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A COMPACTIFICATION OF POLYNOMIAL COALGEBRAS

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Abstract. Coalgebras of polynomial functors on the category of sets have proven useful in theoretical computer science for modelling various types of data structures and state-transition systems. These coalgebras are shown to have an intrinsic topology with respect to which all coalgebraic morphisms are continuous.

The main purpose of the paper is to describe a certain "Stone space like" construction of the *definable enlargement* of a coalgebra, and show that it has a topological characterization reminiscent of the Stone-Čech compactification.

1. Introduction and Summary

Let $T : \mathbf{Set} \to \mathbf{Set}$ be an endofunctor on the category of sets. A *T-coalgebra* $A = (UA, \alpha_A)$ consists of an underlying set UA and a function

$$UA \xrightarrow{\alpha_A} TUA.$$

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A *T-morphism* $f : A \to B$ between two coalgebras is given by a function $Uf : UA \to UB$ for which the following diagram commutes.

$$\begin{array}{ccc} UA & \xrightarrow{Uf} & UB \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ TUA & \xrightarrow{TUf} & TUB \end{array}$$

(we sometimes just write f for Uf).

The notion of a **Set**-based coalgebra has proven useful in modelling data structures such as lists, streams, stacks and trees; transition systems such as automata; and classes in object-oriented programming languages [11, 8]. Generically, the underlying set UA of A is viewed as a set of *states*, and α_A as a *transition structure.* The usefulness of the notion has motivated the development of a theory of "universal coalgebra" [12, 13], by analogy with, and dual to, the study of abstract algebras.

A functor T is called *polynomial* if it can be built from constant-valued functors and the identity functor by forming products, exponential functors with constant exponent (which we will call *power* functors), and coproducts (i.e. disjoint unions). In that case A is a *polynomial coalgebra.* Such coalgebras may be thought of as being constructed, using products, powers and coproducts, from some fixed sets, with the members of these sets being thought of as "observable" elements. Computationally, the states of a coalgebra are regarded as not being directly accessible, but can only be investigated by performing certain "experiments" in the form of coalgebraic operations that yield observable values.

Example 1: An automaton can be viewed as a coalgebra. Let T be a functor that acts on sets X by $TX = X^I \times O^I$, for some fixed sets I (= inputs) and O (= outputs). A T-coalgebra $X \longrightarrow X^I \times O^I$ is a pair of functions

$$X \longrightarrow X^I, \qquad X \longrightarrow O^I.$$

With X being a set of states, this pair can be equivalently viewed as a pair

$$X \times I \longrightarrow X, \qquad X \times I \longrightarrow O,$$

consisting of a state transition function, and an output function, each parameterized by inputs.

Note that this describes a *deterministic* automaton: for a given input to a given state there is a transition to a uniquely determined next state. To model non-deterministic automata would require a transition function of the type $f : X \times I \to \mathcal{P}X$, where $\mathcal{P}$ denotes the power set operation. Then $f(x, i)$ is the set of all possible next states after inputting i at state x.

However the use of the powerset operation takes us outside the class of polynomial functors. An important feature of a polynomial T is that the category of T-coalgebras and T-morphisms has a final (terminal) object, whereas this is not true of the category of coalgebras for the powerset functor (see [13, Sections 9,10]). On the other hand, the *finitary* powerset construction $\mathcal{P}_\omega$ does admit final coalgebras, where $\mathcal{P}_\omega X$ is the set of all finite subsets of X. It would be of interest to determine whether the results of this paper can be extended to coalgebras whose construction involved $\mathcal{P}_\omega X$.

Example 2: [9, Example 6.5] The structure of lists (sequences) can be modelled coalgebraically. Let S^∞ be the set of finite or infinite lists of members of set S. For $\sigma \in S^\infty$, put $\alpha(\sigma) = \#$ if σ is empty, and otherwise $\alpha(\sigma) = (s, \sigma')$ where s is the head (first element) of list σ, and σ' is the rest of the list (tail). This defines a function

$$S^\infty \xrightarrow{\alpha} \{\#\} + (S \times S^\infty),$$

giving a coalgebra for the functor $TX = \{\#\} + (S \times X)$, where $+$ is the disjoint union of sets. In fact this is a *final* coalgebra: for any other T-coalgebra A there is a unique T-morphism $f : A \to (S^\infty, \alpha)$. Given state x_0 of A, if $\alpha_A(x_0) = (s_1, x_1)$, consider $\alpha_A(x_1)$. If $\alpha_A(x_1) = (s_2, x_2)$, consider $\alpha_A(x_2)$, and so on. This process generates a list $f(x_0) = (s_1, s_2, \dots) \in S^\infty$. If $\alpha_A(x_n) = \#$ for some n, then the process terminates and $f(x_0)$ is a finite list. Otherwise it is infinite.

This example illustrates the role of disjoint unions of sets in situations where there are different cases in the definition of a function. Note that we can put $S^\infty = S^* + S^{\mathbb{N}}$, where S^* is the set of finite lists, and $S^{\mathbb{N}}$ the set of infinite ones.

An approach to the model theory of polynomial coalgebras was developed in [6]. This identified a certain class of *observable formulas* as being suitable for specifying classes of coalgebras and characterizing their features. Briefly, an observable formula is a Boolean combination of equations $M_1 \approx M_2$ between terms M_i whose values are observable and state-dependent. Such Boolean combinations are constrained to be either true or false at each state of a coalgebra. They play a role for polynomial coalgebras analogous to that played by ordinary equations in the theory of abstract algebras. In the latter theory there is a famous result of Birkhoff [3] characterizing the equationally definable classes of algebras as being those that are closed under direct products, subalgebras, and homomorphic images. In [5, 4] an analogous structural characterization was obtained for classes of polynomial coalgebras definable by observable formulas. This involved certain "Stone space like" constructions, one of which is the *definable enlargement* A^δ of a coalgebra A.

To describe A^δ, we associate with each formula φ the "truth-set"

$$\varphi^A = \{x : \varphi \text{ is true at state } x \text{ in } A\}.$$

These truth sets form a Boolean set algebra, and the states of A^δ are certain ultrafilters F in this algebra that are *observationally rich*. The richness property requires that for each observable-valued term M there is a single observable value c such that the set

$$(M \approx c)^A = \{x : M \text{ has value } c \text{ at } x\}$$

belongs to F. Intuitively, this means that M takes some constant value on a "large" set of states. A^δ has the remarkable property that in general

$$F \in \varphi^{A^\delta} \quad \text{iff} \quad \varphi^A \in F,$$

which means that φ is true in A^δ at state F iff it is true in A at a set of states that is large in the sense defined by F.

The construction of the transition structure $\alpha^\delta : UA^\delta \to TUA^\delta$ for A^δ and verification of its properties [4] is complex, and involves a modified ultrapower of A [5] that acts as a kind of covering space for A^δ. The purpose of the present paper is to give a characterization of A^δ, using topology, that makes it reminiscent of the Stone-Čech compactification of a topological space.

Any polynomial coalgebra A carries an intrinsic topology, the one with the truth-sets φ^A as base. Any coalgebraic morphism $f : A \to B$ is continuous with respect to the intrinsic topologies on A and B. There is a natural map $\eta_A : A \to A^\delta$ given by

$$\eta_A(x) = \{\varphi^A : x \in \varphi^A\}.$$

η_A is a morphism, hence continuous, and its image is dense in A^δ. η_A is injective iff A is a Hausdorff space.

The definable enlargement A^δ is Hausdorff in its intrinsic topology and satisfies a restricted form of compactness that we call *rich-compactness*. This means that any observationally rich ultrafilter is convergent. Our main result is that η^A is co-universal amongst all T-morphisms from A into rich-compact Hausdorff T-coalgebras, i.e. any morphism $f : A \to B$ with B rich-compact and Hausdorff has a *unique* lifting across η_A to B:

$$\begin{array}{ccc} A & \xrightarrow{\ \eta_A\ } & A^\delta \\ & {\scriptstyle f}\searrow & \vdots\, {\scriptstyle f^\delta} \\ & & B \end{array}$$

This property characterizes A^δ uniquely up to isomorphism, and implies that the inclusion functor from the category of rich-compact Hausdorff T-coalgebras to the category of all T-coalgebras has a left adjoint.

2. Syntax and Semantics

For a fixed set D we write $\bar{D}$ for the constant endofunctor on **Set** that has $\bar{D}X = D$ for all sets X, while $\bar{D}f$ is the identity function on D for all functions f. The identity endofunctor on **Set** will be denoted Id.

We write $X_1 + X_2$ for the coproduct in **Set** of two sets X_1, X_2. This is their disjoint union, and comes equipped with injective *insertion* functions $\iota_j : X_j \to X_1 + X_2$ for $j = 1, 2$. Each element of $X_1 + X_2$ is equal to $\iota_j(x)$ for a unique j and a unique $x \in X_j$.

A functor $T : \mathbf{Set} \to \mathbf{Set}$ is defined to be *polynomial* if it can be obtained in finitely many steps from various constant functors and Id by forming product, coproduct, and power functors with constant exponents. These operations on functors are as follows.

- product functors: $T_1 \times T_2$ acts on sets by $X \mapsto T_1X \times T_2X$, while for a function $f : X \to Y$, $T_1 \times T_2(f)$ is the function

$$T_1(f) \times T_2(f) : T_1X \times T_2X \to T_1Y \times T_2Y$$

  that acts by $(x_1, x_2) \mapsto (T_1(f)(x_1), T_2(f)(x_2))$.
- coproduct functors: $T_1 + T_2$ has $X \mapsto T_1X + T_2X$ on sets; while $T_1 + T_2(f)$ is the function

$$T_1(f) + T_2(f) : T_1X + T_2X \to T_1Y + T_2Y$$

  that acts by $\iota_j(x) \mapsto \iota_j(T_j(f)(x))$.
- power functors: T^D has $X \mapsto (TX)^D$ with constant exponent D, while $T^D(f) : (TX)^D \to (TY)^D$ acts by $g \mapsto T(f) \circ g$.

Any functor formed as part of the construction of T will be called a *component* of T. We will assume from now that

> *T is a particular polynomial functor having at least one constant component $\bar{D}$ for which the set D has more than one element.*

Terms. In general algebra, a term is a symbolic expression built from variables, names of particular elements, and symbols for algebraic operations. A term denotes an element of an algebra in a way that depends on the values assigned to the variables of the term. For example, in the theory of rings, multivariable polynomials are terms.

We associate terms with each component functor S of our functor T. To do so we need a means of associating variables with each component. We fix a set *Var* of variables and define a *context* to be a finite (possible empty) list

$$\Gamma = (v_1 : S_1, \ldots, v_n : S_n)$$

of assignments of components S_i to variables v_i, with the proviso that $v_1, \ldots, v_n$ are all distinct. We may then say that v_i is a variable *of type S in context* Γ.

Axioms

$$\text{(Var)}\quad \frac{v \in Var}{v : S \rhd v : S} \qquad\qquad \text{(Con)}\quad \frac{c \in D}{\emptyset \rhd c : \bar{D}}$$

$$\text{(St)}\quad \frac{}{\emptyset \rhd \mathsf{s} : \mathrm{Id}} \qquad\qquad (T\text{-Tr})\quad \frac{\Gamma \rhd M : \mathrm{Id}}{\Gamma \rhd \mathsf{tr}(M) : T}$$

Weakening

$$\text{(Weak)}\quad \frac{\Gamma, \Gamma' \rhd M : S}{\Gamma, v : S', \Gamma' \rhd M : S} \quad \text{where } v \text{ does not occur in } \Gamma \text{ or } \Gamma'.$$

Product Types

$$\text{(Pair)}\quad \frac{\Gamma \rhd M_1 : S_1 \qquad \Gamma \rhd M_2 : S_2}{\Gamma \rhd \langle M_1, M_2 \rangle : S_1 \times S_2}$$

$$(\text{Proj}_1)\quad \frac{\Gamma \rhd M : S_1 \times S_2}{\Gamma \rhd \pi_1 M : S_1} \qquad\qquad (\text{Proj}_2)\quad \frac{\Gamma \rhd M : S_1 \times S_2}{\Gamma \rhd \pi_2 M : S_2}$$

Coproduct Types

$$(\text{In}_1)\quad \frac{\Gamma \rhd M : S_1}{\Gamma \rhd \iota_1 M : S_1 + S_2} \qquad\qquad (\text{In}_2)\quad \frac{\Gamma \rhd M : S_2}{\Gamma \rhd \iota_2 M : S_1 + S_2}$$

$$\text{(Case)}\ \frac{\Gamma \rhd N : S_1 + S_2 \quad \Gamma, v_1 : S_1 \rhd M_1 : S \quad \Gamma, v_2 : S_2 \rhd M_2 : S}{\Gamma \rhd \mathsf{case}\ N\ \mathsf{of}\ [v_1\ \mathsf{in}\ M_1\ \mathsf{or}\ v_2\ \mathsf{in}\ M_2] : S}$$

Power Types

$$\text{(Abs)}\quad \frac{\Gamma, v : \bar{D} \rhd M : S}{\Gamma \rhd (\lambda v.M) : S^D} \qquad\qquad \text{(App)}\quad \frac{\Gamma \rhd M : S^D \quad \Gamma \rhd N : \bar{D}}{\Gamma \rhd M(N) : S}$$

Figure 1. Axioms and Rules for Generating Terms

A *term-in-context* is an expression of the form

$$\Gamma \rhd M : S,$$

which signifies that M is a *term of type S in context* Γ. This may be abbreviated to $\Gamma \rhd M$ if the type of the term is understood. If S is a constant functor $\bar{D}$, then the term is called *observable*, and D is viewed as a set of observable elements.

Figure 1 gives axioms that legislate certain *base terms* into existence, and rules for generating new terms from given ones. Axiom (Con) states that an observable element is a constant term of its type, while the term s in axiom (St) is a special parameter which will be interpreted as the "current" state in a coalgebra. The symbol *tr* in axiom (T-Tr) is interpreted as the transition structure α_A. The rules for products, coproducts and powers are the standard ones for introduction and transformation of terms of those types.

Our main focus in this paper is on terms whose context is empty. These are called *ground* terms, and have the form $\emptyset \rhd M : S$, which may be abbreviated to $M : S$, or just to the raw term M. Bindings of variables in terms occur in lambda-abstractions and case terms: the v in the consequent of rule (Abs) and the v_j's in the consequent of (Case) are bound in those terms. It is readily shown that in any term $\Gamma \rhd M$, all free variables of M appear in the list Γ. Thus a ground term has no free variables. It may however contain the state parameter s, which behaves as a variable in that it can denote any state of a coalgebra.

In a T-coalgebra A, each term $\Gamma \rhd M : S$ is interpreted as a function

$$[\![\, \Gamma \rhd M \,]\!]_A : UA \times S_1UA \times \cdots \times S_nUA \to SUA.$$

Thus given a state $x \in UA$ and values $a_i \in S_iUA$ for the variables $v_i : S_i$ in Γ, we obtain a value $[\![\, \Gamma \rhd M \,]\!]_A(x, a_1, \dots, a_n)$ in SUA for the term $\Gamma \rhd M$. A ground term M of type S is interpreted as a function $[\![\, M \,]\!]_A : UA \to SUA$.

The formal definition of $[\![\, \Gamma \rhd M \,]\!]_A$ is given by induction on the rules of formation of $\Gamma \rhd M$, and is set out in Section 4 of [6]. Here we give a more informal explanation of the interpretations of various kinds of terms.

- *Variables*: $[\![\, v : S \rhd v \,]\!]$ is the projection $UA \times SUA \to SUA$.
- *Constants*: $[\![\, c : \bar{D} \,]\!] : UA \to D$ is the constant function with value c.
- *State Parameter*: $[\![\, \mathsf{s} : \mathrm{Id} \,]\!]_A$ is the identity function on UA.
- *Transitions*: for M : Id, if M takes as value the state $x \in UA$, then $\mathsf{tr}(M)$ takes the value $\alpha_A(x) \in TUA$. In particular, $[\![\, \mathsf{tr}(\mathsf{s}) \,]\!]_A$ is the transition structure α_A itself.

- *Products*
  (i) *Pairs*: the value of $\langle M_1, M_2 \rangle : S_1 \times S_2$ is the pair (a_1, a_2), where a_j is the value of $M_j : S_j$.
  (ii) *Projections*: the value of $\pi_j M : S_j$ is obtained by applying the projection function $\pi_j : S_1 A \times S_2 A \to S_j A$ to the value of $M : S_1 \times S_2$.
- *Coproducts*
  (i) *Injections*: the value of $\iota_j M : S_1 + S_2$ is obtained by applying the insertion function $\iota_j : S_j A \to S_1 A + S_2 A$ to the value of $M : S_j$.
  (ii) *Cases*: if $N : S_1 + S_2$ has value $\iota_j(a)$ in $S_1 A + S_2 A$, then the value of the term $\mathsf{case}\ N\ \mathsf{of}\ [v_1\ \mathsf{in}\ M_1\ \mathsf{or}\ v_2\ \mathsf{in}\ M_2]$ is the value that $M_j : S$ gets when $v_j : S_j$ is given value a.
- *Powers*
  (i) *Evaluation*: the value of the term $M(N) : S$ is the result of applying the function interpreting $M : S^D$ to the value of $N : \bar{D}$.
  (ii) *Abstraction*: $\lambda v M : S^D$ is interpreted as the function that assigns to an element c of D the value that $M : S$ gets when $v : \bar{D}$ is given value c.

An *equation* is an expression $M_1 \approx M_2$, where M_1 and M_2 are terms of the same type. If M_1 and M_2 are ground terms with $[\![M_1]\!]_A(x) = [\![M_2]\!]_A(x)$, then we say that the equation $M_1 \approx M_2$ is *true at state x in the coalgebra A*. The *truth-set* of the equation in A is the set

$$(M_1 \approx M_2)^A = \{x \in UA : [\![M_1]\!]_A(x) = [\![M_2]\!]_A(x)\}$$

of all states in A at which the equation is true.

We will confine our attention to *ground observable* (GO) equations, whose terms are ground and both of type $\bar{D}$ for some set D. A *formula* is an expression φ constructed from GO equations by the logical connectives $\neg$ and $\wedge$. Truth sets for formulas are defined by inductively putting

$$(\neg\varphi)^A = UA - \varphi^A$$
$$(\varphi_1 \wedge \varphi_2)^A = \varphi_1^A \cap \varphi_2^A,$$

thereby giving the connectives their standard Boolean interpretations. A set of states of the form φ^A may be called *definable*.

The collection

$$\boldsymbol{Def}^A = \{\varphi^A : \varphi \text{ is a formula}\}$$

of all definable sets of A-states is closed under intersections and complements, so forms a Boolean algebra of subsets of UA.

Some examples of terms and formulas, and their interpretations, may be found in [6, 5].

The formulas that we have defined play a role in the theory of polynomial coalgebras comparable to that played by standard equations in the theory of abstract algebras. This claim is justified by a number of results from [6] about ways in which GO terms and formulas characterize properties of coalgebras. We will need some of these results, so we review them here.

A relation $R \subseteq UA \times UB$ is a T-*bisimulation* from A to B if there exists a transition structure $\rho : R \to TR$ on R such that the projections from (R, ρ) to A and B are T-morphisms [1, p. 363]:

$$\begin{array}{ccccc}
UA & \xleftarrow{\ \pi_1\ } & R & \xrightarrow{\ \pi_2\ } & UB \\
\big\downarrow{\scriptstyle \alpha_A} & & \big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \alpha_B} \\
TUA & \xleftarrow[T\pi_1]{} & TR & \xrightarrow[T\pi_2]{} & TUB
\end{array}$$

Now in [6, Corollary 5.3] it is shown that truth of a formula at a state is invariant under bisimulation. But a function $f : UA \to UB$ is a T-morphism from A to B iff its graph

$$R_f = \{(a, f(a)) : a \in UA\}$$

is a bisimulation from A to B [13, Theorem 2.5]: a coalgebraic morphism is essentially a functional bisimulation. Hence [6, Corollary 5.3] yields the following result.

Lemma 2.1. *If $f : A \to B$ is a morphism of T-coalgebras and φ a formula, then*

$$x \in \varphi^A \quad \textit{iff} \quad f(x) \in \varphi^B$$

for any A-state x. Hence $\varphi^A = f^{-1}\varphi^B$.

Coalgebraic morphisms can be characterized as functions preserving the values of certain terms:

Lemma 2.2. [6, Theorem 6.7] *A function* $f : UA \to UB$ *is a* T*-morphism* $A \to B$ *if, and only if, for all ground terms* M*,*

(1) *if* M *is of type* Id*, then* $f(\llbracket M \rrbracket_A(x)) = \llbracket M \rrbracket_B(f(x))$*; and*
(2) *if* M *is of observable type, then* $\llbracket M \rrbracket_A(x) = \llbracket M \rrbracket_B(f(x))$*.*

If M is a ground term of type Id, we write $\varphi[M/\mathsf{s}]$ for the formula that results when M is substituted for the parameter s in φ. The following result shows that $\varphi[M/\mathsf{s}]$ expresses the modal assertion that φ will be true after execution of the state transition $x \mapsto \llbracket M \rrbracket_A(x)$.

Lemma 2.3. [6, Theorem 6.5]

$$\llbracket M \rrbracket_A(x) \in \varphi^A \quad \textit{iff} \quad x \in \varphi[M/\mathsf{s}]^A.$$

Formulas can also be used to characterize the computationally important notion of *bisimilarity* of states. The union of any collection of bisimulations from A to B is a bisimulation [13, Section 5]. Hence there is a largest bisimulation from A to B, which is a relation called *bisimilarity.* We denote this by $\sim$. States x and y are *bisimilar*, $x \sim y$, when xRy for some bisimulation R between A and B. This is intended to capture the notion that x and y are "observationally indistinguishable". Bisimilar states are characterized by the properties of assigning the same values to all GO terms, and of satisfying the same formulas:

Lemma 2.4. [6, Theorem 7.2] *If* A *and* B *are* T*-coalgebras, then for any states* x *in* A *and* y *in* B*, the following are equivalent:*

(1) x *and* y *are bisimilar:* $x \sim y$*.*
(2) $\llbracket M \rrbracket_A(x) = \llbracket M \rrbracket_B(y)$ *for all GO terms* M*.*
(3) $x \in \varphi^A$ *iff* $y \in \varphi^B$ *for all formulas* φ*.*

3. Definable Enlargements

A coalgebra A is a *model* of formula φ if φ is true at every state of A, i.e. if $\varphi^A = UA$. It has been shown in [5, 4] that a class K of coalgebras is the class of all models of some set of formulas iff K is closed under disjoint unions (coproducts), domains and images of coalgebraic morphisms, and *definable enlargements.* This is the analogue, mentioned in the Introduction, of Birkhoff's theorem on equational classes. In this section we discuss the nature of the definable enlargement A^δ of a T-coalgebra A.

We will be dealing with filters and ultrafilters in various Boolean set algebras, and it is necessary to be particularly sensitive about which algebra is intended. If $\mathcal{B}$ is a non-empty collection of subsets of some set X, and $\mathcal{B}$ is closed under intersections and under set complements relative to X, then $\mathcal{B}$ is a subalgebra of the powerset Boolean algebra $\mathcal{P}(X)$ of all subsets of X. A non-empty set $F \subseteq \mathcal{B}$ is a *filter in* $\mathcal{B}$ if $\emptyset \notin F$, F is closed under intersections, and F is closed under supersets in $\mathcal{B}$. The latter condition means that

$$\text{if } Y \in F \text{ and } Y \subseteq Z \in \mathcal{B}, \text{ then } Z \in F.$$

Informally F is thought of as a collection of "large" subsets of X, and we may call a set "F-large" if it belongs to F. F is an *ultrafilter in* $\mathcal{B}$ if exactly one of Y and $X - Y$ belongs to F for all $Y \in \mathcal{B}$, or equivalently, if it is a maximal filter in $\mathcal{B}$.

If $\mathcal{B} = \mathcal{P}(X)$, then an (ultra)filter in $\mathcal{B}$ may be called an *(ultra)filter on the set* X. Note that when $\mathcal{B} \neq \mathcal{P}(X)$ an (ultra)filter in $\mathcal{B}$ need not be an (ultra)filter on X, since the superset or complementation closure condition may be violated by some sets in $\mathcal{P}(X) - \mathcal{B}$.

Our focus is on ultrafilters $F \subseteq \boldsymbol{Def}^A$ in the Boolean algebra $\boldsymbol{Def}^A$ of definable subsets of the state set UA of a coalgebra A. Such an F is a collection of definable sets that contains $(\varphi \wedge \psi)^A$ whenever it contains φ^A and ψ^A; contains ψ^A whenever it contains some $\varphi^A \subseteq \psi^A$; and contains, for each formula φ, exactly one of φ^A and $(\neg\varphi)^A$. F will be called *observationally rich* (or just *rich*) if for each GO term $M : \bar{D}$ there exists some $c \in D$ such that

$$(M \approx c)^A = \{x \in UA : [\![\, M \,]\!]_A(x) = c\} \in F,$$

so M has a constant value on an F-large set. For instance, this condition holds when D is finite: if $D = \{c_1, \ldots, c_n\}$, then since

$$(M \approx c_1)^A \cup \cdots \cup (M \approx c_n)^A = UA \in F,$$

and F is an ultrafilter, we must have $(M \approx c_j)^A \in F$ for some $j \leq n$.

Observationally rich ultrafilters arise naturally from consideration of whether a state is determined by the set of all formulas that are true at that state. For each state x of A, let

$$\eta_A(x) = \{\varphi^A : x \in \varphi^A\}$$

be the collection of all truth sets of formulas true at x. Then $\eta_A(x)$ is an observationally rich ultrafilter in $\boldsymbol{Def}^A$: for each GO term M we have $(M \approx c)^A \in \eta_A(x)$ where $c = [\![M]\!]_A(x)$.

The state set UA^δ of the definable enlargement A^δ is the set of all rich ultrafilters in $\boldsymbol{Def}^A$. The definition of the transition structure

$$UA^\delta \xrightarrow{\alpha_{A^\delta}} TUA^\delta$$

is complicated and depends on the components involved in the formation of T. This is one reason why we have sought in this paper to use topology and category theory to give a characterisation of A^δ that relates it naturally to other familiar mathematical constructions.

The function $x \mapsto \eta_A(x)$ is a T-morphism $\eta_A : A \to A^\delta$. The question of when it is injective has an interesting connection with *bisimilarity* considered as the largest bisimulation from A to itself. If x and y are states of A, then the equivalence of parts (1) and (3) of Lemma 2.4, with $A = B$, asserts that

$$x \sim y \quad \text{iff} \quad \eta_A(x) = \eta_A(y).$$

Thus if η_A is injective, then bisimilar states in A are equal. For such a coalgebra this gives rise to the important principle of *coinduction* [10] which states that to prove $x = y$ it is enough to show that there is some bisimulation on A that relates x to y. A topological criterion for injectivity of η_A is given in Theorem 4.1 below.

In general η_A will not be surjective: there will be rich ultrafilters F that are not of the form $\eta_A(x)$. Such an F can be thought of as a state of A^δ having a logical specification that is consistent with the structure of A but not actually realized in A.

The analysis of A^δ is facilitated by a certain *ultrapower* construction developed in detail in [5]. If $\mathcal{U}$ is an ultrafilter on a set I, then for any set X there is an equivalence relation $=_\mathcal{U}$ on the I-th power X^I of X defined by

$$f =_\mathcal{U} g \text{ iff } \{i \in I : f(i) = g(i)\} \in \mathcal{U}.$$

Each $f \in X^I$ has the equivalence class $f^\mathcal{U} = \{g \in X^I : f =_\mathcal{U} g\}$ and the quotient set

$$X^\mathcal{U} = \{f^\mathcal{U} : f \in X^I\}$$

is called the *ultrapower of X with respect to $\mathcal{U}$*.

Now the transition structure $\alpha_A : UA \to TUA$ lifts to the function

$$(UA)^{\mathcal{U}} \xrightarrow{\alpha_A{}^{\mathcal{U}}} (TUA)^{\mathcal{U}}$$

given by $\alpha_A{}^{\mathcal{U}}(f^{\mathcal{U}}) = (\alpha_A \circ f)^{\mathcal{U}}$. This does not define a T-coalgebra on $(UA)^{\mathcal{U}}$, since for that we need a function of the form $(UA)^{\mathcal{U}} \to T((UA)^{\mathcal{U}})$. This problem can be overcome by removing certain elements from $(UA)^{\mathcal{U}}$. Define $f^{\mathcal{U}} \in (UA)^{\mathcal{U}}$ to be *observable* if for each GO term $M : \bar{D}$ there is an element $c \in D$ such that the set $\{i \in I : [\![M]\!]_A(f(i)) = c\}$ belongs to $\mathcal{U}$. If UA^+ is the set of observable elements of $(UA)^{\mathcal{U}}$, then a transition structure

$$\alpha_A{}^+ : UA^+ \to TUA^+$$

can be constructed, defining a T-coalgebra A^+ that we call the *observational ultrapower of A with respect to $\mathcal{U}$*. Its fundamental feature is that for any formula φ and any $f^{\mathcal{U}} \in UA^+$,

$$f^{\mathcal{U}} \in \varphi^{A^+} \text{ if, and only if, } \{i \in I : f(i) \in \varphi^A\} \in \mathcal{U}$$

[5, Theorem 5.2]. Intuitively this means that φ is true in A^+ at state $f^{\mathcal{U}}$ iff it is true in A at state $f(i)$ for a $\mathcal{U}$-large set of elements $i \in I$. This property is a version of Łoś's Theorem in the standard theory of ultrapowers for models of first-order logic [2, Section 5.2].

When $x = f^{\mathcal{U}}$ and $C \subseteq UA$, we will write $x \in_{\mathcal{U}} C$ to mean that

$$\{i \in I : f(i) \in C\} \in \mathcal{U}.$$

By choosing a suitable ultrafilter $\mathcal{U}$, it can be arranged that the following condition holds:

> *any collection $\mathcal{S}$ of subsets of UA with the finite intersection property has a "nonstandard element in its intersection". This element is an $x \in (UA)^{\mathcal{U}}$ such that for each $C \in \mathcal{S}$, $x \in_{\mathcal{U}} C$.*

When this holds we will say that $\mathcal{U}$ is *enlarging over UA*. There is a well-known construction in the theory of ultrapowers of a suitable enlarging $\mathcal{U}$ over any given set. This is explained in [5, Section 6].

As the reader may suspect, there is a close connection between the two notions of an observable element of $A^{\mathcal{U}}$ and a rich ultrafilter in $\boldsymbol{Def}^A$. To clarify this, for each $x \in UA^+$ put

$$\theta_A(x) = \{\varphi^A : x \in_{\mathcal{U}} \varphi^A\}.$$

$\theta_A(x)$ is an observationally rich ultrafilter, and the function θ_A is a T-morphism $A^+ \to A^\delta$. When $\mathcal{U}$ is enlarging, $\theta_A : UA^+ \to UA^\delta$ is surjective, because if $F \in UA^\delta$ and $x \in_{\mathcal{U}} \varphi^A$ for all $\varphi^A \in F$, then x will be observable and $\theta_A(x) = F$.

Notice that the condition

$$\{i \in I : f(i) \in \varphi^A\} \in \mathcal{U}$$

means that $f^{\mathcal{U}} \in_{\mathcal{U}} \varphi^A$ by definition of $\in_U$, so the above version of Łoś's Theorem can be expressed succinctly as

$$f^{\mathcal{U}} \in \varphi^{A^+} \text{ if, and only if, } f^{\mathcal{U}} \in_{\mathcal{U}} \varphi^A.$$

Lemma 3.1. *For any $F \in UA^\delta$ and any formula φ,*

$$F \in \varphi^{A^\delta} \textit{ if, and only if, } \varphi^A \in F.$$

Proof. Let A^+ be the observational ultrapower of A with respect to an enlarging $\mathcal{U}$. Then there is some $x \in UA^+$ with $F = \theta_A(x)$. By definition, $\varphi^A \in \theta_A(x)$ iff $x \in_{\mathcal{U}} \varphi^A$, which, by the last observation above, holds iff $x \in \varphi^{A^+}$. But θ_A is a T-morphism, so by Lemma 2.1, $x \in \varphi^{A^+}$ iff $\theta_A(x) \in \varphi^{A^\delta}$. □

We will also need the following result about the behaviour of state transitions $F \mapsto [\![M]\!]_{A^\delta}(F)$ in A^δ induced by a term M of type Id (recall the notation $\varphi[M/\mathsf{s}]$ defined just prior to Lemma 2.3).

Lemma 3.2. *If M is a ground term of type* Id*, then for any $F \in UA^\delta$ and any formula φ,*

$$\varphi^A \in [\![M]\!]_{A^\delta}(F) \textit{ if, and only if, } \varphi[M/\mathsf{s}]^A \in F.$$

Proof.

$$\begin{aligned} & \varphi^A \in [\![M]\!]_{A^\delta}(F) \\ \text{iff } & [\![M]\!]_{A^\delta}(F) \in \varphi^{A^\delta} && \text{by Lemma 3.1} \\ \text{iff } & F \in \varphi[M/\mathsf{s}]^{A^\delta} && \text{by Lemma 2.3} \\ \text{iff } & \varphi[M/\mathsf{s}]^A \in F && \text{by Lemma 3.1.} \end{aligned}$$

□

4. The Definable Topology

Any polynomial coalgebra, or more precisely the state set UA of any such coalgebra, may be viewed as a topological space in a natural way. The set $\boldsymbol{Def}^A$ of definable sets of states of A is closed under intersections and set complements, so forms a base for a topology on UA in which each definable set φ^A is *clopen*. We refer to this as the *definable topology*.

A coalgebra may not be a Hausdorff space. If x and y are two states that belong to the same basic open sets φ^A, then by Lemma 2.4 they must be bisimilar, i.e. $x \sim y$ when bisimilarity is taken as a relation from A to A. Thus the Hausdorff separation property amounts to the requirement that bisimilar states be equal: $x \sim y$ implies $x = y$. But it is known that this condition forces A to be a *simple* coalgebra in the sense of having no proper quotients, i.e. every epimorphism $A \to B$ with domain A is an isomorphism [13, Section 8].

Hausdorff separation can also be characterized by reference to the morphism $\eta_A : A \to A^\delta$. By definition, $\eta_A(x) = \eta_A(y)$ iff x and y belong to the same basic open sets, so A is Hausdorff iff $\eta_A(x) = \eta_A(y)$ implies $x = y$. To summarize, we have shown the following:

Theorem 4.1. *For any T-coalgebra A, the following are equivalent.*

(1) *A is Hausdorff in the definable topology.*
(2) *The T-morphism $\eta_A : A \to A^\delta$ is injective.*
(3) *Every T-epimorphism with domain A is an isomorphism.*

Any T-morphism $f : A \to B$ is continuous, since $f^{-1}\varphi^B = \varphi^A$ for all formulas φ (Lemma 2.1), so the inverse of any basic open set in B is open in A. In particular, η_A is continuous.

Theorem 4.2. *The definable enlargement A^δ of any T-coalgebra A is Hausdorff, and the image of η_A is topologically dense in A^δ.*

Proof. For the Hausdorff property, let F and G be distinct states of A^δ. Then there is some definable set φ^A that belongs to one of them, say F, and not the other. Then by Lemma 3.1, $F \in \varphi^{A^\delta}$ while $G \notin \varphi^{A^\delta}$, hence $G \in (\neg\varphi)^{A^\delta}$. Thus φ^{A^δ} and $(\neg\varphi)^{A^\delta}$ are disjoint open sets in A^δ separating F and G.

For density of $\eta_A(UA)$ in UA^δ, let F be any point in UA^δ, and φ^{A^δ} any basic open neighbourhood of F. Then $\varphi^A \in F$ by Lemma 3.1, so $\varphi^A \neq \emptyset$. Choose $x \in \varphi^A$. Then $\varphi^A \in \eta_A(x)$, so $\eta_A(x) \in \varphi^{A^\delta}$ by Lemma 3.1 again, showing that $\varphi^{A^\delta} \cap \eta_A(UA) \neq \emptyset$. This proves that F is in the closure of $\eta_A(UA)$. □

Now we consider criteria for compactness of a coalgebra. Recall that a filter F on a space X is said to *converge* to a point x if it includes the filter of all neighbourhoods of x. In the case that X is the state space UA of a coalgebra A, this is equivalent to requiring that $\eta_A(x) \subseteq F$, since $\eta_A(x)$ is the set of basic open neighbourhoods of x. Correspondingly, if F is a filter in $\boldsymbol{Def}^A$, we will say that F converges to x if $\eta_A(x) \subseteq F$. But now this is equivalent to having $\eta_A(x) = F$, since $\eta_A(x)$ is a maximal filter in $\boldsymbol{Def}^A$.

In general, a space is compact iff every ultrafilter on that space converges. Using this we show:

Theorem 4.3. *A T-coalgebra A is compact if, and only if, every ultrafilter in $\boldsymbol{Def}^A$ converges.*

Proof. Suppose A is compact. If F is an ultrafilter in $\boldsymbol{Def}^A$, then F extends to an ultrafilter G on the set UA. By compactness, G converges to some point x, so $\eta_A(x) \subseteq G$. But then F converges to x, for if there did exist a set φ^A in $\eta_A(x) - F$ we would get $(\neg\varphi)^A \in F \subseteq G$, contradicting the fact that $\varphi^A \in G$.

For the converse, let G be any ultrafilter on UA. Then $F = G \cap \boldsymbol{Def}^A$ is an ultrafilter in $\boldsymbol{Def}^A$. If F converges to x, then $\eta_A(x) \subseteq G$, so G converges to x. This shows that all ultrafilters on UA converge, proving compactness. □

The last result suggests a weaker notion of compactness: a coalgebra A will be called *rich-compact* if every observationally rich ultrafilter in $\boldsymbol{Def}^A$ converges. This means that each member F of UA^δ is equal to $\eta_A(x)$ for some $x \in UA$, i.e. that η_A is surjective. Therefore if A is rich-compact and Hausdorff, η_A is a bijective T-morphism, and hence an isomorphism between A and A^δ.

Theorem 4.4. *For any T-coalgebra A, the definable enlargement A^δ is rich-compact.*

Proof. Let $G \subseteq \boldsymbol{Def}^{A^\delta}$ be an observationally rich ultrafilter in $\boldsymbol{Def}^{A^\delta}$. We need to show that G converges in A^δ. Put

$$F_G = \{\varphi^A : \varphi^{A^\delta} \in G\}.$$

We show that $F_G \in UA^\delta$, i.e. F_G is an observationally rich ultrafilter in $\boldsymbol{Def}^A$.

Note first that if $\varphi^A = \psi^A$, then by Lemma 3.1 $\varphi^{A^\delta} = \psi^{A^\delta}$. Hence the definition of F_G does not depend on how its members are named. More generally, if $\varphi^A \subseteq \psi^A$, then $\varphi^{A^\delta} \subseteq \psi^{A^\delta}$, so closure of G under supersets in $\boldsymbol{Def}^{A^\delta}$ implies closure of F_G under supersets in $\boldsymbol{Def}^A$. If $\varphi^A, \psi^A \in F_G$, then $(\phi \wedge \psi)^{A^\delta} = \phi^{A^\delta} \cap \psi^{A^\delta} \in G$, and so $\phi^A \cap \psi^A = (\phi \wedge \psi)^A \in F_G$. For each φ, one of φ^{A^δ} and $(\neg\phi)^{A^\delta}$ is in G, so one of φ^A and $(\neg\phi)^A$ is in F_G. Thus F_G is an ultrafilter in $\boldsymbol{Def}^A$. If M is a ground term of type $\bar{D}$, then there is some $c \in D$ with $(M \approx c)^{A^\delta} \in G$, hence $(M \approx c)^A \in F_G$, so F_G is observationally rich as desired.

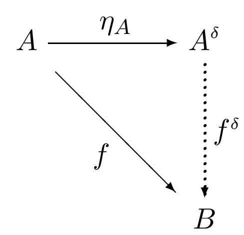
Now we see that if $\varphi^{A^\delta} \in G$, then since $\varphi^A \in F_G$ we get $F_G \in \varphi^{A^\delta}$ by Lemma 3.1. This shows that G is included in the set $\eta_{A^\delta}(F_G)$ of basic open neighbourhoods of F_G in A^δ. Hence $G = \eta_{A^\delta}(F_G)$ by maximality of G, showing that G converges to F_G. □

It has now been established that A^δ is both rich-compact and Hausdorff, and therefore is isomorphic to its own definable enlargement $A^{\delta\delta}$.

5. Characterizing A^δ

Here now is the main result of this paper.

Theorem 5.1. *For any T-coalgebra A and any T-morphism $f : A \to B$ with B rich-compact and Hausdorff, there exists a unique T-morphism $f^\delta : A^\delta \to B$ such that the following diagram commutes.*



Proof. To define f^δ, for each $F \in UA^\delta$ put

$$G_F = \{\varphi^B : f^{-1}\varphi^B \in F\}.$$

Since f is a morphism, $f^{-1}\varphi^B = \varphi^A$ (Lemma 2.1), so

$$G_F = \{\varphi^B : \varphi^A \in F\}.$$

Now G_F is an observationally rich ultrafilter in $\boldsymbol{Def}^B$. That it is an ultrafilter is essentially standard theory using that fact that f^{-1} acts as a Boolean homomorphism from $\boldsymbol{Def}^B$ to $\boldsymbol{Def}^A$. For richness, observe that for each GO term $M : \bar{D}$ there is some $c \in D$ with $(M \approx c)^A \in F$, and hence $(M \approx c)^B \in G_F$.

Since B is rich-compact it follows that there is some $y \in UB$ with $G_F = \eta_B(y)$. Since B is Hausdorff there is only one such y, so we can put $f^\delta(F) = y$. Thus $f^\delta(F)$ is determined uniquely by the requirement that for any formula φ,

$$f^\delta(F) \in \varphi^B \text{ if, and only if, } \varphi^A \in F.$$

Now for each $x \in UA$, since $f^{-1}\varphi^B = \varphi^A$ we get

$$f(x) \in \varphi^B \text{ iff } x \in \varphi^A \text{ iff } \varphi^A \in \eta_A(x).$$

From above this shows that $f(x)$ must be $f^\delta(\eta_A(x))$, so the diagram in the Theorem commutes.

Our main task is to show that f^δ is a T-morphism, and this is done by showing that it satisfies the two conditions of Lemma 2.2 with respect to a ground term M. First, if M is of type Id, we need to show that

$$f^\delta([\![M]\!]_{A^\delta}(F)) = [\![M]\!]_B(f^\delta(F))$$

for any $F \in UA^\delta$. But for any formula φ,

$$\begin{array}{ll} f^\delta([\![M]\!]_{A^\delta}(F)) \in \varphi^B & \\ \text{iff } \varphi^A \in [\![M]\!]_{A^\delta}(F) & \text{by definition of } f^\delta \\ \text{iff } \varphi[M/\mathsf{s}]^A \in F & \text{by Lemma 3.2} \\ \text{iff } f^\delta(F) \in \varphi[M/\mathsf{s}]^B & \text{by definition of } f^\delta \\ \text{iff } [\![M]\!]_B(f^\delta(F)) \in \varphi^B & \text{by Lemma 2.3.} \end{array}$$

This shows that $f^\delta(\llbracket M \rrbracket_{A^\delta}(F))$ and $\llbracket M \rrbracket_B(f^\delta(F))$ belong to the same basic open sets φ^B in B, and so must be equal as B is Hausdorff.

Secondly, if $M : \bar{D}$ we must show that

$$\llbracket M \rrbracket_{A^\delta}(F) = \llbracket M \rrbracket_B(f^\delta(F)).$$

Let $c = \llbracket M \rrbracket_B(f^\delta(F)) \in D$. Then $f^\delta(F) \in (M \approx c)^B$, so $(M \approx c)^A \in F$ by definition of f^δ. Hence $F \in (M \approx c)^{A^\delta}$ by Lemma 3.1, and so $\llbracket M \rrbracket_{A^\delta}(F) = c = \llbracket M \rrbracket_B(f^\delta(F))$.

That completes the proof that f^δ is a T-morphism. Finally, to show that f^δ is uniquely determined by f, let $g : A^\delta \to B$ be any T-morphism having $g \circ \eta_A = f$. Then for any $F \in UA^\delta$, we have $g(F) \in \varphi^B$ iff $F \in \varphi^{A^\delta}$ by Lemma 2.1 as g is a morphism, so

$$g(F) \in \varphi^B \text{ iff } \varphi^A \in F \text{ iff } f^\delta(F) \in \varphi^B.$$

Thus $g(F)$ and $f^\delta(F)$ are inseparable by basic open sets in B, and hence are equal because B is Hausdorff. Therefore $g = f^\delta$. □

The property stated in Theorem 5.1 characterizes the pair (A^δ, η_A) uniquely up to unique isomorphism. If another pair (A_0, η_0) fulfils Theorem 5.1 in place of (A^δ, η_A), with $\eta_0 : A \to A_0$ a T-morphism and A_0 rich-compact and Hausdorff, then there must exist unique T-morphisms

$$A^\delta \xrightarrow{\eta_1} A_0, \qquad A_0 \xrightarrow{\eta_2} A^\delta$$

that are mutually inverse, with $\eta_1 \circ \eta_A = \eta_0$ and $\eta_2 \circ \eta_0 = \eta_A$.

Let **Coal**(T) be the category of T-coalgebras together with their T-morphisms, and let **RHCoal**(T) be its full subcategory of rich-compact Hausdorff coalgebras. Then Theorem 5.1 states that for each T-coalgebra A, the pair (A^δ, η_A) is free over A with respect to the inclusion functor

$$\mathbf{Coal}(T) \hookrightarrow \mathbf{RHCoal}(T).$$

The assignment $A \mapsto A^\delta$ gives rise to a functor **RHCoal**$(T) \longrightarrow$ **Coal**(T) that is left adjoint to this inclusion. Thus, in category-theoretic terms, **RHCoal**(T) is a reflective subcategory of **Coal**(T) [7, Chapter X].

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