $f : X \to M$, where $f(x) = (f(x_\lambda))$, is continuous with $F_\lambda = f^{-1}(E_\lambda)$. Thus, (i) $\Rightarrow$ (ii).

Observe that under the assumptions of (iii), because P is countably paracompact and collectionwise normal, there is a locally finite open family $\{H_\lambda\}$ of P with $E_\lambda \subset H_\lambda$ for each λ (see [9], problem 5.5.17). Furthermore, by normality of P, each H_λ can be taken to be cozero. On letting, $G_\lambda = f^{-1}(H_\lambda)$, we see that (i) is satisfied. It follows that (i), (ii) and (iii) are equivalent.

As the implications (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$(v) are clear, in order to establish the equivalence of the five conditions, it remains to prove (v) $\Rightarrow$ (i).

Let us therefore assume (v). Then $\mathcal{V} = f^{-1}(\mathcal{U})$ for some continuous $f : X \to M$ into a metric space M and an open cover $\mathcal{U}$ of M. (This follows readily from the fact that, in the proof of (i) $\Rightarrow$ (ii), $G_\lambda = f^{-1}(\{y \in M : y_\lambda > 0\})$; cf. Exercise 5.1.J of [9].) Let $\mathcal{F}$ be a locally finite closed refinement of $\mathcal{U}$ and set for each λ

$$G_\lambda = X \setminus \bigcup \{f^{-1}(F) : F \in \mathcal{F}, f^{-1}(F) \cap F_\lambda = \emptyset\}.$$

One readily verifies that $\{G_\lambda\}$ is a locally finite cozero collection of X and (i) holds.

Finally, let us suppose that one of the above conditions holds and that X is normal. By (iv), there is a locally finite cozero cover $\{V_\mu\}$ of X that witnesses the local finiteness of $\{F_\lambda\}$. Consider for each $\lambda \in \Lambda$ a zero set E_λ of F_λ, and let $E = \bigcup E_\lambda$. Observe that each E_λ is a zero set of the normal X, each $E_\lambda \cap V_\mu$ is a zero set of V_μ and, because zero sets are closed under finite unions, $E \cap V_\mu$ is a zero set of V_μ. Thus, by Lemma 2, E is a zero set of X and $\{F_\lambda\}$ is z-conservative. □

Remark 1. It is clear from the proof that the union of a family $\{F_\lambda\}$ of zero sets of any space X satisfying (i) of Proposition 1 is a zero set of X, a fact proved in Lemma 1 of [3] as well as in Lemma 2.3 of [18].

The following result extends Proposition 13 of [3].

Corollary 4. *Suppose a countably paracompact and collectionwise normal space X has a σ-locally finite cover $\{F_\lambda\}$ consisting of zero sets with $Ind_0 \leq n$. Then $Ind_0 X \leq n$.*

Proof. By Proposition 1, $\{F_\lambda\}$ is σ-z-conservative in X, and the result follows from Corollary 2. □

Corollary 5. *Let $\{G_{i,\alpha}\}$ be a σ-locally finite cozero cover of a space X with $Ind_0 G_{i,\alpha} \leq n$. Then $Ind_0 X \leq n$.*

Proof. We can assume that n is finite so that each $G_{i,\alpha}$ is normal. Each $G_{i,\alpha}$ is the union of the interiors of zero sets $F_{i,j,\alpha}$ of X with $F_{i,j,\alpha} \subset G_{i,\alpha}$. By the subset theorem, $Ind_0(F_{i,j,\alpha}) \leq n$. Now, by Proposition 1, $\{F_{i,j,\alpha}\}_{i,j,\alpha}$ is σ-z-conservative and we can apply Corollary 3 to obtain $Ind_0 X \leq n$. □

Corollary 6. *Suppose a paracompact normal space X has a cover by open sets with $Ind_0 \leq n$. Then $Ind_0 X \leq n$. (*[2]*, Proposition 2 on page 129).*

Proof. X has a locally finite cozero cover $\{G_\alpha\}$ with each G_α a subset of some member of the above open cover. Then, by the subset theorem, $Ind_0 G_\alpha \leq n$ and the result follows from Corollary 5. □

Example 1. Consider a first countable Tychonoff space X which contains two discrete and disjoint closed subsets A, B that are not contained in disjoint open sets. X can be, for example, the Niemytzki plane or the square of the Sorgenfrey line. As disjoint zero sets in any space are contained in disjoint cozero sets, it is clear that either A or B is not a zero set. Thus, not every discrete collection of zero sets is z-conservative (cf. [9], exercise 1.5.J., page 49).
A similar pathology is exhibited in the non-normal space $X = \beta\mathbb{R} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$, where $\mathbb{N}$ is a closed G_δ but not a zero set (see Problem 6P of [14]).

Problem 1. Give an example of a locally finite collection of zero sets in a normal space which is not z-conservative.

Let us note that the finite sum theorem for Ind_0 for arbitrary closed subsets does not hold even for completely normal spaces. For each positive integer n, there is a compact and completely normal space S_n with $Ind_0 S_n = n$ which is the union of 2^{n-1} closed subspaces with $Ind_0 = 1$ (see [15] for the case n = 2, or [5] for the general case). However, the finite sum theorem for Ind_0 holds for perfectly κ-normal spaces, and the locally finite sum theorem for arbitrary closed sets is valid for paracompact perfectly κ-normal spaces [7].

Problem 2. Construct a normal space X which is the union of two zero sets X_1, X_2 with $IndX > max\{IndX_1, IndX_2\}$.

Note that Lokucievskii's compact space X is the union of two closed subsets X_i, $i = 1, 2$ with $IndX = 2$ while $IndX_i = 1$ [10]. However, X_1, X_2 are not zero sets of X.

3. A Technical Result

The following result is instrumental in both sections that follow. This should be compared with (the proof of) theorem 2 of [13], lemma 2 of [16] and the Main Lemma of [6].

Lemma 3. *Let $\{G_\alpha : \alpha \in A\}$ be an open cover and $\{F_\alpha : \alpha \in A\}$ a closed cover of a space X such that A is linearly ordered, $G_\alpha \subset F_\alpha$ and $\{F_\beta : \beta \leq \alpha\}$ is locally finite in G_α for each α. Define*

$$S_\alpha = G_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta,\ S = \bigcup_{\alpha\in A} S_\alpha,\ T_\alpha = F_\alpha \setminus \bigcup_{\beta\leq\alpha} G_\beta \text{ and } T = \bigcup_{\alpha\in A} T_\alpha.$$

Then
(i) $X = S \cup T$, $S \cap T = \emptyset$ and $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$.
(ii) $\{F_\beta : \beta \leq \alpha\}$ is locally finite in T_α for each α, and $\{T_\alpha : \alpha \in A\}$ is locally finite in X.
(iii) If each $F_\alpha \setminus G_\alpha$ is normal, then T is normal.
(iv) If H_α is a cozero set of X inside ClS_α for each α, then $H = \bigcup_\alpha H_\alpha$ is a cozero set of X.
(v) If $\{F_\alpha : \alpha \in A\}$ is σ-z-conservative in T and each G_α is a cozero set of X with $Ind_0(F_\alpha \setminus G_\alpha) \leq n$ for each α, then $Ind_0 T \leq n$.
(vi) If T is normal and paracompact, F_α is a zero set of X, G_α is a cozero of X and $Ind_0(F_\alpha \setminus G_\alpha) \leq n$ for each α, then $Ind_0 T \leq n$.

Proof. A point x of X belongs to some G_α. Because A is linearly ordered and $\{F_\beta : \beta \leq \alpha\}$ is locally finite in G_α, the sets $\{\alpha : x \in F_\alpha\}$ and $\{\beta : x \in G_\beta\}$ have first elements, denoted by $\alpha(x)$ and $\beta(x)$, respectively. Evidently, $\alpha(x) \leq \beta(x)$, if $\alpha(x) = \beta(x)$, then x belongs to $S_{\alpha(x)}$ and no other member of $\{S_\alpha, T_\alpha : \alpha \in A\}$,

and if $\alpha(x) < \beta(x)$, then $x \in T_{\alpha(x)} \setminus S$. The validity of (i) is now clear.
Observe that for $x \in T_\alpha$, $\alpha(x) \leq \alpha < \beta(x)$. Hence there is an open neighbourhood $U \subset G_{\beta(x)}$ of x which intersects only finitely many members of $\{F_\beta : \beta \leq \beta(x)\}$. Then U intersects only finitely many members of $\{T_\gamma : \gamma \in A\}$. This proves (ii). To prove (iii), we need only observe that if each $F_\alpha \setminus G_\alpha$ is normal, then $\{T_\alpha : \alpha \in A\}$ is a locally finite cover of T consisting of closed and normal subspaces of T.
To prove (iv), let $f_\alpha : X \to [0,1]$ be continuous with $f_\alpha^{-1}(0,1] = H_\alpha$. Note that for $\alpha \neq \beta$, because H_α and H_β are open subsets of X contained in ClS_α and ClS_β, respectively, and $S_\alpha \cap S_\beta = \emptyset$, we also have $H_\alpha \cap H_\beta = \emptyset$. Hence, for each $x \in X$, $f_\alpha(x) > 0$ for at most one value of α. Thus, we can define $f : X \to [0,1]$ by $f(x) = \sum_{\alpha\in A} f_\alpha(x)$. For $x \in G_\alpha$, $f(x) = \sum_{\beta\leq\alpha} f_\beta(x)$ because for $\alpha < \gamma$, $G_\alpha \cap S_\gamma = \emptyset$ and hence $G_\alpha \cap ClS_\gamma = \emptyset$. By the local finiteness of $\{F_\beta : \beta \leq \alpha\}$ in G_α, the restriction of f to each G_α is continuous. Hence f is continuous and $f^{-1}(0,1] = H$ is a cozero set of X.
To the assumptions of (v) we may add that n is finite so that each $F_\alpha \setminus G_\alpha$ is normal and therefore, by (iii), T is normal. Write $A = \bigcup_{i\in\mathbb{N}} A_i$, where A_i is such that $\{F_\alpha \cap T : \alpha \in A_i\}$ is a z-conservative family of T, and $G_\alpha = \bigcup_{j\in\mathbb{N}} F_{\alpha,j}$, where each $F_{\alpha,j}$ is a zero set of X. Now one readily sees that

$$T \cap \bigcup_{\beta\leq\alpha} G_\beta = \bigcup_{i,j\in\mathbb{N}} \bigcup_{\beta\in A_i,\beta\leq\alpha} T \cap F_{\beta,j}$$

is an F_σ-set of T. Hence $T_\alpha = T \cap (F_\alpha \setminus \bigcup_{\beta<\alpha} G_\beta)$, being a closed G_δ-set, is a zero set of the normal space $F_\alpha \cap T$. Note that, by the closed subset theorem, $Ind_0 T_\alpha \leq n$. Now, for each $i \in \mathbb{N}$, because $\{F_\alpha \cap T : \alpha \in A_i\}$ is z-conservative in T, by the remark following Definition 1, $\{T_\alpha : \alpha \in A_i\}$ is z-conservative in T and hence $T_i = \bigcup_{\alpha\in A_i} T_\alpha$ is a zero set of T and, by Theorem 1, $Ind_0 T_i \leq n$. Finally, by the countable sum theorem, $Ind_0 T \leq n$.
To prove (vi), for each α, we apply (v) to the cozero cover $\{G_\beta \cap G_\alpha : \beta \leq \alpha\}$ and the zero cover $\{F_\beta \cap G_\alpha : \beta \leq \alpha\}$ of the space G_α. Note that the corresponding "T" is $T \cap G_\alpha$, which as an F_σ-set of T is paracompact and normal. Hence, by Proposition 1, $\{F_\beta \cap G_\alpha : \beta \leq \alpha\}$ is z-conservative in $T \cap G_\alpha$. Also, by the cozero subset theorem, for each β, $Ind_0(F_\beta \setminus G_\beta) \cap G_\alpha \leq n$.

Observe that $(F_\beta \cap G_\alpha) \setminus (G_\beta \cap G_\alpha) = (F_\beta \setminus G_\beta) \cap G_\alpha$. Hence, by (v), $Ind_0(T \cap G_\alpha) \leq n$ for each α. Finally, by Corollary 6, $Ind_0 T \leq n$. □

4. Subset Theorems

A collection $\{V_\lambda : \lambda \in \Lambda\}$ is called *order locally finite* (resp. *order closure preserving*) if Λ is linearly ordered and $\{V_\mu : \mu \leq \lambda\}$ is locally finite (resp. closure preserving) for each λ in Λ. Suppose $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$, where $\Lambda = \bigcup_{i\in\mathbb{N}} \Lambda_i$ and, for each i, $\{V_\lambda : \lambda \in \Lambda_i\}$ is locally finite (resp. closure preserving). Give $A = \bigcup_{i\in\mathbb{N}}\{i\} \times \Lambda_i$ the lexicographic order and, for each $\alpha = (i, \lambda)$ in A, define V_α to be V_λ. Then $\{V_\beta : \beta \leq \alpha\}$ is locally finite (resp. closure preserving) for each α. Thus, $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ is order locally finite (resp. order closure preserving). Consequently, every σ-locally finite collection is order locally finite and order closure preserving.

We note a lemma that generalizes a standard result for countable collections.

Lemma 4. *Let E, F be subsets of X, $\{U_\alpha : \alpha \in A\}$ and $\{V_\alpha : \alpha \in A\}$ order closure preserving open collections of X such that $E \cap ClV_\alpha = \emptyset$, $F \cap ClU_\alpha = \emptyset$, $E \subset \bigcup_{\alpha\in A} U_\alpha$ and $F \subset \bigcup_{\alpha\in A} V_\alpha$. Then there are disjoint open sets G, H of X with $E \subset G$ and $F \subset H$.*

Proof. Define $G = \bigcup_{\alpha\in A} G_\alpha$ and $H = \bigcup_{\alpha\in A} H_\alpha$, where $G_\alpha = U_\alpha \setminus \bigcup_{\beta\leq\alpha} ClV_\beta$ and $H_\alpha = V_\alpha \setminus \bigcup_{\beta\leq\alpha} ClU_\beta$. □

Definition 2. A cozero set V of X will be called n-cozero if there are cozero sets V_i and zero sets E_i of X such that $V = \bigcup_{i\in\mathbb{N}} V_i$, $V_i \subset E_i \subset V$ and $Ind_0(E_i \setminus V_i) \leq n$.

Evidently, a cozero subset of a space X with $Ind_0 X \leq n$, and hence any clopen subset of a cozero set of X, is $(n-1)$-cozero.

Theorem 2. *Suppose every binary open cover of X has a σ-locally finite refinement by $(n-1)$-cozero sets. Then $Ind_0 X \leq n$.*

Proof. As every binary open cover of X has a σ-locally finite cozero refinement, it follows from Lemma 4 that X is normal. Let E_1, E_2 be disjoint closed sets of X. These are respectively contained in cozero sets U_1, U_2 with disjoint closures. Let $\mathcal{U}$ be a σ-locally finite refinement of $\{X \setminus ClU_1, X \setminus ClU_2\}$ by $(n-1)$-cozero sets of X. Bearing in mind the opening remarks of this section, we readily see that the $(n-1)$-cozero cover $\mathcal{U}$ is refined by three σ-locally finite covers $\{G_\alpha : \alpha \in A\}$, $\{F_\alpha : \alpha \in A\}$ and $\{H_\alpha : \alpha \in A\}$ where A is linearly ordered and, for each $\alpha \in A$, G_α and H_α are cozero and F_α is a zero set of X, $G_\alpha \subset F_\alpha \subset H_\alpha$, $Ind_0(F_\alpha \setminus G_\alpha) \leq n-1$ and $\{H_\beta : \beta \leq \alpha\}$ is locally finite in X. Observe that, by Proposition 1, $\{F_\alpha : \alpha \in A\}$ is σ-z-conservative in any closed subspace of the normal space X, and $\{F_\beta : \beta \leq \alpha\}$ is z-conservative for each α. Let us now adopt the notation of Lemma 3 and define

$$V_1 = U_1 \cup \bigcup\{S_\alpha : S_\alpha \cap ClU_1 \neq \emptyset\}, \text{ and}$$
$$V_2 = U_2 \cup \bigcup\{S_\alpha : S_\alpha \cap ClU_1 = \emptyset\}.$$

Note that each S_α is a cozero set of X because each $\{F_\beta : \beta \leq \alpha\}$ is z-conservative. One easily sees from Lemma 3(i) and (iv) that V_1, V_2 are disjoint cozero sets of X respectively containing E_1, E_2. Moreover, $L = X \setminus V_1 \cup V_2$ is a closed subset of the normal T and, by Lemma 3(v), $Ind_0 T \leq n-1$. Hence $Ind_0 L \leq n-1$ and therefore $Ind_0 X \leq n$. □

Corollary 7. *Let X be a subspace of Y such that every binary open cover of X has a σ-locally finite refinement consisting of clopen subsets of traces on X of cozero sets of Y. Then $Ind_0 X \leq Ind_0 Y$.*

Proof. We can assume that $Ind_0 Y = n$ is finite. The property formulated above for binary open covers of X is, in fact, enjoyed by binary open covers of every closed set of X. Hence, arguing by induction on n, if G is a cozero set of Y, which is $(n-1)$-cozero, then $X \cap G$, and any clopen subset of it, is an $(n-1)$-cozero set of X. Now the result is an immediate consequence of Theorem 2. □

Let us recall that a regular space X is called σ-totally paracompact if every base $\mathcal{B}$ of X, as a cover of X, has a σ-locally finite refinement consisting of open sets U for which $U \subset V$ and $BdU \subset BdV$ for some V in $\mathcal{B}$ ([10], page 165).

Corollary 8. *Let X be a σ-totally paracompact subspace of Y. Then $Ind_0 X \leq Ind_0 Y$.*

Proof. Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{B}$ be a base of X refining $\mathcal{U}$ and consisting of traces on X of cozero sets of Y. Then the σ-totally paracompact space X has a σ-locally finite cover $\mathcal{V}$ consisting of clopen subsets of members of $\mathcal{B}$. Thus, the result is an immediate consequence of Corollary 7. □

Let us recall that a subset X of a space Y that is the union of a locally finite in X collection of cozero sets of Y is called D-open in Y. Y is called *supernormal* if any two separated subsets of Y are contained in disjoint D-open subsets of Y [19]. Observe that in a totally normal space every open set is D-open, so that each of the following two results extends a subset theorem for totally normal spaces [3].

Lemma 5. *If X is D-open in Y, then $Ind_0X \leq Ind_0Y$.*

Proof. X has a locally finite cover consisting of cozero sets G_α of Y. By the subset theorem, $Ind_0G_\alpha \leq Ind_0Y$, and the rest follows from Corollary 5. □

Corollary 9. *For a subspace X of a supernormal space Y, $Ind_0X \leq Ind_0Y$.*

Proof. We may assume that $Ind_0Y = n$ is finite and argue by induction on n. Consider disjoint closed sets E, F of X. Observe that X is a subset of $Z = Y \setminus Cl_YE \cap Cl_YF$ and $Cl_YE \cap Z$, $Cl_YF \cap Z$ are disjoint closed sets of Z and separated subsets of Y. Hence they are contained in disjoint D-open sets G, H of Y. Evidently, these are subsets of the normal space Z, and there are cozero sets U, V and a zero set A of Z such that $E \subset Cl_YE \cap Z \subset U \subset A \subset V \subset G$. By Lemma 5, $Ind_0G \leq n$ and hence $Ind_0V \leq n$. Hence there is a cozero set S and a zero set T of V such that $E \subset Cl_YE \cap Z \subset S \subset T \subset U$ and $Ind_0(T \setminus S) \leq n - 1$. Note that T is a zero set of A and hence of Z and S is a cozero set of V and hence of Z. It is now clear that $L = X \cap S$ is a cozero set of X and $M = X \cap T$ is a zero set of X with $E \subset L \subset M \subset X \setminus F$ and, by the obvious induction hypothesis, $Ind_0(M \setminus L) = Ind_0X \cap (T \setminus S) \leq n - 1$. Hence $Ind_0X \leq n$, as wanted. □

5. Equality of inductive dimensions

Recall that a regular space X is called order totally paracompact (henceforth abbreviated to OTP) if for every open base $\mathcal{B}$ of X, there is an open cover $\{G_\alpha : \alpha \in A\}$ of X such that A is linearly ordered and, for each α, G_α is a clopen subset of some member of $\mathcal{B}$ and $\{G_\beta : \beta \leq \alpha\}$ is locally finite in G_α (cf. [12], [13]). A completely paracompact space is σ-totally paracompact and a σ-totally paracompact space is OTP ([10], page 165), but apparently no example is known of an OTP space which is not σ-totally paracompact [13].

Definition 3. We call a completely regular space X strongly order paracompact, SOP for short, if for every cozero base $\mathcal{B}$, X has a cozero cover $\{G_\alpha : \alpha \in A\}$ and a zero cover $\{F_\alpha : \alpha \in A\}$ such that A is linearly ordered and, for each α, $G_\alpha \subset F_\alpha$, $\{F_\beta : \beta \leq \alpha\}$ is locally finite in G_α and, for some member B_α of $\mathcal{B}$, $G_\alpha = F_\alpha \cap B_\alpha$.

Evidently, an open cover of an SOP space has an open refinement $\{G_\alpha : \alpha \in A\}$ such that A is linearly ordered and, for each α, $\{G_\beta : \beta \leq \alpha\}$ is locally finite in G_α. Hence every SOP space is paracompact and normal ([10], page 165).

Lemma 6. *A closed subspace Y of an SOP space X is SOP.*

Proof. A cozero base of Y is the trace on Y of some cozero base of X. □

Theorem 3. *For an SOP space X, $ind_0 X = Ind_0 X$.*

Proof. Let $ind_0 X = n$, where $0 \leq n < \infty$. It suffices to show $Ind_0 X \leq n$.
Let E_1, E_2 be disjoint zero sets of X, respectively contained in cozero sets U_1, U_2 with disjoint closures. Let $\mathcal{B}$ be a base of X consisting of cozero sets B_λ inside zero sets A_λ such that $ind_0(A_\lambda \backslash B_\lambda) \leq n-1$ and A_λ is disjoint from one of ClU_1, ClU_2 . By Lemma 6 and the obvious induction hypothesis, $Ind_0(A_\lambda \setminus B_\lambda) \leq n-1$.
Let $\{G_\alpha : \alpha \in A\}$ be a cozero cover and $\{F_\alpha : \alpha \in A\}$ a zero cover of X such that A is linearly ordered and, for each α, $G_\alpha \subset F_\alpha$, $\{F_\beta : \beta \leq \alpha\}$ is locally finite in G_α and, for some $\lambda(\alpha)$, $G_\alpha = F_\alpha \cap B_{\lambda(\alpha)}$. Evidently, we can assume that $F_\alpha \subset A_{\lambda(\alpha)}$ so that $F_\alpha \setminus G_\alpha \subset A_{\lambda(\alpha)} \setminus B_{\lambda(\alpha)}$ and hence $Ind_0(F_\alpha \setminus G_\alpha) \leq n-1$. Now, with the notation of Lemma 3, the closed subset T of X is paracompact normal and $Ind_0 T \leq n-1$. Exactly as in the proof of Theorem 2, we can define disjoint cozero sets V_1, V_2 of X respectively containing E_1, E_2 and such that $L = X \setminus V_1 \cup V_2$ as a closed subset of T. Hence $Ind_0 L \leq n-1$ and therefore $Ind_0 X \leq n$. □

Recall that a space where the closure of every open set is a zero set is called perfectly κ-normal, according to [23] or a member of **Oz**, according to [1]. A space where the closure of every cozero set is a zero set is said to be almost Ščepin [8].

Proposition 2. *An almost Ščepin OTP space X is SOP.*

Proof. Let $\mathcal{B}$ be a cozero base of X. Let $\{G_\alpha : \alpha \in A\}$ be an open cover of X such that A is linearly ordered and, for each α, G_α is a clopen subset of some member B_α of $\mathcal{B}$ and $\{G_\beta : \beta \leq \alpha\}$ is locally finite in G_α. Then G_α is a cozero set and $F_\alpha = ClG_\alpha$ is a zero set of X, $\{F_\beta : \beta \leq \alpha\}$ is locally finite in G_α and $G_\alpha = F_\alpha \cap B_\alpha$. Thus, X is SOP. □

It follows from the previous two results that the equality $ind_0 = Ind_0$ holds for all almost Ščepin OTP spaces, which generalizes Theorem 3 of [7].

Proposition 3. *A regular and completely paracompact space X is SOP.*

Proof. Let $\mathcal{B}$ be a cozero base of X. As X is completely paracompact, there are star-finite open covers $\mathcal{V}_i$ of X such that $\bigcup_{i\in\mathbb{N}} \mathcal{V}_i$ contains a cover $\mathcal{U}$ which refines $\mathcal{B}$. Write $\mathcal{V}_i = \{V_{i,j,\alpha} : j \in \mathbb{N}, \alpha \in A\}$, where $V_{i,j,\alpha} \cap V_{i,j,\beta} = \emptyset$ for $\alpha \neq \beta$, and set $X_{i,\alpha} = \bigcup_j V_{i,j,\alpha}$. Then $\{X_{i,\alpha} : \alpha \in A\}$ is a discrete clopen cover of X for each i. For each triple (i, j, α) with $V_{i,j,\alpha} \in \mathcal{U}$, we fix a $B_{i,j,\alpha} \in \mathcal{B}$ that contains $V_{i,j,\alpha}$, and set $G_{i,j,\alpha} = B_{i,j,\alpha} \cap X_{i,\alpha}$ and $F_{i,j,\alpha} = X_{i,\alpha}$. Clearly, $\{G_{i,j,\alpha}\}$ is a cozero cover and $\{F_{i,j,\alpha}\}$ is a σ-locally finite, and therefore order locally finite, zero cover of X. It follows that X is SOP. □

Remark 2. For τ uncountable, $J(\tau)$, the hedgehog with τ spines, is metric σ-totally paracompact (and therefore SOP), but not completely paracompact (see [22], page 86).

Evidently, Theorem 3 includes both of the known cases of the equality $ind_0 = Ind_0$, i.e. for completely paracompact spaces [3, 15] as well as for OTP hereditarily perfectly κ-normal spaces [7].

6. A Product Theorem

Recall that a *cozero (resp. zero) rectangle* of a product $X \times Y$ is a subset of the form $G \times H$, where G, H are cozero (resp. zero) sets of X, Y, respectively. Also, $X \times Y$ is called *(piecewise) rectangular* if every every finite cozero cover of it has a σ-locally finite refinement consisting of (clopen subsets of) cozero rectangles. These notions are due to B.A. Pasynkov and, if the product is normal, rectangularity is equivalent to J. Nagata's notion of F-product.

For further information, see [21] where a proof is given of the product theorem for Pasynkov's dimension function Id if the product is piecewise rectangular. The product theorem for Ind^* when the product is rectangular was announced without proof in [4].

Theorem 4. *Let a non-empty product $X \times Y$ be normal and piecewise rectangular. Then $Ind_0 X \times Y \leq Ind_0 X + Ind_0 Y$.*

Proof. We can assume $Ind_0 X = m < \infty$ and $Ind_0 Y = n < \infty$. Consider a cozero rectangle $G \times H$ of $X \times Y$ and let G_i, H_i be cozero and E_i, F_i zero sets of X, Y, respectively, such that $G = \bigcup_{i \in \mathbb{N}} G_i$, $G_i \subset E_i \subset G$, $Ind_0(E_i \setminus G_i) \leq m-1$, $H = \bigcup_{i \in \mathbb{N}} H_i$, $H_i \subset F_i \subset H$ and $Ind_0(F_i \setminus H_i) \leq n-1$. Observe that because $X \times Y$ is normal, every zero rectangle of the product is piecewise rectangular. Hence, by a trivial induction argument, $Ind_0(E_i - G_i) \times F_i \leq m-1+n$ and $Ind_0 E_i \times (F_i - H_i) \leq m+n-1$. Now, by the finite sum theorem, $Ind_0(E_i \times F_i - G_i \times H_i) \leq m+n-1$. Thus, every cozero rectangle, and hence every clopen subset of it, is $(n+m-1)$-cozero. Now Theorem 2 implies that $Ind_0 X \times Y \leq Ind_0 X + Ind_0 Y$. □

In conclusion, we note one more case where the product is rectangular.

Proposition 4. *Let X be metrizable and Y perfectly κ-normal. Then $X \times Y$ is perfectly κ-normal and rectangular.*

Proof. For each open set G of the product and each open set U of X,

$$V(U) = Int(Cl(\bigcup\{V \text{ open in } Y : U \times V \subset G\}))$$

is a cozero set of Y. Let $\{U_{i,\alpha} : i \in \mathbb{N}, \alpha \in A\}$ be a σ-locally finite base for X. To prove the rectangularity of the product, it remains to observe that a regular open set G of the product is the union of all cozero rectangles $U_{i,\alpha} \times V(U_{i,\alpha})$ that are contained in G. Finally, Theorem 3 of [23] asserts that the product is also perfectly κ-normal. □

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