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THE TOPOLOGY OF COMPACT GROUPS

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Abstract. This paper is a slightly expanded version of a plenary addresss at the 2002 Summer Topology Conference in Auckland, New Zealand. The aim in this paper is not to survey the vast literature on compact groups, or even the topological group structure of compact groups which is described in considerable detail in the 1998 book "The Structure of Compact Groups: A Primer for the Student – A Handbook for the Expert" by Karl Heinrich Hofmann and this author. Rather, the much more modest purpose in this article is to focus on point-set topology and, in a gentle fashion, describe the topological structure of compact groups, most of which can be extracted or derived from that large book.

1. Fundamental Facts

Portions of the material presented here can be found in various papers and books. The book ([3],[4]) by Karl Heinrich Hofmann and the author is a comfortable reference. However, this is not meant to imply that all the results in [3] were first proved by Hofmann and/or Morris, although often the presentation and approach in the book are new.

Definition 1.1. Let G be a group with a topology τ. Then G is said to be a *topological group* if the maps $G \to G$ given by $g \mapsto g^{-1}$ and $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1.g_2$ are continuous.

Example 1.2. Let G be any group with the discrete topology. Then G is a topological group. In particular, if G is any finite group with the discrete topology then G is a compact topological group.

Terminology. The term *compact group* will be used as a shorthand for compact Hausdorff topological group.

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Notation. If G is any topological group, then $|G|$ will be used to denote the underlying topological space of G.

Definition 1.3. A topological space X is said to be *homogeneous* if for each ordered pair (x, y) of points from X, there exists a homeomorphism ϕ of X into itself such that $\phi(x) = y$.

Proposition 1.4. *If G is any topological group, then the topological space $|G|$ is homogeneous.*

Proof. Let (x, y) be any ordered pair of points from G. Then the mapping $g \mapsto yx^{-1}g$, for each $g \in G$, is a homeomorphism of G onto itself which maps x onto y. □

Example 1.5. The compact unit interval $[0, 1]$ is not the underlying space for any topological group as it is not homogeneous. Indeed, for any positive integer n, the closed unit ball in $\mathbb{R}^n$ is not the underlying space for any topological group.

Definition 1.6. Let $\mathbb{T}$ denote the multiplicative group of complex numbers of modulus 1, that is the unit circle in the complex plane, with the topology it inherits from the plane. Then it is readily verified that $\mathbb{T}$ is a compact group. (Of course $\mathbb{T}$ is $\mathbb{S}^1$.) The topological group $\mathbb{T}$ is called the *circle group.*

The circle group is certainly the most important abelian compact group.

It is reasonable to ask if the unit sphere in each $\mathbb{R}^n$ is the underlying space of a compact group.

Firstly we observe that the multiplicative group $\mathbb{H}$ of quaternions with the usual euclidean topology is a topological group. And the subgroup $\{q : q \in \mathbb{H},\ |q| = 1\}$ with the subspace topology is a compact group and its underlying topological space is the sphere $\mathbb{S}^3$ and is called the *3-sphere group.* Theorem 6.95 of [3] contains the statement that

$\mathbb{T}$ *and the 3-sphere group are the only topological groups whose underlying spaces are spheres.*

The proof is far less elementary than one might expect. It depends in particular on cohomology theory and the structure theory of compact connected Lie groups.

In conclusion on this point we mention that the 3-sphere group is also known as SU(2), but more on that later.

At this stage we are still in the business of finding examples of compact groups. Some general constructions are helpful. But before this, some observations on the separation properties of topological groups are desirable.

Notation. let us denote the identity element in a topological group by **1**.

Proposition 1.7.

(i) *Let G be a topological group. Then the conditions (a), (b), (c) and (d) below are equivalent:*
   (a) *G is a T_0-space;*
   (b) *$\{1\}$ is a closed subset of G;*
   (c) *G is a T_1-space;*
   (d) *G is a Hausdorff or T_2-space.*

(ii) *Every topological group is a regular space. So every T_0 topological group is a T_3-space.*

(iii) *Every topological group is a completely regular space. So every T_0 topological group is a $T_{3_{1/2}}$-space.*

(iv) *There exist Hausdorff topological groups which are not normal spaces.*

Since our business is compact groups, not general topological groups, we shall not prove these standard results here. A.A. Markov [6] introduced free topological groups to prove (iv) in the above Proposition.

Now we turn to operations which allow us to find more compact groups.

Example 1.8.

(i) Let G be a topological group and H a subgroup of G with the subspace topology. Then H is a topological group. Further, if G is a compact group and H is a closed subgroup, then H is a compact group; that is, *every closed subgroup of a compact group is a compact group.*

(ii) Let N be a normal subgroup of a topological group G. Then the quotient group G/N with the quotient topology is a topological group. The topological group G/N is a Hausdorff topological group if and only if N is a closed subgroup. Further, if G is a compact group and N is a closed normal subgroup, then G/N is a compact group, that is, *every Hausdorff quotient group of a compact group is a compact group.*

(iii) If $\{G_i,\ i \in I\}$, is a family of topological groups for some index set I, then the product group $\prod_{i\in I} G_i$ with the Tychonoff product topology is a topological group. Further if each G_i is a compact group, then $\prod_{i\in I} G_i$ is a compact group, that is, *every product of compact groups is a compact group.*

We are now able to produce a large family of infinite compact groups.

Example 1.9. If $\{G_i,\, i \in I\}$, is any family of finite discrete topological groups, for any index set I, then $G = \prod_{i\in I} G_i$ is a compact group. In fact, G is a totally disconnected compact group.

The underlying space of a countably infinite product of non-trivial finite topological groups is homeomorphic to the Cantor space, and more generally an infinite product of non-trivial finite topological groups is homeomorphic to a *Cantor cube*, that is an infinite product of 2-point discrete spaces.

The surprising result in this area is:

Theorem 1.10. *The underlying topological space of every infinite totally disconnected compact group is a Cantor cube.*

This result is again not entirely trivial, though it can be obtained by elementary means as well as by an application of the Countable Layer Theorem in the structure theory of compact groups. (See [5].)
The important thing to note about the above theorem is that it ends the study of the *topology* of totally disconnected compact groups.

The topology of a totally disconnected compact group G is completely determined by the cardinality of G. Further, the only possible cardinalities of infinite totally disconnected compact groups are 2^m, for some infinite cardinal number m, and all such cardinalities do in fact occur.
This does not end the study of the topological group structure of totally disconnected compact groups - as evidenced by recent books on profinite groups. (See [10], [9])

2. Connectivity in Compact Groups

We shall see that Theorem 1.10 does more than describe the topology of a special class of compact groups.

Definition 2.1. If G is any topological group, then the smallest connected set containing 1 is said to be the *identity component* of G and is denoted by $\mathbf{G_0}$.

It is readily verified that
for any topological group G, the identity component G_0 is a closed normal subgroup of G. Further, if G is a compact group, then G_0 is a compact group too.

The following proposition is almost obvious.

Proposition 2.2. *Let G be a topological group. Then the quotient group G/G_0 is a totally disconnected topological group. Further, if G is a compact group, then G/G_0 is a totally disconnected compact group.*

Now we state a powerful result which significantly reduces the task of describing the topology of a general compact group.

Theorem 2.3. *(*[3]*, 10.40) Let G be any compact group and H a closed subgroup which contains G_0. Then G contains a compact totally disconnected subspace D such that $(g, d) \mapsto gd : H \times D \to G$ is a homeomorphism. In particular, if G is any compact group then it is homeomorphic to the product group $G_0 \times G/G_0$.*

Theorem 1.10 and Theorem 2.3 together reduce the study of the topology of compact groups to the study of the topology of connected compact groups.

So let us look at a rich source of examples of connected compact groups.

Example 2.4. The multiplicative group of all nonsingular $n \times n$ matrices with complex number entries is called the *general linear group over* $\mathbb{C}$ and is denoted by $\mathrm{GL}(n, \mathbb{C})$; the subgroup of matrices of determinant 1 is called the *special linear group over* $\mathbb{C}$ and is denoted by $\mathrm{SL}(n, \mathbb{C})$. The *unitary group* $\mathrm{U}(n)$ consists of those matrices A in $\mathrm{GL}(n, \mathbb{C})$ with $A = (a_{jk})$ and $A^{-1} = (b_{jk})$, where b_{jk} is the complex conjugate of a_{kj}, $1 \leq j, k \leq n$. The *orthogonal group* $\mathrm{O}(n)$ is those matrices A in $\mathrm{GL}(n, \mathbb{C})$ with $A = (a_{jk})$, where $A^{-1} = (c_{jk})$ for $c_{jk} = a_{kj}$, $1 \leq j, k \leq n$. The *special unitary group* $\mathrm{SU}(n) = \mathrm{SL}(n, \mathbb{C}) \cap \mathrm{U}(n)$. The *special orthogonal group* $\mathrm{SO}(n) = \mathrm{SL}(n, \mathbb{C}) \cap \mathrm{O}(n)$.

The group $\mathrm{GL}(n, \mathbb{C})$ and its subgroups can be regarded as subsets of $\mathbb{C}^{n^2}$ and so have induced topologies. It is easily verified that with these topologies they are topological groups. Further $\mathrm{U}(n)$, $\mathrm{SU}(n)$*, and* $\mathrm{SO}(n)$ *are metrizable arcwise connected compact groups, for each n. The groups* $\mathrm{O}(n)$ *are compact groups. The group* $\mathrm{O}(3)$ *has two components.*
The matrix groups we have just described are much more than a rich source of examples. To explain this we need some definitions.

Definition 2.5. Let G and H be topological groups. Then G and H are said to be *topologically isomorphic* denoted by $G \cong H$ if there is a map $f : G \to H$ which is both a homeomorphism and a group isomorphism. Such a map is called a *topological isomorphism.*

We can readily verify that $\mathbb{T} \cong \mathrm{U}(1) \cong \mathrm{SO}(2)$.

Definition 2.6. Let G be a topological group. Then G is said to have *no small subgroups* or to be an *NSS-group* if there exists a neighbourhood of the identity which contains no non-trivial subgroup of G.

For each positive integer n, the compact groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, each finite discrete group and $\mathbb{T}$ are NSS-groups.

Note that if B is a Banach space, then the underlying abelian group with its topology is a topological group - and it is an NSS-group.

We now state a version of the Peter-Weyl Theorem.

Theorem 2.7. [Peter-Weyl] ([3], 2.29) *Let G be any compact group. Then G is topologically isomorphic to a (closed) subgroup of a product of unitary groups.*

In Corollary 6 of [1] it is proved that a metrizable subgroup of an infinite product of topological groups is topologically isomorphic to a subgroup of a countable subproduct of those groups. From this one obtains Corollary 2.8.

Corollary 2.8. *Let G be any metrizable compact group. Then G is topologically isomorphic to a (closed) subgroup of a countable product of unitary groups.*

Definition 2.9. For any cardinal number m, the product $\mathbb{T}^m$ is said to be a *torus group.*

Of course every torus group is a compact group.

Definition 2.10. Let X be a topological space, $\mathcal{B}$ the set of all bases B for the topology of X, and $\operatorname{card} B$ the cardinality of the set B. Then the *weight* of the space X is the minimum of the set $\{\operatorname{card} B : B \in \mathcal{B}\}$.

An extension of the above argument yields Corollary 2.11.

Corollary 2.11. *Let G be any abelian compact group. Then G is topologically isomorphic to a (closed) subgroup of a torus group $\mathbb{T}^m$, where $m = w(G)$, the weight of the compact group G.*

Definition 2.12. A compact group G is said to be a *Lie group* if it is topologically isomorphic to a (closed) subgroup of $\mathrm{U}(n)$, for some positive integer n.

Theorem 2.13. [Hilbert 5 for Compact Groups] *A compact group G is a Lie group if and only if it satisfies one (and hence both) of the equivalent conditions:*

(i) G is an NSS-group;
(ii) The topological space $|G|$ is locally euclidean (that is, a neighbourhood of 1 in G is homeomorphic to a neighbourhood of 0 in $\mathbb{R}^n$, for some positive integer n).

Proof. By Corollary 2.40 of [3], G is a Lie group if and only if it has no small subgroups. A compact Lie group is locally euclidean because it is a linear Lie group (see [3], p. 134, Theorem 5.31, Proposition 5.33.) A locally euclidean compact group is a compact Lie group by Theorem 9.57 of [3]. □

Topologists will of course appreciate the topological characterization of compact Lie groups contained in (ii) above.

The work of Montgomery, Zippin and Gleason (see [7]) in the 1950s characterized noncompact Lie groups by conditions (i) and (ii) above.

Earlier we reduced the study of the topology of compact groups to the study of the topology of *connected* compact groups. Next we reduce the study to that of the topology of *abelian* connected compact groups and what we will call *semisimple* groups.

Definition 2.14. Let g, h be elements of a group G. Then $g^{-1}h^{-1}gh \in G$ is said to be a *commutator* and the smallest subgroup of G containing all commutators is called the *commutator subgroup* and denoted by G'.

Theorem 2.15. ([3], Theorem 9.2 & Proposition 9.4) *If G is any connected compact group, then G' is connected and (i) every element of G' is a commutator, (ii) G' is a compact group, and (iii) $G'' = G'$.*

The result (i) is remarkable. The results (ii) and (iii) are not valid without connectivity. (See [3] Exercise E6.6 & the example following Proposition 9.4].

Definition 2.16. A connected compact group G is said to be *semisimple* if $G' = G$.

Corollary 2.17. *If G is any connected compact group, then G' is semisimple.*

The Borel-Scheerer-Hofmann Theorem ([3], 9.39) says that a connected compact group G is the semidirect product of its commutator subgroup by a connected abelian compact subgroup of G. This implies, our next Theorem, the Topological Decomposition Theorem ([3], Corollary 10.39).

Theorem 2.18. *Let G be any connected compact group. Then G is homeomorphic to the product $G' \times G/G'$.*

Corollary 2.19. *If G is any compact group then G is homeomorphic to $G/G_0 \times (G_0)' \times G_0/(G_0)'$, where G/G_0 is homeomorphic to a Cantor cube.*

In Theorem 1.10 above, we have already completely described the topology of the totally disconnected compact group G/G_0. Observing that for any connected compact group G, the quotient group G/G' is abelian and connected, we will next look more carefully at the topological structure of *abelian* connected compact groups. Later we will examine the structure of semisimple groups.

3. Compact abelian groups

As is well-known, there is a wonderful duality theory for abelian compact groups, indeed for locally compact abelian groups, known as Pontryagin-van Kampen duality. We will quickly outline the duality for abelian compact groups and indicate its relevance to us.

Definition 3.1. If A is any abelian group, then the group $\mathrm{Hom}(A, \mathbb{T}) \subseteq \mathbb{T}^A$ of all group homomorphisms of A into the circle group $\mathbb{T}$ (no continuity involved!) given the induced topology from the Tychonoff product $\mathbb{T}^A$ is called the *dual group or character group* of the abelian group A and is written $\widehat{A}$. Its elements are called *characters*.

As the dual group is clearly a closed subset of $\mathbb{T}^A$, we obtain:

Proposition 3.2. *The dual group of any abelian group is an abelian compact group.*

Definition 3.3. If G is any abelian compact group, then the abelian group (without topology) $\mathrm{Hom}(G, \mathbb{T})$ of all continuous homomorphisms of G into $\mathbb{T}$ is called the *dual group or character group* of the abelian compact group G and is written $\widehat{G}$.

So if G is an abelian compact group, then its dual group, $\widehat{G}$, is an abelian group and the dual group of that dual group, $\widehat{\widehat{G}}$, is again an abelian compact group.

Further there is a natural evaluation map from G into its second dual, namely $\eta : G \to \widehat{\widehat{G}}$, where for each $g \in G$, $\eta(g) = \eta_g : \widehat{G} \to \mathbb{T}$ and for each $\gamma \in \widehat{G}$, $\eta_g(\gamma) = \gamma(g) \in \mathbb{T}$.

Theorem 3.4. ([3], Theorems 2.32, 7.63) *If G is any abelian compact group then the evaluation map $\eta : G \to \widehat{\widehat{G}}$ is a topological group isomorphism.*

The Pontryagin-van Kampen Duality Theorem above implies that no information about G is lost in going to the dual group. But the dual group is just an abelian group without topology. From this we deduce the fact that every piece of information about G can be expressed in terms of *algebraic* information about its dual group. Let us give a a few examples:

Proposition 3.5. ([3], Theorem 7.76, Theorem A4.16, Corollary 8.5) *Let G be an abelian compact group.*

(i) the weight $w(G)$ equals the cardinality of its dual group;
(ii) G is metrizable if and only if its dual group is countable;
(iii) G is connected if and only if its dual group is torsion-free (that is it has no nontrivial finite subgroups);
(iv) G is torsion-free if and only if its dual group is divisible;
(v) G is totally disconnected if and only its dual group is a torsion group (that is each element has finite order).

Note: (i) a compact group is 0-dimensional (that is has a basis of clopen subsets) if and only if it is totally disconnected; (ii) the weight of a compact group equals its local weight.

It is not yet obvious that duality adds to knowledge of the structure of abelian compact groups. But we can also use duality to prove that *an abelian compact group is connected if and only if it is divisible.* (See [3], Corollary 8.5.) So any compact topology on the additive group of real numbers or the multiplicative group of complex numbers of modulus 1 which makes it into a topological group must be connected.

Duality also gives us some interesting abelian compact groups such as $\widehat{\mathbb{Q}}$, where $\mathbb{Q}$ is the additive group of rational numbers. And $\widehat{\mathbb{Q}}$ is clearly a torsion-free abelian connected compact group. Indeed *an abelian connected compact group is torsion-free if and only if it is topologically isomorphic to $\widehat{\mathbb{Q}}^m$, for some cardinal number m.* And using duality, we can prove:

Proposition 3.6. ([3], Proposition 8.21) *If G is an abelian connected compact group of weight $w(G)$ and $m = w(G) + \aleph_0$, then there exists a continuous homomorphism of $\widehat{\mathbb{Q}}^m$ onto G.*

Recall that a topological space is said to be *dyadic* if it is a continuous image of the Cantor cube 2^m, for some cardinal number m. Note that every compact metric space is dyadic, and so in particular $\widehat{\mathbb{Q}}$ is dyadic. This Proposition together with Theorem 1.10, yields:

Theorem 3.7. *If G is an abelian compact group, then $|G|$ is a dyadic space.*

Another result which can be proved using duality is the following:

Theorem 3.8. ([3], Proposition 8,21 and Theorem 7.76) *Let G be an abelian connected compact group. If $w(G) > \aleph_0$, then G has $[0,1]^{w(G)}$ as a subspace.*

In due course we will see that the "abelian" restriction in Theorems 3.7 and 3.8 are unnecessary.
Using Theorems 3.8, 1.10 & 2.3 and Corollary 2.11 we obtain:

Corollary 3.9. *If G is an infinite abelian compact group, then*
(i) G has the Cantor cube $2^{w(G)}$ as a subspace;
(ii) the cardinality of G is $2^{w(G)}$.

We digress to mention that there are duality theories for nonabelian compact groups too, such as Krein-Tannaka duality. But to the best of our knowledge such dualities have not advanced knowledge of the structure of compact groups at all.
On the other hand it is well-known that Lie algebras are a powerful tool in understanding Lie groups, and [3] successfully uses an extension of this approach to expose the structure of general compact groups.

Some time ago we saw that every metrizable abelian connected compact group is topologically isomorphic to a subgroup of a torus group. One might ask is such a group in fact topologically isomorphic to a torus group?

We immediately see that it is not, since we now know that $\widehat{\mathbb{Q}}$ is a metrizable abelian connected compact group which is torsion-free, while a torus group is not torsion-free. However, we have the following:

Proposition 3.10. *If G is a metrizable abelian arcwise connected compact group, then G is topologically isomorphic to a torus group.*

What happens in the nonmetrizable case? The so-called *Torus Proposition* says that every arcwise connected abelian compact group is a torus group. In 1974 Shelah proved the following surprising result.

Theorem 3.11. ([3], Theorem 8.48)

(i) Assume that the axioms of ZFC, Zermelo–Fraenkel Set Theory with the Axiom of Choice, and the Diamond Principle $\diamond$ are valid. Then every compact arcwise connected abelian group is a torus group.

(ii) *Assume the axioms of ZFC, Martin's Axiom, and* $\aleph_1 < 2^{\aleph_0}$*. Then given any uncountable cardinal* $\aleph$ *there exists a compact arcwise connected abelian group* G *of weight* $w(G) = \aleph$ *which is not a torus group.*

(iii) *If ZFC is consistent, then ZFC+ Torus Proposition and ZFC+* $\neg$ *Torus Proposition are consistent; that is, the Torus Proposition is undecidable in ZFC.*

Let us digress again – this time to Banach spaces and a moment's criticism of the way introductory topology courses are sometimes taught. It is standard to begin teaching topology with examples of what are in fact metric spaces and indeed often Banach spaces. Many examples of interesting topological spaces are given. But sometimes exactly the same example is given over and over again. If what we are teaching is topology and producing examples of topological spaces, the following beautiful result should be remembered.

Proposition 3.12. *If* B *is any infinite-dimensional separable Banach space, then* B *is homeomorphic to* $\mathbb{R}^{\aleph_0}$*.*

So all infinite-dimensional separable Banach spaces are homeomorphic.

My preferred approach to introducing topology is available on the web – see [8].

The situation for abelian connected compact groups is about as far as one can get from the Banach space result just mentioned.

Theorem 3.13. ([3]8.58) *If* G *is an abelian connected compact group, let* $[|G|, \mathbb{T}]$ *be the group of all homotopy classes of maps* $f : |G| \to \mathbb{T}$*. Then* G *is topologically isomorphic to* $\widehat{A}$ *where* $A = [|G|, \mathbb{T}]$*.*

This has the following remarkable corollary.

Corollary 3.14. ([3], Proposition 8.61) *If abelian connected compact groups* G_1 *and* G_2 *are homeomorphic, then they are topologically isomorphic.*

Finally on abelian compact groups, we note the following:

Proposition 3.15. ([3], Theorem 8.62) *Let* G *be an abelian compact group, then* $\pi_n(G) = 1$*, for* $n = 2, 3, \ldots$*.*

4. Semisimple compact groups

We have no time to discuss in detail the Lie algebra/exponential approach to exposing the structure of general compact groups. However, we simply touch upon it if only to have available the necessary notation.

Definition 4.1. A *one parameter subgroup* of a topological group G is a continuous homomorphism $X\colon \mathbb{R} \to G$, that is an element X of $\mathrm{Hom}(\mathbb{R}, G)$. The topological space $\mathrm{Hom}(\mathbb{R}, G)$ which is obtained by endowing $\mathrm{Hom}(\mathbb{R}, G)$ with the topology of uniform convergence on compact subsets of $\mathbb{R}$ will be denoted by $\mathcal{L}(G)$ and called the *Lie algebra* of G. The *exponential function* $exp_G : \mathcal{L}(G) \to G$ is defined by $\exp_G(X) = X(1)$, for $X \in \mathcal{L}(G)$.

Theorem 4.2. ([3], Theorem 9.60) *Let G be a compact group and $G_a = exp(\mathcal{L}(G))$. Then G_a is the arc component of G and it is a dense subgroup of G_0.*

We state the Sandwich Theorem for Semisimple Connected Compact Groups. This tells us that each semisimple connected compact group is *almost* a product of simple simply connected Lie groups.
This is important here as
every compact group is homeomorphic to the product of a Cantor cube, a connected abelian compact group and a semisimple connected compact group.

Theorem 4.3. ([3], Theorem 9.19 p. 450ff, notably Corollary 9.20) *Let G be a semisimple connected compact group. Then there is a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups and there are surjective continuous homomorphisms q and f*

$$\prod_{j\in J} S_j \xrightarrow{f} G \xrightarrow{q} \prod_{j\in J} S_j/Z(S_j)$$

where each finite discrete abelian compact group $Z(S_j)$ is the centre of S_j and

$$\prod_{j\in J} S_j \xrightarrow{qf} \prod_{j\in J} S_j/Z(S_j)$$

is the product of the quotient morphisms $S_j \to S_j/Z(S_j)$.

Simply connected is defined here in a manner suitable for topological groups, not depending on arcwise connectedness.

Definition 4.4. ([3], Definition A2.6) A topological space X is called *simply connected* if it is connected and has the following universal property: For any covering map $p\colon E \to B$ between topological spaces, any point $e_0 \in E$ and any continuous function $f\colon X \to B$ with $p(e_0) = f(x_0)$ for some $x_0 \in X$ there is a continuous map $g\colon X \to E$ such that $p \circ g = f$ and $g(x_0) = e_0$.

$$\begin{array}{ccc} X & \xrightarrow{g} & E \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

Theorem 4.5. ([3], Theorem 9.29)

(i) Every simply connected compact group is semisimple;
(ii) Every simply connected abelian compact group is singleton;
(iii) Every simply connected connected compact group is topologically isomorphic to a product of simply connected simple compact Lie groups.

Now (i)–(iv) of the final Theorem 4.6 follows Theorems 2.18, 3.7, 3.8 and 1.10 together with Theorem 9.49 of [3] while (v) is proved in [2].

Theorem 4.6. *Let G be an infinite compact group. Then*

(i) G has the Cantor cube $2^{w(G)}$ as a subspace;
(ii) The cardinality of G is $2^{w(G)}$;
(iii) $|G|$ is a dyadic space;
(iv) If G is connected and $w(G) > \aleph_0$, then G has $[0,1]^{w(G)}$ as a subspace;
(v) If G is connected and $w(G) > \aleph_0$, then G contains a homeomorphic copy of every compact group K with $w(K) \leq w(G)$.

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