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IDEAL REFLECTIONS

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ABSTRACT. A reflection theorem is a result of the form "if all small subsets of a space have property $\mathcal{P}$ then the space itself has $\mathcal{P}$". Typical "small" sets would be those of cardinality $\leq \aleph_1$, the meager sets, or closed nowhere dense sets. We abstract the properties of "small" necessary to ensure reflection of the countable chain condition, separability and the Lindelöf property. We investigate when first and second countability reflect in meager sets.

1. INTRODUCTION

A *reflection theorem* is a result of the form "if all small subsets of a space (in some class $\mathcal{C}$ of spaces) have property $\mathcal{P}$ then the space itself has $\mathcal{P}$". If a reflection theorem holds for a property $\mathcal{P}$, then one says "$\mathcal{P}$ reflects in small subsets (for the class $\mathcal{C}$)". Classically, "small" has meant "of size $\leq \omega_1$". For example Hajnal and Juhasz [10] showed that second countability reflects in size $\leq \omega_1$ subsets; and Dow [7] proved that metrizability reflects in size $\leq \omega_1$ subsets for compact spaces. Reflection results are also important in set theory, hence the emphasis on cardinality.

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However, in the topological context other definitions of “small” also make sense. In particular, closed nowhere dense subsets, or meager subsets, are traditionally considered small. Katetov [11] showed that compactness reflects in closed nowhere dense sets for spaces with no isolated points. Mills and Wattel [12] extended this result to show (in particular) the Lindelöf property and countable compactness both reflect in closed nowhere dense sets for spaces with no isolated points (or, in their terminology, that these properties are *nowhere densely generated*). See also Blair [4] for some related results.

In this paper we continue the study of reflection in “small” subsets, paying particular attention to reflection in closed nowhere dense subsets and related notions of smallness. Recall that a *Luzin space* is a topological space which is uncountable and has no isolated points, in which every nowhere dense set is countable. Let $\mathcal{I}$ be a functor associating with each topological space X a collection $\mathcal{I}(X)$ of subsets of X (the “small” subsets of X). It turns out that many results on reflection in subsets in $\mathcal{I}$ are of the form “property $\mathcal{P}$ does not reflect in subsets in $\mathcal{I}$ if and only if there is space X such that X does not have $\mathcal{P}$ and all sets in $\mathcal{I}(X)$ are countable.” In other words, failure of reflection is witnessed by a very strong counterexample along the lines of a Luzin space. Existence, non-existence, or set-theoretic independence of the strong example may then be verified. For example, first countable non-separable Luzin spaces exist under CH, but there are no Luzin spaces under MA+¬CH.

The paper is structured as follows. In Section 2 we formulate and prove a metatheorem which captures the general form of the result that non-reflection is equivalent to the existence of a strong example. In Section 3 we consider the (hereditary) Lindelöf property, the (hereditary) countable chain condition (ccc) and (hereditary) separability. We then determine the status of the strong example for each of these properties and a wide choice of small sets $\mathcal{I}$. For these properties, the metatheorem applies because the property reflects in subsets of size $\leq \omega_1$. In contrast to these properties, reflection of first countability and second countability in small subsets related to nowhere dense sets behaves quite unlike reflection in subsets of size $\leq \omega_1$, even though the results are of the standard form. We present results on reflection of first and second countability in Sections 4 and Section 5 respectively.

Note that the results quoted above on reflection in closed nowhere dense sets held for spaces with no isolated points. Without restricting the number of isolated points, no interesting properties reflect in nowhere dense sets, meager sets and so on. To see this, note that there are no nonempty meager subsets of a discrete space, and the only nonempty meager subset of a space with only one non-isolated point is the non-isolated point. Thus we will put some restriction on the number of isolated points, usually that there are at most countably many isolated points, or sometimes that there are only finitely many. We denote the set of isolated points of a space X by $I(X)$, and we denote $|I(X)|$ by $i(X)$.

All spaces are assumed to be T_3. We will also investigate reflection in discrete sets, and in the closures of discrete sets. The relationship between such sets and nowhere dense sets is captured in the following well-known observation.

Proposition 1.1. *Let D be a discrete subset of a space X. Then $D \setminus I(X)$ is nowhere dense.*

A space is *linearly Lindelöf* if every increasing open cover has a countable subcover, and *cosmic* if it has a countable network. All other undefined terms will be found in [13] or [9].

2. A Metatheorem

In this section we prove a general reflection theorem.

Let $\mathcal{P}$ be a topological property, let $\mathcal{I}$ be a functor associating with each topological space X a family $\mathcal{I}(X)$ of subsets of X, and let $\mathcal{C}$ be a class of spaces. Let A1 to A3 be the following assumptions on $\mathcal{P}$ and $\mathcal{I}$:

A1 (Countability Property): If $A \in \mathcal{I}(X)$ and A is countable, then A has property $\mathcal{P}$.

A2 (Covering Property): Suppose $X_0 \subseteq X$. If $A \subseteq X_0$, then there exists $\widehat{A} \subseteq X$ such that:

(i) $A \subseteq \widehat{A}$;

(ii) if $\widehat{A}$ has $\mathcal{P}$ in X then A has $\mathcal{P}$ in X_0; and

(iii) if $A \in \mathcal{I}(X_0)$ then $\widehat{A} \in \mathcal{I}(X)$.

A3 (Non Reflection in $\leq \aleph_1$ sized subsets): If X does not have property $\mathcal{P}$, then there exists $X_0 \subseteq X$ with $|X_0| = \omega_1$ such that:
- (i) X_0 does not have $\mathcal{P}$;
- (ii) every subset of X_0 with $\mathcal{P}$ is countable;
- (iii) there is a $J \subseteq X_0$ with $I(X_0) \setminus J$ countable and $\widehat{J} \in \mathcal{I}(X)$.

Theorem 2.1. *If $\mathcal{P}$ is a topological property and $\mathcal{I}$ is a functor satisfying A1–A3 (for spaces with countably many isolated points) then $\mathcal{P}$ reflects in $\mathcal{I}$ (for spaces with countably many isolated points) if and only if there does not exist a space Y such that Y has only countably many isolated points, Y does not have property $\mathcal{P}$, and all sets in $\mathcal{I}(Y)$ are countable. Further, any counterexample to reflection will contain a subspace of this special form.*

Proof. For necessity, note that if Y is such a space then, by A1, Y is an example showing that $\mathcal{P}$ does not reflect in $\mathcal{I}$.

For sufficiency, suppose that X does not have $\mathcal{P}$ but every $A \in \mathcal{I}(X)$ does have $\mathcal{P}$. Let Y be a subset of X with cardinality ω_1 as in A3. Thus Y does not have $\mathcal{P}$. Let $A \in \mathcal{I}(Y)$. By A2, there is some $B \in \mathcal{I}(X)$ with $A \subseteq B$. By hypothesis, B has $\mathcal{P}$ in X, so A has $\mathcal{P}$ in Y, so A is countable. It remains only to show that Y has only countably many isolated points. By A3(iii) there is some J with $I(X_0) \setminus J$ countable and $\widehat{J} \in \mathcal{I}(X)$. By hypothesis, $\widehat{J}$ has $\mathcal{P}$, so J has $\mathcal{P}$, so J is countable, and thus $I(X_0)$ is also countable. □

Notice that if any property $\mathcal{P}$ satisfies A1–A3, then hereditary $\mathcal{P}$ also satisfies A1–A3.

3. Some applications of the metatheorem

In what follows, we will apply our metatheorem to the following properties $\mathcal{P}$ and functors $\mathcal{I}$:

Properties: the (hereditary) countable chain condition (ccc); the (hereditary) Lindelöf property; the linearly Lindelöf property; and (hereditary) separability.

Functors: meager; nowhere dense; closed nowhere dense; closure of a discrete set; and discrete.

All of the above properties and functors satisfy A1, since any countable set has each of the properties. For A2, notice that if A is nowhere dense (discrete) in $X_0 \subseteq X$ then A is nowhere dense (discrete) in X. When $\mathcal{I}$ is the functor of meager sets, of nowhere dense sets or of discrete sets, we may take $\widehat{A} = A$, and it is clear that A2 will be satisfied. When $\mathcal{I}$ is the functor of closed nowhere dense sets or closures of discrete sets, we take $\widehat{A} = \overline{A}^X$. This certainly gives $\widehat{A} \in \mathcal{I}(X)$ whenever $A \in \mathcal{I}(X_0)$. When $\mathcal{P}$ is a hereditary property or ccc, knowing that $\widehat{A}$ has $\mathcal{P}$ will clearly imply that A has $\mathcal{P}$. Thus A2 easily holds for all combinations except for the properties of being separable or (linearly) Lindelöf, and the functors of closed nowhere dense sets and of closures of discrete sets. Below we will check A3, determine the status of the strong counterexample provided by the metatheorem, and investigate the six exceptional cases.

3.1. The countable chain condition.

Theorem 3.1. *The ccc and the hereditary ccc both reflect in meager sets, in nowhere dense sets and in closed nowhere dense sets for spaces with countably many isolated points. Both properties reflect in discrete subsets and in closures of discrete subsets for arbitrary spaces.*

Proof. As remarked above, A1 and A2 are satisfied, so we need only verify A3.

We deal first with reflection in meager, nowhere dense and closed nowhere dense sets. Let X be a space with countably many isolated points which is not (hereditarily) ccc. Take a family $\{ U_\alpha : \alpha \in \omega_1 \}$ of disjoint open sets in (a subspace of) X. Without loss of generality $U_\alpha \cap I(X) = \varnothing$ for each α. Choose x_α in U_α for each α. Then $X_0 = \{ x_\alpha : \alpha \in \omega_1 \}$ is a discrete subspace of X so it does not have the (hereditary) ccc, every (hereditarily) ccc subset of X_0 is countable, and, putting $J = X_0$, $\widehat{J} \in \mathcal{I}(X)$ so A3(iii) holds.

Similarly, A3 holds for discrete subsets and for closures of discrete subsets without the restriction on $i(X)$.

The strong example given by the metatheorem is an uncountable space which is not (hereditary) ccc in which every discrete subset is countable: there are no such spaces, so the (hereditary) ccc reflects in each of the functors as claimed. □

3.2. Separability. As we remarked before, if $\mathcal{I}$ is the functor of meager sets, nowhere dense sets or discrete sets and $\mathcal{P}$ is the property of separability or hereditary separability then A1 and A2 are easily satisfied with $\widehat{A} = A$ for $A \subseteq X_0$. When $\mathcal{P}$ is hereditary separability, A1 and A2 are also satisfied when $\mathcal{I}$ is the functor of closed nowhere dense sets or closures of discrete sets, taking $\widehat{A} = \overline{A}^X$ for $A \subseteq X_0$. For A3, if X is a non-separable space then we construct a left-separated subspace $X_0 = \{ x_\alpha : \alpha \in \omega_1 \}$ in the standard way. This clearly satisfies A3(i) and (ii). Putting $J = I(X_0) \setminus I(X)$, J is discrete and nowhere dense in X, so $\widehat{J} \in \mathcal{I}(X)$. Further, if $\widehat{J}$ has $\mathcal{P}$ then J has $\mathcal{P}$. Thus A3(iii)(b) is satisfied for the above combinations of $\mathcal{P}$ and $\mathcal{I}$ for spaces where $\mathcal{I}(X)$ is countable.

The canonical example given by the metatheorem for non-reflection in meager sets, nowhere dense sets and closed nowhere dense sets is (almost) a Luzin space, while the canonical example for non-reflection in discrete and closures of discrete sets contains an L-space and vice versa.

Proposition 3.2. *There exists a non-separable Luzin space if and only if there exists a non-separable space with countably many isolated points in which all nowhere dense sets are countable.*

Proof. For the non-trivial implication, suppose that X has countably many isolated points and every nowhere dense set countable. Let Z be the space obtained by resolving each isolated point of X into $\mathbb{Q}$. In other words, if we put $Y = X \setminus I(X)$ then $Z = Y \cup (I(X) \times Q)$, a basic neighbourhood of $y \in Y$ is $(U \cap Y) \cup ((U \cap I(X)) \times \mathbb{Q})$ for U a neighbourhood of y in X, and a basic neighbourhood of (x, q) is $\{x\} \times V$ where V is a neighbourhood of q in $\mathbb{Q}$. Then Z is a Luzin space. □

Proposition 3.3. *There exists an L-space if and only if there exists a non-separable space in which all (closures of) discrete subsets are countable.*

Proof. If X is an L-space, then it contains a left-separated subspace which is an L-space in which all closures of discrete subsets are countable.

Conversely, suppose X is a non-separable space in which all discrete subsets are countable. Then X contains a left-separated subspace $X_0 = \{ x_\alpha : \alpha \in \omega_1 \}$. If X_0 were not hereditarily Lindelöf then there would be an uncountable $\Lambda \subseteq \omega_1$ with $\{ x_\alpha : \alpha \in \Lambda \}$

right-separated. But then this subset would be uncountable and discrete, a contradiction. So X_0 is a non-separable L-space. □

Combining the above observations we have the following results.

Theorem 3.4. *The property of (hereditary) separability reflects in meager subsets and in nowhere dense subsets for spaces with countably many isolated points, if and only if there does not exist a non-separable Luzin space.*

The property of hereditary separability reflects in closed nowhere dense subsets for spaces with countably many isolated points if and only if there does not exist a non-separable Luzin space.

The property of (hereditary) separability reflects in discrete subsets if and only if there does not exist an L-space.

The property of hereditary separability reflects in closures of discrete subsets if and only if there does not exist an L-space.

This leaves only the two exceptional cases: reflection of separability in closures of discrete subsets and in closed nowhere dense sets.

Theorem 3.5. *Separability reflects in closures of discrete subsets for spaces with countably many isolated points if and only if there does not exist an L-space.*

Proof. Let X be an L-space. Then X is not separable. But any discrete set in X is countable, and dense in its closure. So closures of discrete sets are separable. Thus separability does not reflect.

Conversely, let X be a non-separable space with all closures of discrete sets separable. Then all closures of discrete sets are ccc, and by Theorem 3.1, X is hereditarily ccc. Since X is non-separable it contains a left-separated subset, which is hereditarily ccc, and hence must be hereditarily Lindelöf. This then is an L-space. □

The status of reflection of separable in closed nowhere dense sets sets remains unclear. If there is a non-separable space X with only countably many isolated points, all of whose closed nowhere dense sets are separable, then as in the preceding proof, there is a left-separated subspace in type ω_1, X_0 say, which is hereditarily ccc. Then X_0 is hereditarily Lindelöf, and hence an L-space. Thus:

Proposition 3.6. *If there are no L-spaces then separability reflects in closed nowhere dense subspaces.*

It is not clear that the X_0 discussed above is Luzin. Nor is it clear (to the authors) how to construct a space like X just from the existence of an L-space. Restricting our attention to compact spaces improves the situation.

Proposition 3.7. *If there are no non-separable Luzin spaces, then separability reflects in closed nowhere dense, and closures of discrete subsets for compact spaces with countably many isolated points.*

Proof. Let X be a compact space with only countably many isolated points and all closed nowhere dense subsets separable. Let C be a closed nowhere dense set. Then C is separable, and all its closed subsets are closed nowhere dense in X, and so too are separable. Thus, C is a compact space each of whose closed subsets is separable. By a theorem of Arhangel'skii [3] it follows all subsets of C are separable. Thus all closed nowhere dense subsets of X are hereditarily separable, and by Theorem 3.4 X is (hereditarily) separable. □

3.3. The Lindelöf property. As with separability, if $\mathcal{I}$ is the functor of meager sets, nowhere dense sets or discrete sets and $\mathcal{P}$ is the (hereditary) Lindelöf property then A1 and A2 are satisfied, and likewise when $\mathcal{I}$ is the functor of closed nowhere dense sets or closures of discrete sets and $\mathcal{P}$ is the hereditary Lindelöf property. For A3, if X is a non-Lindelöf space then it contains a right-separated subspace $X_0 = \{ x_\alpha : \alpha \in \omega_1 \}$. This satisfies A3(i) and (ii). Putting $J = I(X_0) \setminus I(X)$ gives a discrete subset of X, so $\widehat{J} \in \mathcal{I}(X)$. Thus A3(iii) is also satisfied if $I(X)$ is countable.

The canonical example for non-reflection of the (hereditary) Lindelöf property in meager and nowhere dense sets is (or contains) a non-Lindelöf space with countably many isolated points and with every nowhere dense set countable. We shall see below that there are no such spaces. Likewise, we shall see that the canonical example for non-reflection of the hereditary Lindelöf property in closures of discrete spaces cannot exist. Finally, the canonical example for non-reflection of the (hereditary) Lindelöf property in discrete subsets contains an S-space and any S-space is a canonical example. The existence of S-spaces is consistent with and independent of ZFC.

Proposition 3.8. *There exists a space X with countably many isolated points such that X is not (hereditarily) Lindelöf and every discrete subset of X is countable if and only if there exists an S-space.*

Proof. Suppose that X is such a space. Then X contains a right-separated subspace $X_0 = \{ x_\alpha : \alpha \in \omega_1 \}$. If X_0 contained a non-separable subspace, there would be an uncountable $\Lambda \subseteq \omega_1$ such that $\{ x_\alpha : \alpha \in \Lambda \}$ is left-separated and hence discrete, a contradiction. Thus X_0 is an S-space.

Conversely, suppose Y is an S-space. Then every discrete subset of Y is separable, hence countable. In particular, $I(Y)$ is countable. □

Theorem 3.9. *The (hereditary) Lindelöf property reflects in meager sets and in nowhere dense sets for spaces with countably many isolated points.*

The hereditary Lindelöf property reflects in closed nowhere dense sets and in closures of discrete sets for spaces with countably many isolated points.

The (hereditary) Lindelöf property reflects in discrete subsets for spaces with countably many isolated points if and only if there does not exist an S-space.

Proof. We have already seen that A1, A2 and A3 hold for each of these combinations of $\mathcal{P}$ and $\mathfrak{I}$, and that the canonical counterexample for reflection in discrete sets exists if and only if there is an S-space. It remains only to show that the canonical example cannot exist in any of the other cases.

Let Y be the canonical example given by the metatheorem, so Y does not have $\mathcal{P}$, Y has only countably many isolated points, and every set in $\mathfrak{I}(Y)$ is countable. Resolving all the isolated points into $\mathbb{Q}$, we may assume that $I(Y) = \varnothing$. Since Y is not (hereditarily) Lindelöf it contains a right-separated subset $Y_0 = \{ y_\alpha : \alpha \in \omega_1 \}$. We claim that $I(Y_0)$ is dense in Y_0, for if not then we can put $\alpha = \min\{ \beta : y_\beta \notin \overline{I(Y_0)} \}$, and y_α is isolated in Y_0 but not in $\overline{I(Y_0)}$, a contradiction. But then $\overline{I(Y_0)}$ is uncountable and the closure of a discrete set, hence in $\mathfrak{I}(Y)$ whether $\mathfrak{I}$ is meager, nowhere dense, closed nowhere dense or closure of discrete. This is a contradiction. □

It is a classical theorem (Mills and Wattel, [12]) that the Lindelöf property reflects in closed nowhere dense sets for spaces without isolated points. The proof does not follow the Metatheorem route. It is an open problem of Arhangel'skii whether or not the Lindelöf property reflects in closures of discrete sets. Since the property "linearly Lindelöf" reflects in closures of discrete sets [1], any counterexample must be linearly Lindelöf but not Lindelöf.

3.4. Reflection in Closed Discrete Subspaces. Recall that a space is countably compact if and only if every closed discrete subset is finite. It follows that countable compactness reflects in closed discrete subspaces. But it also follows that very little else will reflect in closed discrete subsets: all closed discrete subsets of the long line are finite and hence ccc (Lindelöf, separable, second countable), while the long line is not ccc (Lindelöf, separable, second countable).

4. Reflection of first countability

Let p be a free ultra-filter on ω. Then $\omega \cup \{p\}$ as a subspace of $\beta\omega$ has a countable infinity of isolated points, is not first countable, but has all meager subsets second countable. Hence in this section we will be dealing with spaces which are *dense in themselves*, in other words which have no isolated points.

Van Douwen [6] showed that there exists a T_3, dense in itself countable space in which every nowhere dense subset is closed discrete, which is therefore not first countable. Thus, first countability does not reflect in nowhere dense (or closed nowhere dense, or closures of discrete) subsets. In this section we consider classes of spaces in which first countability does reflect in meager subspaces. Most of the results depend on finding a **dense** meager subset.

Proposition 4.1. *If every meager subset of a dense in itself space X is first countable, and X has a dense meager subspace, then X is first countable.*

Proof. Note that if D is dense and meager in X and $x \in X$ then $D \cup \{x\}$ is dense and meager in X, and, since X is T_3, if D is dense in X and $x \in D$ then $\chi(x, D) = \chi(x, X)$. □

In our search for dense meager subsets, we look at the type of neighborhood base of each point.

Let $\mathcal{B}(x) = \{ B(x, \alpha) : \alpha \in \chi(x, X) \}$ be a neighborhood base at x. We say that $\mathcal{B}(x)$ satisfies **P1** if, for every $A \subseteq \mathcal{B}(x)$ with

$|A| < \chi(x,X)$ we have $\operatorname{int}(\bigcap A) \neq \varnothing$. We say that $\mathcal{B}(x)$ satisfies **P2** if, for every $A \subseteq \mathcal{B}(x)$ with $|A| < \operatorname{cf}(\chi(x,X))$ we have $\operatorname{int}(\bigcap A) \neq \varnothing$. We say that X is a P1-space if every point has a local basis which satisfies P1.

Proposition 4.2. (1) *If a point x has a neighborhood base satisfying P1 then it has a linearly ordered local π-base of size* $\operatorname{cf}(\chi(x,X))$.

(2) *If a point x has a linearly ordered local π-base of size* $\operatorname{cf}(\chi(x,X))$ *then every neighbourhood base at x satisfies P2. In particular, if $\chi(x,X)$ is a regular cardinal then every neighbourhood base at x satisfies P1.*

Proof. (1) Suppose $\mathcal{B}(x) = \{\, B_\alpha : \alpha \in \chi(x,X)\,\}$ is a neighborhood base at x satisfying P1. Then $\{\, \operatorname{int}(\bigcap_{\beta<\alpha} B_\beta) : \alpha < \chi(x,X)\,\}$ is a linearly ordered local π-base, which contains a linearly ordered local π-base of size $\operatorname{cf}(\chi(x,X))$.

(2) Suppose that $\mathcal{P} = \{\, P(\alpha) : \alpha < \operatorname{cf}(\chi(x,X))\,\}$ is a linearly ordered local π-base at x. Let $\mathcal{B}$ be a neighbourhood base at x and let $\mathcal{A} \subseteq \mathcal{B}$ with $|\mathcal{A}| < \operatorname{cf}(\chi(x,X))$. For each $A \in \mathcal{A}$ choose $\mu_A < \operatorname{cf}(\chi(x,X))$ with $P(\mu_A) \subseteq A$, and put $\mu = \sup\{\, \mu_A : A \in \mathcal{A}\,\}$. Then $P(\mu) \subseteq \operatorname{int}(\bigcap \mathcal{A})$, so $\operatorname{int}(\bigcap \mathcal{A}) \neq \varnothing$ as required. □

Proposition 4.3. *If every meager subset of a dense in itself P1 space X is first countable, then for each $x \in X$ we have* $\operatorname{cf}(\chi(x,X)) = \omega$.

Proof. Suppose that there is some $x \in X$ with $\operatorname{cf}(\chi(x,X)) > \omega$. Put $\kappa = \operatorname{cf}(\chi(x,X))$. Let $\mathcal{P} = \{\, P(\alpha) : \alpha < \kappa\,\}$ be a linearly ordered local π-base at x. Inductively choose $\beta_\alpha < \kappa$ and $x_\alpha \in P(\beta_\alpha)$ so that $x_\alpha \notin \overline{P(\beta_{\alpha+1})}$. Then $D = \{\, x_\alpha : \alpha < \kappa\,\}$ is discrete so (since X has no isolated points) $D \cup \{x\}$ is meager. Every neighbourhood of x contains cofinally many of the points x_α, so $D \cup \{x\}$ is not first countable. □

In the next few results we will show that first countability reflects in meager subsets by constructing dense meager subsets and appealing to Proposition 4.1. The nowhere dense sets from which the meager sets are constructed will be the boundaries of a dense family of open sets. In order for these boundaries to be non-empty, we require a certain amount of connectedness: what we shall require is that there is a π-base of connected sets.

Definition 4.4. Let X be a dense in itself space. A Δ-*tree* for X is a tree T, with the following properties:

- the αth level $T(\alpha)$, is a disjoint family of open sets in X;
- $T(0) = \{X\}$;
- for each $a \in T(\alpha)$, the collection
$$M(a) = \{\, b \in T(\alpha+1) : a < b \,\}$$
is a maximal disjoint collection of open sets with $\overline{b} \subseteq a$ for each $b \in M(a)$ and $|M(a)| \geq 2$.
- if α is a limit and $\mathfrak{B}_\alpha$ is the set of branches in $T_\alpha = \bigcup_{\beta<\alpha} T(\beta)$. Then $T(\alpha) = \{\, \mathrm{int}(\bigcap B) : B \in \mathfrak{B}_\alpha \,\} \smallsetminus \{\varnothing\}$.

Note that every dense in itself T_3 space has such a tree. We define the Δ-*height* of X to be the least height of a Δ-tree for X.

Lemma 4.5. *Let X be dense in itself with a π-base of connected sets, and and let T be a Δ-tree for X of height α_0. For each $\alpha < \alpha_0$ let $D_\alpha = \partial(\bigcup T(\alpha))$. Then $\bigcup_{\alpha<\alpha_0} D_\alpha$ is dense. In particular, if α_0 is countable then X has a dense meager subspace.*

Proof. Let U be a member of the π-base of connected sets. Suppose, for a contradiction, that $U \cap \partial a = \varnothing$ for all $a \in T$. Inductively, we can choose some $a_\alpha \in T(\alpha)$ with $a \cap U \neq \varnothing$ and therefore (since $a \cap U$ is clopen in U) $U \subseteq a$: at limit levels α we have $U \subseteq \bigcap_{\beta<\alpha} a_\beta$ so $a_\alpha = \mathrm{int}(\bigcap_{\beta<\alpha} a_\beta)) \in T(\alpha)$ and $U \subseteq a_\alpha$. Proceeding in this way we obtain elements of the tree of arbitrary height, in particular we obtain $a_{\alpha_0} \in T(\alpha_0)$, contradicting the definition of α_0. So U must meet ∂a for some a in some $T(\alpha)$, and hence U meets D_α.

Finally note that each D_α is the boundary of an open set, hence nowhere dense, so if α_0 is countable then $\bigcup_{\alpha<\alpha_0} D_\alpha$ is meager. $\square$

Recall that a space X has a G_δ diagonal if it has a sequence $(\mathcal{G}_n)_{n\in\omega}$ of open covers such that for each x, $\bigcap_{n\in\omega} \mathrm{st}(x, \mathcal{G}_n) = \{x\}$.

Theorem 4.6. *First countability reflects in meager subsets for dense in themselves spaces with a G_δ diagonal and a π-base of connected sets.*

Proof. Let X be a dense in itself space with a G_δ diagonal and a π-base of connected sets. Let $\mathcal{G} = (\mathcal{G}_n)_{n\in\omega}$ be a G_δ diagonal sequence. We can clearly construct a Δ-tree T for X with the property that if $a \in T(n)$ then $a \subseteq U$ for some $U \in \mathcal{G}_n$. But then, since X has no isolated points, if $a_n \in T(n)$ for all $n \in \omega$ then $\operatorname{int}(\bigcap_{n\in\omega} a_n) = \varnothing$. Thus X has Δ-height ω. By Lemma 4.5, X has a dense meager subset, so by Proposition 4.1 first countability reflects in meager subsets of X. □

Proposition 4.7. *Assume that Souslin's Hypothesis (SH) holds. Let X be a dense in itself ccc space with a π-base of connected sets. Then X has a dense meager subset.*

Proof. Let T be a Δ-tree for X. For every $a, b \in T$ with $a < b$ we have $\overline{b} \subseteq a$, so since X is ccc every chain in T is countable. Further, let $a, b \in T$ be incompatible. We have $a \in T(\alpha)$ and $b \in T(\beta)$ for some α, β and without loss of generality $\alpha \leq \beta$. There is some $b' \in T(\alpha)$ with $b' \leq b$. Since $a \not\leq b$ we have $a \neq b'$ so $a \cap b' = \varnothing$, so $a \cap b = \varnothing$. Thus every anti-chain in T is a collection of disjoint open sets in X, hence countable. Thus, if T had uncountable height it would be a Souslin Tree. By SH, there are no such trees. So T has countable height, and by Lemma 4.5 X has a dense meager subset. □

Theorem 4.8. *Let $\mathcal{C}$ be the class of dense in themselves spaces with a π-base of connected sets. Then the following are equivalent.*

(i) *Every ccc space in $\mathcal{C}$ has a dense meager subset.*
(ii) *First countability reflects in meager subsets for ccc spaces in $\mathcal{C}$.*
(iii) *Every hereditarily Lindelöf space in $\mathcal{C}$, all of whose meager subsets are second countable, is first countable.*
(iv) *There are no Souslin trees.*

Proof. The implication (i) implies (ii) follows from Proposition 4.1, and (ii) implies (iii) is trivial. The implication (iv) implies (i) is Proposition 4.7. It remains only to prove that if there is a Souslin tree then there is a hereditarily Lindelöf space in $\mathcal{C}$, which is not first countable but every meager subset of which is second countable.

Let L be a compact, connected Souslin line with no separable open subsets. Then it is known [14] that the meager subsets of L are second countable. Remove one point from L, call the result L'. Note that L' is first countable, of weight ω_1, hereditarily Lindelöf, locally connected, and all its meager subsets are second countable.

Dow [8] has proved that every ccc space of π-weight no more than ω_1 has a remote point in its Stone–Čech compactification. (A point $p \in \beta X \setminus X$ is a remote point if and only if for every nowhere dense subset N of X we have $p \notin \overline{N}^{\beta X}$).

In particular L' has a remote point p. Let $X = L' \cup \{p\}$, as a subspace of $\beta L'$. Then X is T_3, hereditarily Lindelöf, and is not first countable (points in a Stone–Cech remainder never have countable character). As L' is locally connected, X has a π-base of connected sets. Further, every meager subset of X is either a meager (and second countable) subspace of L', or is the disjoint sum of a meager subset of L' and the single point p, and hence is second countable because p is remote. Thus X is as required. □

5. Reflection of second countability

In this section we consider reflection of second countability. To start with we restrict our attention to first countable spaces. Then we combine the results obtained with the results on reflection of first countability from the preceding section.

Again, we will restrict ourselves to spaces with only finitely many isolated points: if we allow infinitely many isolated points then no positive results hold, as the following example shows.

Proposition 5.1. *There is a first countable cosmic space X with countably many isolated points, all of whose meager subsets are second countable but which is not itself second countable.*

Proof. Let B the the "bowtie" space obtained by refining the usual topology on $\mathbb{R} \times \mathbb{R}$ by declaring sets of the form $B(x, s) = \{(x, 0)\} \cup \{\,(x', y) : |y| < s|x - x'|\,\}$ to be open for all $x, s \in \mathbb{R}$ with $s > 0$. Let $D = \mathbb{Q} \times (\mathbb{Q} \setminus \{0\})$, let $R = \mathbb{R} \times \{0\}$ and let X be the space with underlying set $R \cup D$ and topology obtained by further refining the topology inherited from B by declaring all the points of D to be isolated. Then X is first countable, cosmic, and not second countable. Further, every nowhere dense subset of X is contained in R, and hence every meager subset is second countable. □

5.1. Reflection of second countability for first countable spaces.

Definition 5.2. Let Y be a subspace of a space X. Then Y is *second countable in X* if there is a countable family $\mathcal{B}$ of open sets in X such that for every $y \in Y$, if $y \in U$ and U is open in X then there is some $B \in \mathcal{B}$ with $y \in B \subseteq U$. Such a family $\mathcal{B}$ is called a *base for Y in X*.

Proposition 5.3. *Let Y be a subspace of a space X. If Y is second countable in X and $\mathcal{B}$ is a base for Y in X then there is a countable $\mathcal{B}' \subseteq \mathcal{B}$ which is a base for Y in X.*

Proof. Exactly as for the corresponding result for bases of second countable spaces. □

Proposition 5.4. *If Y is a dense subspace of a (T_3) space X and Y is second countable then Y is second countable in X.*

Proof. Let $\mathcal{B}$ be a countable family of regular open subsets of X whose traces onto Y form a countable base for Y. Suppose $y \in U$ where $y \in Y$ and U is regular open in X. Pick $B \in \mathcal{B}$ so that $y \in B \cap Y \subseteq U \cap Y$. Then $y \in B = \operatorname{int}(\overline{B \cap Y}) \subseteq \operatorname{int}(\overline{U \cap Y}) = U$. So $\mathcal{B}$ is a countable base for Y in X. □

Theorem 5.5. *Let $\mathcal{C}$ be the class of first countable dense in themselves spaces. Then second countability reflects in meager subsets for spaces in $\mathcal{C}$ if and only if there is no space in $\mathcal{C}$ which is not second countable but which has every meager subset countable.*

Proof. If there is a space in $\mathcal{C}$ which is not second countable but which has every meager set countable, then every meager subset is second countable so second countability does not reflect in meager subsets for $\mathcal{C}$.

Conversely, let X be a space in $\mathcal{C}$ which is not second countable but in which every meager subset is second countable. If X is not separable, then by our results on reflection of separability X has a subspace which is non-separable (hence non-second-countable), all of whose meager subsets are countable (and hence second countable). So we are done.

Assume then that X is separable. Let D be a countable dense set and for each $x \in X$ let $\mathcal{B}(x)$ be a countable neighbourhood base at x. Since X is not second countable we can inductively choose points x_α such that $\bigcup_{\beta<\alpha} \mathcal{B}(x_\beta)$ is not a base for $\{ x_\beta : \beta \leq \alpha \}$ in X. We can assume $D = \{ x_\alpha : \alpha < \omega \}$. Note that $\bigcup_{\alpha<\omega_1} B(x_\alpha)$ is a base for $Y = \{ x_\alpha : \alpha < \omega_1 \}$ in X. By construction Y is not second countable in X. And since Y is dense in X, by Proposition 5.4 Y is not second countable.

We will show that all meager subsets of Y are countable. For a contradiction, suppose that there is some uncountable subset A of Y which is nowhere dense in Y. Let $C = \overline{A}^X$. Then C is closed nowhere dense in X. Hence $C \cup D$ is meager in X, so second countable, and hence (since it is dense in X) $C \cup D$ is second countable in X. But then since $A \subseteq C \cup D$, A is second countable in X. Thus there is a countable subset of $\bigcup_{\alpha<\omega_1} \mathcal{B}(x_\alpha)$ which is a basis for A in X, which contradicts the construction of the points x_α and the fact that A is uncountable. □

The question of whether it is consistent that second countable reflects in closed nowhere dense subsets of first countable spaces remains open. Cairns [5] and Alas et al. [1] have (respectively) shown that second countability does consistently reflect in closed nowhere dense (respectively, closures of discrete) sets for compact spaces.

5.2. Reflection of second countability for spaces with arbitrary character. From the previous results and noting that every Souslin line contains a Luzin subspace, we have the following.

Corollary 5.6. *Let $\mathcal{C}$ be the class of spaces with a dense subset which has a π-base of connected sets. If there are no Luzin spaces then second countability reflects in meager subsets for spaces in $\mathcal{C}$. If there is a Souslin tree then second countability does not reflect in meager subsets for spaces in $\mathcal{C}$.*

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