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EXTENDING PSEUDO-REGULARITY FOR LIPSCHITZ FUNCTIONS TO ASPLUND GENERATED SPACES

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Abstract. Locally Lipschitz functions on separable Banach spaces are generically pseudo-regular. We extend a slightly weaker property to Banach spaces which are Asplund generated spaces.

In the 1970's and 80's there was considerable study of the differentiability properties of continuous convex functions on Banach spaces, [8]. This work has been of interest in the classification of these spaces. Well established classes where such functions have discernible differentiability properties are the Asplund and Asplund generated spaces, [3]. But this work has also been useful for its applications in nonlinear optimisation, [1]. The next logical stage of this research has been to determine differentiability properties of locally Lipschitz functions on Banach spaces. This investigation is much more complicated and has been centred around the techniques developed by Clarke, [2]. The study is naturally simpler in separable spaces. The research took a giant leap forward in 1990 with the work of David Preiss [9], who established the dense Fréchet differentiability of locally Lipschitz functions on Asplund spaces.

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In this paper we consider the pseudo-regular property which, if it holds at every point for a locally Lipschitz function on an Asplund space, implies that the function is generically strictly differentiable, [7, p.135] and [8, p.107]. On separable Banach spaces locally Lipschitz functions are generically pseudo-regular, which implies that the differentiable points which are not strictly differentiable are first category, [4]. In this paper we attempt a generalisation of this property for the wider class of Asplund generated spaces.

A real valued function ψ on an open subset A of a Banach space X is *locally Lipschitz* if given $x \in A$ there exists $K > 0$ and $\delta > 0$ such that

$$|\psi(y) - \psi(z)| \leq K\|y - z\| \quad \text{for all } y, z \in B(x; \delta).$$

Important tools in the study of the differentiability of such a function ψ are the *Dini directional derivative* at $x \in A$ in the direction $y \in X$,

$$\psi^+(x)(y) := \limsup_{\lambda \to 0^+} \frac{\psi(x + \lambda y) - \psi(x)}{\lambda}$$

and the *Clarke directional derivative* at $x \in A$ in the direction $y \in X$

$$\psi^\circ(x)(y) := \limsup_{\substack{z \to x \\ \lambda \to 0^+}} \frac{\psi(z + \lambda y) - \psi(z)}{\lambda}.$$

We say that ψ is *pseudo-regular at* $x \in A$ *in the direction* $y \in X$ if

$$\psi^\circ(x)(y) = \psi^+(x)(y)$$

and is *pseudo-regular at* $x \in A$ if it is pseudo-regular at x in all directions y. In general, given $y \in X$, ψ is pseudo-regular in the direction y at the points of a residual subset of A and if X is separable then ψ is pseudo-regular at the points of a residual subset of A, [4, p.208]. It is not known whether this property extends in general beyond separable spaces. However we can define a property slightly weaker than pseudo-regularity and show that it holds generically for all locally Lipschitz functions on spaces belonging to a large class which includes the separable spaces.

Now the Clarke directional derivative has useful continuity properties: given $x \in A$, $\psi^\circ(x)(y)$ is sublinear in y and given $y \in X$, $\psi^\circ(x)(y)$ is upper semi-continuous in x. So we are able to define the *Clarke subdifferential* at $x \in A$

$$\partial^\circ \psi(x) := \{f \in X^\star : f(y) \le \psi^\circ(x)(y) \quad \text{for all } y \in X\}$$

a non-empty weak* compact convex subset of $X^\star$ and the *Clarke subdifferential mapping* $x \mapsto \partial^\circ \psi(x)$ a locally bounded weak* upper semi-continuous set-valued mapping. The Dini directional derivative does not have comparable useful continuity properties. We overcome the continuity deficiencies of the Dini directional derivative as follows.

Given $p \in \mathbb{N}$ we define the *approximate Dini directional derivative* at $x \in A$ in the direction $y \in X$ as

$$\begin{cases} \psi_p^+(x)(y) := \sup\limits_{0<\lambda<1/p} \dfrac{\psi(x+\lambda y) - \psi(x)}{\lambda} & \text{for } \|y\| = 1 \\ \\ \psi_p^+(x)(\alpha y) := \alpha \psi_p^+(x)(y) & \alpha \ge 0. \end{cases}$$

We observe that it has useful continuity properties: given $x \in A$, $\psi_p^+(x)(y)$ is continuous in y and given $y \in X$, because $\psi_p^+(x)(y)$ is the supremum of continuous functions, it is lower semi-continuous in x. Further,

$$\psi^+(x)(y) = \lim_{p\to\infty} \psi_p^+(x)(y).$$

For our purposes we consider the associated directional derivative at $x \in A$ in the direction $y \in X$ defined by

$$\psi_p^\square(x)(y) := \psi^\circ(x)(y) - \psi_p^+(x)(y)$$

which for given $y \in X$ is upper semi-continuous in x.

Our result is derived from an analysis of the associated subdifferential of ψ at $x \in A$ defined as

$$\partial_p^\square \psi(x) := \{f \in X^\star : f(y) \le \max\{0, \psi_p^\square(x)(y)\} \quad \text{for all } y \in X\}.$$

It is not difficult to see that this is a weak* compact convex subset of $X^\star$ and that $0 \in \partial_p^\square \psi(x)$ for all $x \in A$. Furthermore, it follows from the continuity properties of $\psi_p^\square(x)(y)$ that the associated subdifferential mapping $x \mapsto \partial_p^\square \psi(x)$ is a locally bounded weak* upper semi-continuous set-valued mapping. We define

$$\partial^\square \psi(x) := \bigcup_{p\in\mathbb{N}} \partial_p^\square \psi(x).$$

Now $0 \in \partial^{\Box}\psi(x)$ for all $x \in A$ and if ψ is pseudo-regular at $x \in A$ then $\partial^{\Box}\psi(x) = \{0\}$. But as the converse does not hold in general, we are led to say that ψ is *virtually pseudo-regular* at $x \in A$ if $\partial^{\Box}\psi(x) = \{0\}$. Virtual pseudo-regularity is a generalisation of pseudo-regularity and simple examples show that it is somewhat weaker.

We now proceed to define the class of spaces on which we will establish our virtual pseudo-regularity property. A real-valued function ψ on an open subset A of a Banach space X is *Gâteaux differentiable* at $x \in A$ if there exists a continuous linear functional $\psi'(x)$ on X and given $\epsilon > 0$ and $y \in X$ there exists $\delta(\epsilon, y) > 0$ such that

$$\left|\frac{\psi(x+\lambda y)-\psi(x)}{\lambda} - \psi'(x)(y)\right| < \epsilon \quad \text{for all } \lambda \in (0,\delta),$$

and is *Fréchet differentiable* at $x \in A$ if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\left|\frac{\psi(x+\lambda y)-\psi(x)}{\lambda} - \psi'(x)(y)\right| < \epsilon \quad \text{for all } \lambda \in (0,\delta)$$

and $y \in X, \|y\| = 1$.

A Banach space Z is an *Asplund (weak Asplund)* space if every continuous convex function on an open convex subset A of Z is Fréchet (Gâteaux) differentiable at the points of a residual subset of A. A Banach space X where there exists an Asplund space Z and a continuous linear mapping $T : Z \to X$ such that $X = \overline{T(Z)}$ is called an *Asplund generated space*. All weakly compactly generated spaces are Asplund generated spaces, [3, p.12]. The class of such spaces is a tractable subclass of the weak Asplund spaces, [3, p.17] and we aim to show that on a space of this class locally Lipschitz functions are generically virtually pseudo-regular.

We use the following structural properties of an Asplund generated space $X = \overline{T(Z)}$. Since T has dense range then the conjugate mapping $T^\star = X^\star \to Z^\star$ is one-to-one. Then $\|.\|_\rho : X^\star \to \mathbb{R}$ defined by

$$\|f\|_\rho = \|T^\star f\| = \sup\{|f(Tz)| : z \in S(Z)\},$$

where $S(Z)$ is the unit sphere in the Asplund space Z, is a norm for X^*.

Given a non-empty bounded subset E of the dual $X^\star$ of a Banach space X, a *weak* slice* of E is a nonempty subset of E of the form

$$S\ell(E,x,\delta) \equiv \{f \in E : f(x) > M - \delta\}$$

where $M \equiv \sup\{f(x) : f \in E\}$ and $\delta > 0$.

Asplund generated spaces inherit useful properties from the Asplund space which generates them.

Proposition 1. [5, Theorem 2.1, p.359] *Consider an Asplund generated space $X = \overline{T(Z)}$. For any nonempty bounded subset E of $X^\star$ there exists a weak* slice determined by Tz for some $z \in Z$, with arbitrarily small $\|.\|_\rho$-diameter.*

A locally Lipschitz function ψ on a non-empty open subset A of an Asplund generated space $X = \overline{T(Z)}$ is said to be $T(Z)$-*differentiable* at $x \in A$ if there exists a continuous linear functional f_x on X and given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\left|\frac{\psi(x+\lambda Tz) - \psi(x)}{\lambda} - f_x(Tz)\right| < \epsilon$$

for all $0 < \lambda < \delta$ and $z \in S(Z)$. We call f_x the $T(Z)$-*derivative* of ψ at x. Since $T(Z)$ is dense in X then f_x is a Gâteaux derivative of ψ at x.

Proposition 2. [5, Theorem 2.2, p.360] *A locally Lipschitz function ψ on a nonempty open subset A of an Asplund generated space $X = \overline{T(Z)}$ is $T(Z)$-differentiable at the points of a dense G_δ subset D of A and for each $x \in A$,*

$$\partial^\circ\psi(x) = \bigcap\{\overline{\text{co}}^{w^\star}\psi'(V \cap D) : V \text{ an open neighbourhood of } x\}.$$

To establish our Theorem we make the following deductions from Proposition 2.

Lemma. *Consider a locally Lipschitz function ψ on a nonempty open subset A of an Asplund generated space $X = \overline{T(Z)}$.*

(i) Given any weak* slice of $\partial^\circ\psi(A)$ there exists $x \in A$ where ψ has $T(Z)$-derivative f_x in that slice.

(ii) Given $\epsilon > 0$ and $p \in \mathbb{N}$, for any $x \in A$ where ψ has $T(Z)$-derivative f_x, there exists an open neighbourhood V of x where $V \subset U$ such that

$$f_x + \partial_p^{\square}\psi(V) \subset \partial^{\circ}\psi(V) + 2\epsilon B_\rho(X^\star),$$

where $B_\rho(X^\star)$ is the closed unit ball of $X^\star$ with the $\|.\|_\rho$-norm.

Proof. (i) Is immediate from Proposition 2.

(ii) There exists $\delta \in (0, 1/p)$ such that

$$\left|\frac{\psi(x+\lambda Tz)-\psi(x)}{\lambda} - f_x(Tz)\right| < \epsilon \text{ for } z \in S(Z) \text{ and } \lambda \in (0,\delta).$$

But then

$$\left|\frac{\psi(y+\lambda Tz)-\psi(y)}{\lambda} - f_x(Tz)\right| \leq \frac{2\|y-x\|}{\lambda} + \left|\frac{\psi(x+\lambda Tz)-\psi(x)}{\lambda} - f_x(Tz)\right| < 2\epsilon$$

for all $y \in V := B\big(x; \frac{\epsilon\delta}{4}\big)$ and all $\lambda \in \left(\frac{\delta}{2}, \delta\right)$ and $z \in S(Z)$. Therefore

$$f_x(Tz) - \psi_p^+(y)(Tz) \leq f_x(Tz) - \sup_{\frac{\delta}{2}<\lambda<\delta} \frac{\psi(y+\lambda Tz)-\psi(y)}{\lambda} \leq 2\epsilon$$

for all $y \in V$ and $z \in S(Z)$.

Then for $f \in f_x + \partial_p^{\square}\psi(y)$ we have

$$\begin{aligned} f(Tz) &\leq f_x(Tz) + \max\left\{0, \psi^{\circ}(y)(Tz) - \psi_p^+(y)(Tz)\right\} \\ &\leq \max\left\{f_x(Tz), \psi^{\circ}(y)(Tz) + f_x(Tz) - \psi_p^+(y)(Tz)\right\} \\ &\leq \max\left\{f_x(Tz), \psi^{\circ}(y)(Tz) + 2\epsilon\right\} \text{ for all } z \in S(Z). \end{aligned}$$

Since $T(Z)$ is dense in X, this implies that

$$f \in \partial^{\circ}\psi(V) + 2\epsilon B_\rho(X^\star). \qquad \square$$

We are now in a position to establish our theorem.

Theorem. *A locally Lipschitz function ψ on an open subset A of an Asplund generated space $X = \overline{T(Z)}$ has the property that $\partial^{\square}\psi(x) = \{0\}$ at the points of a residual subset of A.*

Proof. Consider a nonempty subset U of A and consider ψ defined on U.

By Proposition 1, given $\epsilon > 0$ there exists a weak* slice $S\ell(\partial^\circ\psi(U), Tz, \delta_1)$ for some $z \in S(Z), \delta_1 > 0$ and $M = \sup\{f(Tz) : f \in \partial^\circ\psi(U)\}$, with $\|.\|_\rho$-diam $(S\ell(\partial^\circ\psi(U), Tz, \delta_1)) < \epsilon$. Now consider a weak* slice $S\ell(\partial^\circ\psi(U), Tz, \delta_2)$ where $0 < \delta_2 < \delta_1$.
By Lemma (i), there exists $x \in U$ where ψ has $T(Z)$-derivative $f_x \in S\ell(\partial^\circ\psi(U), Tz, \delta_2)$. Choose $0 < 2r < \min(\epsilon, \delta_1 - \delta_2)$. By Lemma (ii) there exists an open neighbourhood V of x where $V \subset U$ and

$$f_x + \partial_p^{\square}\psi(V) \subset \partial^\circ\psi(V) + 2rB_\rho(X^\star) \subset \partial^\circ\psi(U) + 2rB_\rho(X^\star).$$

Now ψ is pseudo-regular in the direction Tz at the points of a residual subset P_{Tz} of V, [3, p.208]. So for $y \in V \cap P_{Tz}$ we have

$$\psi^\circ(y)(Tz) - \psi_p^+(y)(Tz) \le 0$$

and it follows from the definition of $\partial_p^{\square}\psi(y)$ that

$$f_y(Tz) = 0 \quad \text{ for all } f_y \in \partial_p^{\square}\psi(y).$$

Now $f_x(Tz) > M - \delta_2$ so $(f_x + f_y)(Tz) > M - \delta_2$ for all $f_y \in \partial_p^{\square}\psi(y)$. Since the mapping $x \mapsto \partial_p^{\square}\psi(x)$ is weak* upper semi-continuous there exists an open neighbourhood W of y where $W \subseteq V$ such that $(f_x + f)(Tz) > M - \delta_2$ for all $f \in \partial_p^{\square}\psi(W)$. It follows that

$$(f_x + \partial_p^{\square}\psi(W)) \cap (\partial^\circ\psi(U)\backslash S\ell(\partial^\circ\psi(U), Tz, \delta_1) + 2rB_\rho(X^\star)) = \emptyset$$

so we conclude that

$$(f_x + \partial_p^{\square}\psi(W)) \subset S\ell(\partial^\circ\psi(U), Tz, \delta_1) + 2rB_\rho(X^\star).$$

That is, $\|.\|_\rho$-diam $(\partial_p^{\square}\psi(W)) < 3\epsilon$.

Then given $p \in \mathbb{N}$ and $\epsilon > 0$, the set

$$O_\epsilon^p := \bigcup \big\{\text{open sets } W \subset A \text{ where } \|.\|_\rho^\star\text{-diam } \partial_p^{\square}\psi(W) < \epsilon\big\}$$

is open and dense in A and so $D_p := \cap_\epsilon O_\epsilon^p$ is a dense G_δ subset of A.

We conclude that $\partial^{\square}\psi(x)$ is singleton at the points of a dense G_δ subset $D := \cap_{p\in\mathbb{N}} D_p$ of A. Since $0 \in \partial^{\square}\psi(x)$ for all $x \in A$ we deduce that $\partial^{\square}\psi(x) = \{0\}$ for all $x \in D$. $\square$

It is important to notice that in general

$$\begin{aligned}\partial^{\square}\psi(x) &:= \bigcup_{p\in\mathbb{N}} \partial^{\square}_p\psi(x)\\ &\neq \{f \in X^\star : f(y) \leq \max\{0, \psi^\circ(x)(y) - \psi^+(x)(y)\} \text{ for } y \in X\}.\end{aligned}$$

as is shown by the following example.

Example. Consider $(c_0, \|.\|_\infty)$ with a countable family of countable Schauder bases $\{B_n\}$ where for each $n \in \mathbb{N}$, B_n consists of elements e_{nk} with only n nonzero unit entries. Then $\|e_{nk} - e_{m\ell}\|_\infty = 1$ for all $m, n, k, \ell \in \mathbb{N}$ where $n \neq m$ or $k \neq \ell$.
Define ψ on c_0 by

$$\psi(x) = \max\{0, \frac{1}{n} - (1 + \frac{1}{n})\|\frac{1}{n}e_{nk} - x\|_\infty\} \quad \text{for all } n, k \in \mathbb{N}\}.$$

(A camp of Indian teepees centred around 0.)
Now ψ has Lipschitz constant 2 on c_0.
Given $p \in \mathbb{N}$, $\psi^+(0)(e_{pk}) = 1$ for all $k \in \mathbb{N}$ and

$$\psi'(\frac{1}{p(p+1)}e_{pk})(e_{pk}) = 1 + \frac{1}{p} \quad \text{for all } k \in \mathbb{N}.$$

Since the closed linear span of each B_p generates c_0,

$$\{f \in X^\star : f(y) \leq \max\{0, \psi^\circ(x)(y) - \psi^+_p(x)(y)\} \text{ for } y \in X\} = \{0\}.$$

But ψ is Gâteaux differentiable at 0 with $\psi'(0) = 0$, so $\{f \in X^\star : f(y) \leq \max\{0, \psi^\circ(x)(y) - \psi^+(x)(y)\} \text{ for } y \in X\} = B(X^\star)$.

Of course, since $(c_0, \|.\|_\infty)$ is separable, ψ is pseudo-regular at the points of a residual subset D of c_0. So

$$\{f \in X^\star : f(y) \leq \max\{0, \psi^\circ(x)(y) - \psi^+(x)(y)\} \text{ for } y \in X\} = \{0\}$$

for all $x \in D$, and is therefore equal to $\partial^{\square}\psi(x)$ for all $x \in D$.

This leads us to ask the following question.

Given a locally Lipschitz function ψ on an open subset A of an Asplund generated space $X = \overline{T(Z)}$, is it true that there exists a residual subset D of A such that

$$\{f \in X^\star : f(y) \leq \max\{0, \psi^\circ(x)(y) - \psi^+(x)(y)\} \text{ for } y \in X\} = \{0\}$$

for all $x \in D$?

If so then it would have immediate differentiability consequences.

A locally Lipschitz function ψ on an open subset A of a Banach space X is said to be *strictly differentiable* at $x \in A$ if it is Gâteaux differentiable at x and

$$\lim_{\substack{z \to x \\ \lambda \to 0^+}} \frac{\psi(z + \lambda y) - \psi(z)}{\lambda} = \psi'(x)(y) \quad \text{for all } y \in X.$$

It is well known that for a continuous function on the real line the set of points where it is differentiable but not strictly differentiable is of the first category. But also for a locally Lipschitz function on an open subset of a separable Banach space, the set of points where it is Gâteaux differentiable but not strictly differentiable is of the first category, [4, p.210]. If the stronger result were to hold then we would have an extension of this differentiability result beyond separable spaces.

We wish to thank the referee for his careful scrutiny of our result and for alerting us to the situation which we illustrate in our example above. In so far as they differ, the results outlined here supersede those given in [6], which are in error on some points.

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