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A COUNTABLE α-NORMAL NON-REGULAR SPACE

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ABSTRACT. We construct a non-regular countable Hausdorff space in which dense parts of any two closed disjoint sets can be separated.

A.V. Arhangel'skii and L. Ludwig formulated separation axioms that generalize normality by separating dense subsets of closed disjoint sets instead of the whole sets [AL]. We recall the weaker one here, which is relevant for countable spaces.

Definition. A topological space X is called *α-normal* whenever for each pair of closed disjoint subsets A, $B \subset X$ there are open sets U, $V \subset X$ such that $\overline{A \cap U} = A$, $\overline{B \cap V} = B$ and $U \cap V = \emptyset$.

It is easy to observe that every α-normal T_1 space is Hausdorff. And there exists an α-normal non-regular T_1 space of cardinality $\aleph_1$ ([M], Example 2). Its construction is based on properties of the filter generated by closed unbounded subsets of a regular cardinal: if from a closed unbounded set C the set of all points isolated in C is removed, the remainder is nowhere dense in C, but still an element of the filter. Therefore the filter can be used to define a neigbourhood base in a non-regularity point, while α-normality is preserved.

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In this paper a different method is used to produce a countable α-normal non-regular Hausdorff space.

The following is a straightforward observation.

Lemma 1. *If X is a T_1 space and $x \in X$ is such that $X \setminus \{x\}$ is α-normal and for every closed $F \not\ni x$ there are open disjoint U, $V \subset X$ such that $x \in U$ and $\overline{F \cap V} = F$ then X is α-normal.*

The next two lemmas describe a construction scheme of a space with the desired properties.

Lemma 2. *Suppose there is a countable infinite regular space X and a filter base $\mathcal{P}$ consisting of nonempty closed subsets of X such that*

(*) *for every closed $F \subset X$ there exists $P \in \mathcal{P}$ with $F \cap P$ nowhere dense in F.*

Then there is a countable Hausdorff α-normal non-regular space.

Proof. Starting with X and a point $\infty \notin X$, we associate to this pair an α-normal space Y. Put

$$Y = X \cup (X \times \omega) \cup \{\infty\},$$

and topologize it in the following way. Let all the points of $X \times \omega$ be isolated, let $\{U_{O,n};\ O \text{ open in } X \ \&\ n \in \omega\}$, where

$$U_{O,n} = O \cup (O \times \{m \in \omega;\ m \geq n\}),$$

be an open base in points of X in Y and let $\{O_P;\ P \in \mathcal{P}\}$, where

$$O_P = \{\infty\} \cup (P \times \omega),$$

be an open base in ∞.

To see that Y is Hausdorff, note that $\mathcal{P}$ is a free filter base and, moreover, as every $P \in \mathcal{P}$ is closed, (*) yields separation of ∞ and points of X. The set X is closed in Y, but

$$(\forall P \in \mathcal{P})\ \overline{O_P} \cap X = P \neq \emptyset,$$

hence Y is not a regular space. We shall show that Y and ∞ satisfy assumptions of Lemma 1. Let $F \subset Y$ be closed, $\infty \notin F$; we may assume $F \subset X$. Now (*) gives $P \in \mathcal{P}$ such that $F \cap P$ is nowhere dense in F. Put $U = U_{X \setminus P,0}$. Then O_P and U separate the point ∞ from a dense part of F.

As α-normality of $Y \setminus \{\infty\}$ easily follows from regularity of X, Lemma 1 finishes the proof. $\square$

Lemma 3. *Let X be a countable regular scattered space. Denote*

$$\mathcal{D} = \{D \subset X;\ D \text{ is discrete}\}.$$

Suppose there is a family $\{O_D;\ D \in \mathcal{D}\}$ of open subsets of X such that

(i) $(\forall D \in \mathcal{D})\ D \subset O_D$,

(ii) $(\forall \mathcal{D}' \subset \mathcal{D})\ |\mathcal{D}'| < \omega \Rightarrow X \neq \bigcup\{O_D;\ D \in \mathcal{D}'\}$.

Then X and

$$\mathcal{P} = \Big\{X \setminus \bigcup_{D \in \mathcal{D}'} O_D;\ \mathcal{D}' \subset \mathcal{D}\ \&\ |\mathcal{D}'| < \omega\Big\}$$

satisfy assumptions of Lemma 2.

Proof. Since X is scattered, for every nonempty closed $F \subset X$ there is a discrete $D \subset X$ such that $\overline{D} = F$. Consequently, $F \cap P$ is nowhere dense in F for $P = X \setminus O_D$. □

To complete the construction of a countable α-normal non-regular space it suffices to find a countable scattered space with the properties described in Lemma 3. In the following example, an ordinal number will be used both as a point and as a set of all its predecessors.

Example. Let X be the countable ordinal $\omega^\omega = \sup\{\omega^n;\ n \in \omega\}$ with the usual topology – X is a regular scattered space of Cantor-Bendixson height ω. For $\alpha \in X$ we shall denote by $ht(\alpha)$ its *height in X*, i.e.

$$ht(\alpha) = n \Leftrightarrow \alpha \in X^{(n)} \setminus X^{(n+1)},$$

where

$$X^{(0)} = X,$$
$$X^{(n+1)} = \big(X^{(n)}\big)'$$

(A' stands for the derived set of A). In the space ω^ω it means that $ht(\alpha) = 0$ iff α is an isolated ordinal, $ht(\alpha) = 1$ iff α is a limit ordinal which is not a limit of limit ordinals etc.

Now, choose a discrete $D \subset X$ and $\alpha \in D$. If $ht(\alpha) = 0$, put $O_D(\alpha) = \{\alpha\}$. Suppose $ht(\alpha) = n + 1$. Put

$$\alpha_D = \sup\big((\alpha \cap D) \cup \{\beta < \alpha;\ ht(\beta) \geq n + 1\}\big)$$

and note that $\alpha_D < \alpha$. Define

$$\alpha_0 = \min\{\beta > \alpha_D;\ ht(\beta) = n\},$$
$$\alpha_1 = \min\{\beta > \alpha_0;\ ht(\beta) = n\}.$$

Hence $\alpha_0 < \alpha_1 < \alpha$ and we may put $O_D(\alpha) = (\alpha_1, \alpha]$. Finally, let $O_D = \bigcup\{O_D(\alpha);\ \alpha \in D\}$. Obviously, $D \subset O_D$.

Let us prove that X and $\{O_D;\ D \in \mathcal{D}\}$, where $\mathcal{D} = \{D \subset X;\ D \text{ discrete}\}$, satisfy condition (ii) of Lemma 3.

First, consider $\beta = \omega^k \in X$ and suppose that $\beta \in O_D$ for some $D \in \mathcal{D}$. Keeping the notation used in the construction of O_D, we shall check that $\beta \in D$.

Assume $\beta \in O_D(\alpha)$, where $\alpha > \beta$. As $ht(\beta) = k$, $ht(\alpha) = n+1 > k$, and since $(\forall\gamma < \beta)\ ht(\gamma) < k \leq n$, one can see that $\alpha_0 \geq \beta$, in particular, $\beta \notin O_D(\alpha) = (\alpha_1, \alpha]$ – a contradiction.

Suppose now there is $\mathcal{D}' \subset \mathcal{D}$, $|\mathcal{D}'| = k < \omega$, such that $X = \bigcup\{O_D;\ D \in \mathcal{D}'\}$. It follows that $\omega^k + 1 \subset \bigcup\{O_D;\ D \in \mathcal{D}'\}$. Let k be the smallest natural number with this property.

If $k = 1$, the assumption implies that $\omega + 1 = O_D$ for some discrete $D \subset \omega + 1$ such that $\omega \in D$, which is clearly impossible. Thus $k \geq 2$.

Take $D_0 \in \mathcal{D}'$ such that $\omega^k \in D_0$ and $\min\big(O_{D_0}(\omega^k)\big)$ is the smallest possible. Recall that for $\alpha = \omega^k$, $I = (\alpha_0, \alpha_1] \neq \emptyset$, $I \cap O_{D_0} = \emptyset$, $ht(\alpha_0) = ht(\alpha_1) = k - 1$ and $I \cap \{\beta \in X;\ ht(\beta) \geq k-1\} = \{\alpha_1\}$. It is now easy to see that I and the ordinal $\omega^{k-1} + 1$ are order isomorphic. For such an isomorphism $\varphi : I \to \omega^{k-1} + 1$ and for each $D \in \mathcal{D}' \setminus \{D_0\}$ the image $\widetilde{D} = \varphi[D \cap I]$ is discrete and the following holds true.

Claim. $\varphi[O_D \cap I] = O_{\widetilde{D}}$.

Proof. For every $\beta \in I$, $ht(\beta) = ht(\varphi(\beta))$. Consequently, if $\beta \in D \cap I$ then $O_D(\beta) = O_{D\cap I}(\beta) \subset I$ and $\varphi[O_D(\beta)] = O_{\widetilde{D}}(\varphi(\beta))$. Let us check that $O_D \cap I = O_{D\cap I}$, in other words, if $\alpha_1 \in O_D(\beta)$, then $\beta = \alpha_1$.

Assume $\alpha_1 \in O_D(\beta)$ for some $\beta > \alpha_1$. Then $\beta \leq \omega^k$. But for $\beta < \omega^k$, $ht(\beta) \leq k - 1 = ht(\alpha_1)$, hence $\alpha_1 \notin O_D(\beta)$. If $\alpha_1 \in O_D(\omega^k)$, then $\omega^k \in D$ and $\min\big(O_D(\omega^k)\big) \leq \alpha_1 < \alpha_1 + 1 = \min\big(O_{D_0}(\omega^k)\big)$ – contradicting the choice of D_0. We may conclude that:

$$\varphi[O_D \cap I] = \varphi[O_{D\cap I}] = \varphi\Big[\bigcup_{\beta\in D\cap I} O_{D\cap I}(\beta)\Big] = \bigcup_{\beta\in D\cap I} \varphi[O_D(\beta)]$$
$$= \bigcup_{\beta\in D\cap I} O_{\widetilde{D}}(\varphi(\beta)) = \bigcup_{\gamma\in\widetilde{D}} O_{\widetilde{D}}(\gamma) = O_{\widetilde{D}}.$$

□

As $I \subset \bigcup\{O_D;\ D_0 \neq D \in \mathcal{D}'\}$, the subspace $\omega^{k-1}+1$ is covered by $k-1$ sets $O_{\widetilde{D}}$, which contradicts minimality of k. □

On the other hand, for countable spaces of small weight α-normality implies regularity. Recall the definition

$$\mathfrak{p} = \min\big\{|\mathcal{P}|;\ \mathcal{P} \subset [\omega]^\omega \ \&\ \mathcal{P} \text{ is closed under finite intersections} \ \&\ \neg(\exists X \in [\omega]^\omega)\,(\forall P \in \mathcal{P})\ X \subset^* P\big\},$$

where $[\omega]^\omega = \{A \subset \omega;\ |A| = \omega\}$ and $A \subset^* B$ means that $A \setminus B$ is finite. It is well known that $\aleph_1 \le \mathfrak{p} \le \mathfrak{c}$; moreover, $\mathfrak{p}$ can be any regular cardinal between $\aleph_1$ and $\mathfrak{c}$. For details on small cardinals see, e.g., [vD].

Proposition. *If X is a countable α-normal T_1 space, $w(X) < \mathfrak{p}$, then X is regular.*

Proof. Let X satisfy the assumptions, $|X| = \aleph_0$. Suppose the space is non-regular and consider $x \in X$ and a closed $F \subset X$ witnessing it. Fix an open base $\mathcal{B}$ in x such that $|\mathcal{B}| < \mathfrak{p}$.

It follows that $\mathcal{F} = \{\overline{B}\cap F;\ B \in \mathcal{B}\}$ is a centered family of infinite sets, $|\mathcal{F}| < \mathfrak{p}$. Hence there is an infinite $P \subset F$ such that $(\forall B \in \mathcal{B})$ $P \subset^* \overline{B}$. As X is Hausdorff, it is easy to see that P is closed and discrete. Apply α-normality to $\{x\}$ and P: let V be open, $B \in \mathcal{B}$, $B \cap V = \emptyset$, $\overline{P \cap V} = P$. But then $P \subset V$, consequently $P \subset^* \overline{B} \cap V = \emptyset$ – a contradiction. □

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