Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 27, No. 2, 2003
Pages 661–673

CONNECTED URYSOHN SUBTOPOLOGIES

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Abstract. We show that each second countable Urysohn space which is not Urysohn-closed can be condensed onto a connected Urysohn space and as a corollary we characterize those countable Urysohn spaces which have connected Urysohn subtopologies. We also answer two questions from [12] regarding condensations onto connected Hausdorff spaces.

1. Introduction and Preliminary Results

Recall that a space is *Urysohn* if distinct points have disjoint closed neighbourhoods and a space is *feebly compact* if every locally finite family of open sets is finite. During the 1960's and 70's many papers appeared in which countable connected Urysohn spaces were constructed (see for example [6],[7],[8] and [9]); such spaces were in some sense considered oddities. In a previous paper [12], we proved that each disconnected Hausdorff space with a countable network can be condensed (that is to say, there is a continuous bijection) onto a connected Hausdorff space if and only if it is not feebly compact. Here we prove an analogous result for Urysohn spaces:

2000 *Mathematics Subject Classification.* Primary 54D05; Secondary 54D10, 54D25.

Key words and phrases. Second countable connected Urysohn space, countable network, condensation, Urysohn family, Urysohn filter, Urysohn-closed space.

*Research supported by Consejo Nacional de Ciencia y Tecnología (CONACYT), Mexico, grant no. 38164E.

A disconnected Urysohn space with a countable network can be condensed onto a connected Urysohn topology if and only if it is not Urysohn-closed. In the case of countable spaces, we show a little more by proving that a countable Urysohn space can be condensed onto a connected Urysohn space if and only if it is not feebly compact. Thus every Urysohn topology on a countable set which is not feebly compact contains a connected Urysohn subtopology, showing that in some sense, countable connected Urysohn spaces exist in abundance.

The closure of a set A in a space (X, τ) will be denoted by $\mathrm{cl}_\tau(A)$ or by $\mathrm{cl}(A)$ or $\overline{A}$ when no confusion is possible. Similarly we use $\mathrm{int}_\tau(A)$ or $\mathrm{int}(A)$ to denote the interior of A in (X, τ). Recall from [13; Problem 12E] that an *open filter* on a topological space (X, τ) is a filter in τ. Following [2], we say that an open filter $\mathcal{F}$ is *free* if it has no accumulation point, that is to say, $\cap\{\mathrm{cl}_\tau(F) : F \in \mathcal{F}\} = \emptyset$. All terms and notation not defined here can be found in [4] or [13].

If D is a dense subspace of a Urysohn space X and $x \in X \setminus D$, then the trace of the open neighbourhood system at x is a free open filter $\mathcal{F}$ on D. It follows that for each $d \in D$ there are disjoint closed neighbourhoods of x and d and so each $d \in D$ has a closed neighbourhood missing the closure of some element of $\mathcal{F}$. Hence the following definition from [10]:

A *Urysohn filter* $\mathcal{F}$ on X is an open filter with the property that if $x \in X$ is not a cluster point of $\mathcal{F}$, then there is a closed neighbourhood U of x and an element $F \in \mathcal{F}$ such that $U \cap \overline{F} = \emptyset$. A Urysohn space is said to be *Urysohn-closed* if it is closed in every Urysohn space in which it is embedded. The following simple result from [10] is now obvious:

Theorem 1.1. *A Urysohn space is Urysohn-closed if and only if every Urysohn filter has a cluster point.*

Remark 1.2. Clearly, a Urysohn topology which is weaker than a Urysohn-closed topology is Urysohn-closed and an open and closed subset of a Urysohn-closed space is Urysohn-closed. However, unlike the case of an H-closed space, a regular closed subset of a Urysohn-closed space need not be Urysohn-closed (see [5; Example 5]).

A Urysohn filter is an open filter, but as noted in the first paragraph of [10], the concepts are subtly different and the existence of a free open filter in a Urysohn space does not imply the existence of a free Urysohn filter (again see [5; Example 5]). The existence of a free Urysohn filter ensures the existence of a proper Urysohn extension, but in order to condense onto a connected Urysohn space we need something more.

For an infinite cardinal κ, a *Urysohn family of size* κ in X is a family $\mathcal{U}$ of κ mutually disjoint regular closed sets with the property that if $x \in X$, there is a closed neighbourhood of x which meets only finitely many elements of $\mathcal{U}$.

Clearly, a Urysohn family is locally finite and if the space is regular, then every locally finite family of open sets gives rise to a Urysohn family of the same cardinality. Furthermore, the existence of a Urysohn family implies the existence of a free Urysohn filter (but the converse is false as we shall see in Example 2.2). To see this, note that if $\{U_n : n \in \omega\}$ is a Urysohn family and $A \subseteq \omega$ is infinite, then $\mathcal{H} = \{\cup\{\mathrm{int}(U_m) : m \in A \text{ and } m \geq n\} : n \in \omega\}$, is a free Urysohn filter base. We make use of this fact in the sequel.

2. Condensing onto connected Urysohn spaces

The proof of our main result is divided into a number of subsidiary results. Given a Urysohn space with a countable network, we first condense it onto a second countable Urysohn space which is not Urysohn-closed; this in its turn can be condensed onto a dense-in-itself Urysohn space which is not Urysohn-closed and finally onto a connected Urysohn space.

Lemma 2.1. *A Urysohn Lindelöf space which is not Urysohn-closed has a Urysohn family.*

Proof. Let X be a Lindelöf Urysohn space which is not Urysohn-closed and suppose that $\mathcal{G}$ is a free Urysohn filter on X. For each $x \in X$, choose an open neighbourhood V_x of x and $G_x \in \mathcal{G}$, so that $\mathrm{cl}(V_x) \cap \mathrm{cl}(G_x) = \emptyset$. $\mathcal{V} = \{V_x : x \in X\}$ is an open cover of X which must have a countable subcover, $\{V_{x_n} : n \in \omega\}$. Let $G_{x_n} = F_n$ and then $\mathcal{F} = \{F_n : n \in \omega\}$ is a free Urysohn filter base on (X, τ). With no loss of generality, we may assume that $F_n \supseteq F_{n+1}$ for each $n \in \omega$.

Choose $z_0 \in F_0$; since z_0 is not a cluster point of $\mathcal{F}$ and $\mathcal{F}$ is a free Urysohn filter base, there is some $F_{m_1} \in \mathcal{F}$ and a regular closed neighbourhood U_0 of z_0 such that $U_0 \cap \overline{F}_{m_1} = \emptyset$; without loss of generality we assume that $U_0 \subseteq \overline{F}_0$.

Having chosen elements $F_0 \supseteq F_{m_1} \supseteq \cdots \supseteq F_{m_n}$ of $\mathcal{F}$, points $z_j \in F_{m_j}$ for each $j \in \{1, \ldots, n-1\}$ and regular closed neighbourhoods $U_j \subseteq \overline{F}_{m_j}$ of z_j for each $j \in \{1, \ldots, n-1\}$, whose closures are mutually disjoint and with the property that $U_j \cap \overline{F}_{m_k} = \emptyset$ whenever $j < k \leq n$, we proceed as follows: Pick $z_n \in F_{m_n}$; since z_n is not a cluster point of $\mathcal{F}$, there is some $m_{n+1} > m_n$ and a regular closed neighbourhood V_n of z_n such that $V_n \cap \overline{F}_{m_{n+1}} = \emptyset$. Then $F_{m_{n+1}} \subseteq F_{m_n}$ and since $U_j \cap \overline{F}_{m_n} = \emptyset$ for each $j < n$, by setting $U_n = V_n \cap \overline{F}_{m_n}$ we have that $U_n \cap U_k = \emptyset$ for each $k < n$. Proceeding in this way, we construct a family $\mathcal{U} = \{U_n : n \in \omega\}$ of mutually disjoint regular closed sets such that $U_n \subseteq \overline{F}_{m_n}$ for each $n \in \omega$. Since $\mathcal{F}$ is a free Urysohn filter base, if $x \in X$, there is some $k \in \omega$ and a closed neighbourhood V of x, such that $V \cap \overline{F}_k = \emptyset$. Clearly then, whenever $m_n > k$, $V \cap U_n = \emptyset$ and it follows that $\mathcal{U}$ is a Urysohn family. $\square$

Without the condition of being Lindelöf, the above theorem is false, even if the space is neither feebly compact nor Urysohn-closed. The following example is inspired by Example 5 of [5].

Example 2.2. Let K denote the Cantor set, $Y = K \setminus \{1\}$ and let D_1, D_2 and D_3 be three mutually disjoint dense subsets of the ordered space $X = \omega_1 \times Y$ with the lexicographic order topology μ, whose union is X. Define a new topology τ on X as follows: $U \in \tau$ provided that for each $x \in U \cap D_k$ there is an open μ-neighbourhood I of x such that $I \cap D_k \subseteq U$ if $k \in \{1, 2\}$ or $I \subseteq U$ if $k = 3$. It is straightforward to check that (X, τ) is a Urysohn space which is neither feebly compact nor Urysohn-closed (for example, the family of all final μ-open intervals of X is a free Urysohn filter base on (X, τ)) but which possesses no Urysohn family.

Theorem 2.3. *A Urysohn space with a countable network which is not Urysohn-closed can be condensed onto a second countable Urysohn space which is not Urysohn-closed.*

Proof. Let (X,τ) be a Urysohn space with a countable network $\mathcal{M} = \{M_k : k \in \omega\}$ which is not Urysohn-closed. Since X is Lindelöf, it follows from Lemma 2.1 that there is a Urysohn family $\{\mathrm{cl}_\tau(W_n) : n \in \omega\}$ in X, where $W_n \in \tau$ for each n. Consider the family $\mathcal{P}$ of all pairs of elements $M_1, M_2 \in \mathcal{M}$ such that there exist open sets $U_1 \supseteq M_1$ and $U_2 \supseteq M_2$ whose closures are disjoint. $\mathcal{P} \neq \emptyset$, since each pair of distinct points can be separated by open sets whose closures are disjoint. For each such pair $\{M_1, M_2\} \in \mathcal{P}$, choose open sets $U_1 \supseteq M_1, U_2 \supseteq M_2$ with disjoint closures and let $\mathcal{B}$ be the collection of all the U_i so chosen; Since $\mathcal{M}$ is countable it follows that $\mathcal{B}$ is countable. Furthermore, for each $x \in X$ there is a closed neighbourhood U_x of x and such that $U_x \cap \mathrm{cl}(W_n) = \emptyset$ for all but finitely many n and then there is $M \in \mathcal{M}$ such that $x \in M \subseteq \mathrm{int}_\tau(U_x)$. For each $M \in \mathcal{M}$ such that there exists $U \in \tau$ with the property that $M \subseteq U$ and $\mathrm{cl}_\tau(U) \cap \mathrm{cl}(W_n) = \emptyset$ for all but finitely many n, we choose $U_M \in \tau$ such that $M \subseteq U_M$ and $\mathrm{cl}_\tau(U_M) \cap \mathrm{cl}(W_n) = \emptyset$ for all but finitely many n; let $\mathcal{C}$ be the family of all such U_M chosen in this way. Let σ be the topology on X with subbase

$$\begin{aligned}\mathcal{S} = \{W_n : n \in \omega\} &\cup \{X \setminus \mathrm{cl}_\tau(W_n) : n \in \omega\} \cup \mathcal{B} \\ &\cup \mathcal{C} \cup \{X \setminus \mathrm{cl}_\tau(U) : U \in \mathcal{B}\} \cup \{X \setminus \mathrm{cl}_\tau(U) : U \in \mathcal{C}\}.\end{aligned}$$

Clearly $\sigma \subseteq \tau$ and since $\mathcal{S}$ is countable, it follows that (X,σ) is second countable. If $U \in \mathcal{B}$, then $\mathrm{cl}_\sigma(U) \supseteq \mathrm{cl}_\tau(U)$, but since $X \setminus \mathrm{cl}_\tau(U) \in \sigma$ it follows that $\mathrm{cl}_\sigma(U) = \mathrm{cl}_\tau(U)$. Now since distinct points $x_1, x_2 \in X$ have disjoint closed τ-neighbourhoods, V_1, V_2 respectively, it follows that there are $M_1, M_2 \in \mathcal{M}$ such that $x_1 \in M_1 \subseteq \mathrm{int}_\tau(V_1)$ and $x_2 \in M_2 \subseteq \mathrm{int}_\tau(V_2)$. Thus $\{M_1, M_2\} \in \mathcal{P}$ and it follows that there are $U_1, U_2 \in \mathcal{B}$ such that $x_i \in M_i \subseteq U_i$ $(i = 1, 2)$. However, $\mathrm{cl}_\sigma(U_i) = \mathrm{cl}_\tau(U_i)$ and so U_1, U_2 have disjoint σ-closures, implying that (X,σ) is Urysohn. A similar argument applies to show that $\{W_n : n \in \omega\}$ is a Urysohn family in (X,σ) which is thus not Urysohn-closed. □

Before proceeding, we require some more terminology.

If $\mathcal{G} = \{\mathcal{G}_\alpha : \alpha \in \kappa\}$ is a family of free open filter bases on a space (X,τ) and $A = \{x_\alpha : \alpha \in \kappa\}$ is a subset of X, then we define a topology on X as follows: $U \in \sigma$ if and only if $U \in \tau$ and whenever $x_\alpha \in U$ then $U \supseteq G$ for some $G \in \mathcal{G}_\alpha$. It is easy to check that this defines a topology σ on X such that $\sigma \subseteq \tau$ and that $\mathcal{G}_\alpha$ converges to x_α in (X,σ). Informally, we say that σ *is defined by* requiring that $\mathcal{G}_\alpha$ converges to x_α.

Theorem 2.4. *A separable Lindelöf Urysohn space (X, τ) which is not Urysohn-closed can be condensed onto a dense-in-itself Urysohn space which is not Urysohn-closed.*

Proof. By Lemma 2.1, there is a Urysohn family $\mathcal{U} = \{U_n : n \in \omega\}$ in X. We denote the (countable) set of isolated points of X by $D = \{d_n : n \in \omega\}$. There are two possibilities:

1) If only finitely many of the U_n meet D, then infinitely many (and we assume all) of the U_n are dense-in-themselves. Let $\{K_j : j \in \omega\}$ be an infinite family of disjoint infinite subsets of ω and for $j \in \omega$ let $\mathcal{G}_j$ denote the open filter base

$$\{\cup\{\text{int}(U_m) : m \in K_j \text{ and } m \geq n\} : n \in \omega\}.$$

As noted in the remark at the end of Section 1, each $\mathcal{G}_j$ is a free Urysohn filter base and we define a new topology σ on X by requiring that $\mathcal{G}_{j+1}$ converge to d_j; that is to say, $U \in \tau$ is a σ-neighbourhood of d_j if and only if $U \supseteq G$ for some $G \in \mathcal{G}_{j+1}$. It is straightforward to check that (X, σ) is a dense-in-itself space and we will show that it is Urysohn. To this end, note that if $A \subseteq X$ is such that A misses all but finitely many elements of $\mathcal{U}$, then $\text{cl}_\tau(A) = \text{cl}_\sigma(A)$; we make use of this fact repeatedly in the sequel. Suppose that $x, z \in X$; there are three cases to consider.

(a) If $x, z \notin \text{cl}_\tau(D)$, then there are disjoint closed τ-neighbourhoods $U, V \subseteq X \setminus D$ of x, z respectively and also closed τ-neighbourhoods S and T of x and z respectively, which for some $m \in \omega$ are disjoint from $\cup\{U_n : n \geq m\}$. It follows that $U \cap S$ and $V \cap T$ are disjoint σ-closed neighbourhoods of x and z respectively.

(b) If $x \in \text{cl}_\tau(D)$ and $z \notin \text{cl}_\tau(D)$, then we can choose τ-open neighbourhoods U and V of x and z respectively such that

(*i*) $\text{cl}_\tau(U) \cap \text{cl}_\tau(V) = \emptyset$,

(*ii*) $\text{cl}_\tau(V) \cap D = \emptyset$, and

(*iii*) for some $m \in \omega$,

$$\text{cl}_\tau(U) \cap (\cup\{U_n : n \geq m\}) = \text{cl}_\tau(V) \cap (\cup\{U_n : n \geq m\} = \emptyset.$$

Let $A = \{n \in \omega : d_n \in U\}$ and then we claim that $W_1 = U \cup \cup\{\text{int}(U_n) : n \in K_j \text{ for some } j \in A \text{ and } n \geq m\}$ and $W_2 = V$ are σ-neighbourhoods of x and z respectively whose σ-closures are disjoint. Clearly $W_1, W_2 \in \sigma$ and by (*ii*) and (*iii*), $\text{cl}_\tau(V) = \text{cl}_\sigma(V)$ and $\text{cl}_\tau(U) = \text{cl}_\sigma(U)$.

Finally, note that $\mathrm{cl}_\sigma(\cup\{\mathrm{int}(U_n) : n \in K_j$ for some $j \in A$ and $n \geq m\}) = \mathrm{cl}_\tau(\cup\{\mathrm{int}(U_n) : n \in K_j$ for some $j \in A$ and $n \geq m\}) \cup \mathrm{cl}_\sigma(\{d_j : j \in A\}) = \mathrm{cl}_\tau(\cup\{\mathrm{int}(U_n) : n \in K_j$ for some $j \in A$ and $n \geq m\}) \cup \mathrm{cl}_\sigma(U \cap D) = \mathrm{cl}_\tau(\cup\{\mathrm{int}(U_n) : n \in K_j$ for some $j \in A$ and $n \geq m\}) \cup \mathrm{cl}_\tau(U \cap D) \subseteq \mathrm{cl}_\tau(\cup\{\mathrm{int}(U_n) : n \in K_j$ for some $j \in A$ and $n \geq m\}) \cup \mathrm{cl}_\tau(U)$ and the result follows from (i).

(c) If $x, z \in \mathrm{cl}_\tau(D)$, then we choose open τ-neighbourhoods U and V of x and z respectively whose τ-closures are disjoint and such that for some $m \in \omega$,

$$\mathrm{cl}_\tau(U) \cap (\cup\{U_n : n \geq m\}) = \mathrm{cl}_\tau(V) \cap (\cup\{U_n : n \geq m\} = \emptyset.$$

Let $A = \{n \in \omega : d_n \in U\}$ and $B = \{n \in \omega : d_n \in V\}$. It follows that

$$U \cup \cup\{\mathrm{int}(U_n) : n \in K_j \text{ for some } j \in A\},$$

$$V \cup \cup\{\mathrm{int}(U_n) : n \in K_j \text{ for some } j \in B\}$$

are open σ-neighbourhoods of x and z respectively whose σ-closures are disjoint. We omit the details which are similar to the previous case.

Finally note that (X, σ) is not Urysohn-closed, since $\{U_m : m \in K_0\}$ is a Urysohn family in (X, σ).

2) If, on the other hand, infinitely many of the U_n meet D, then by choosing $a_n \in D \cap U_n$, we obtain an infinite open and closed discrete subset $A = \{a_n : n \in \omega\} \subseteq D$. Define a new topology η on X by requiring that $(A, \eta|A)$ be homeomorphic to the rationals, $\mathbb{Q}$. It is clear that $\eta \subseteq \tau$ and that (X, η) is a separable Lindelöf Urysohn space which is not Urysohn-closed since $(A, \eta|A)$ is an open and closed subspace homeomorphic to the rationals. Let $\{V_n : n \in \omega\}$ be a Urysohn family in $(A, \eta|A)$; case 1) now applies and we are done. □

If in the proof of the previous theorem, the topology τ has a countable network (hence is Lindelöf and separable), then so does σ. Thus we have proved:

Corollary 2.5. *Every Urysohn space with a countable network which is not Urysohn-closed can be condensed onto a dense-in-itself Urysohn space with a countable network which is not Urysohn-closed.*

Combining this result with Theorem 2.3 we obtain:

Corollary 2.6. *A Urysohn space with a countable network which is not Urysohn-closed can be condensed onto a second countable dense-in-itself Urysohn space which is not Urysohn-closed.*

In [12] it was shown that each non-compact regular Lindelöf space with a G_δ-diagonal can be condensed onto a connected Hausdorff space and the question was asked as to whether every non-compact regular Lindelöf space (or each non-H-closed Lindelöf T_2-space) can be condensed onto a connected Hausdorff space. These are parts of Problems 3.10 and 3.11 of [12]. The following very simple example shows that the condition of separability in Theorem 2.4 cannot be omitted and at the same time answers both of the above questions in the negative.

Example 2.7. Let C be the one-point compactification of a discrete space of cardinality $\kappa > 2^{\mathfrak{c}}$ and let (X, τ) be the disjoint topological union of a countable discrete space $\mathbb{N}$ and C. Then X is a regular Lindelöf space which cannot be condensed onto a dense-in-itself Hausdorff space.

Proof. Suppose that σ is a Hausdorff topology on X weaker than τ. First note that since $|X| > 2^{\mathfrak{c}}$, (X, σ) is not separable and hence $\mathbb{N}$ is not dense in (X, σ). But then, $A = X \setminus \mathrm{cl}_\sigma(\mathbb{N})$ is a non-empty open subset of $(C, \sigma|C)$. However, since $(C, \tau|C)$ is compact Hausdorff, it follows that $(C, \tau|C) = (C, \sigma|C)$ and so the open set A contains isolated points, showing that (X, σ) is not dense-in-itself. □

The space X above is neither first countable nor separable and so the following questions (the first of which constitutes part of Problems 3.10 and 3.11 of [12]) remain open.

Problem 2.8. Can every non-compact regular (respectively, non-H-closed Hausdorff) first countable Lindelöf space be condensed onto a connected Hausdorff space?

Problem 2.9. Can every non-compact regular (respectively, non-H-closed Hausdorff) separable Lindelöf space be condensed onto a connected Hausdorff space?

With regard to the latter problem, we note that an argument similar to that of Theorem 2.4, can be used to condense a non-H-closed, separable Lindelöf Hausdorff space onto a dense-in-itself space with the same properties.

The next theorem should be compared with that of Theorem 3.4 of [12], where a connected second countable Hausdorff subtopology was constructed.

Theorem 2.10. *A second countable Urysohn space* (X, τ) *which is dense-in-itself but not Urysohn-closed can be condensed onto a connected second countable Urysohn space.*

Proof. Since (X, τ) is not Urysohn-closed, by Lemma 2.1 there exists a Urysohn family $\mathcal{U} = \{U_n : n \in \omega\}$ in (X, τ). Furthermore, by Corollary 2.2 of [1] each set $\mathrm{int}_\tau(U_n)$ has a countable dense regular subspace Q_n which, since it is dense-in-itself, must be homeomorphic to the rationals $\mathbb{Q}$ whose metric topology we denote by μ.

Denote by h_n some homeomorphism from Q_n onto $\mathbb{Q}$. Let ρ be a connected second countable Urysohn topology on ω (for example that of [9]). Given $T \in \mu$ and $W \subseteq \omega$, we define

$$O(T, W) = \cup\{\mathrm{int}_\tau(\mathrm{cl}_\tau(h_n^{-1}[T])) : n \in W\}.$$

For each $T \in \mu$ and $W \in \rho$, it is immediate that $O(T, W) \in \tau$. We denote $\cup\{U_n : n \in \omega\}$ by Z and define a topology σ on X as follows:

$$\sigma = \{U \in \tau : \text{for all } x \in U \cap \mathrm{int}_\tau(Z) \text{ there exist } T \in \mu \text{ and } W \in \rho \text{ such that } x \in O(T, W) \subseteq U\}.$$

Obviously, $\sigma \subset \tau$ and we omit the straightforward verification that σ is indeed a topology. We proceed to prove that (X, σ) is a Urysohn space. To this end, suppose x, y are distinct elements of X.

1) $x, y \in Z$. If $x \in U_n$ and $y \in U_m$, where $m \neq n$, then there are mutually disjoint closed ρ-neighbourhoods W_1, W_2 of n and m respectively, and open τ-neighbourhoods $U, V \in \tau$ of x, y respectively, whose closures are disjoint. Furthermore, since $\mathcal{U}$ is a Urysohn family, we can assume that $U \cap Z \subseteq U_n$ and $V \cap Z \subseteq U_m$. Then $U' = h_n[U \cap Q_n]$ and $V' = h_m[V \cap Q_m]$ are open subsets of $\mathbb{Q}$ and the sets $U \cup O(U', \mathrm{int}_\rho(W_1))$ and $V \cup O(V', \mathrm{int}_\rho(W_2))$ are disjoint σ-neighbourhoods of x and y respectively whose closures are disjoint. If on the other hand, $x, y \in U_n$ for some $n \in \omega$, then there are neighbourhoods $U, V \in \tau$ of x, y respectively whose closures are disjoint and again we assume that $U \cap Z \subseteq U_n \supseteq V \cap Z$. Defining U' and V' as before we have that $U \cup O(U', \omega)$ and $V \cup O(V', \omega)$ are disjoint σ-neighbourhoods of x and y respectively whose closures are disjoint.

2) $x \in Z$, say $x \in U_n$, and $y \in X \setminus Z$. There exist neighbourhoods $U, V \in \tau$ of x, y respectively whose closures are disjoint and have the properties that $U \cap Z \subseteq U_n$, $V \cap U_m = \emptyset$ for all m and $\mathrm{cl}_\tau(V)$ meets only finitely many of the sets U_m, say $A = \{m : U_m \cap \overline{V} \neq \emptyset\}$. We choose an open neighbourhood W of n in (ω, ρ) whose closure misses the finite set $A \setminus \{n\}$. Then as before, defining $U' = h_n[U \cap Q_n]$ we have that $U \cup O(U', W)$ and V are open sets in (X, σ) containing x, y respectively and whose closures are disjoint.

3) If $x, y \in X \setminus Z$, then choose τ-open sets U and V such that $\mathrm{cl}_\tau(U) \cap \mathrm{cl}_\tau(V) = \emptyset$ and such that $A = \{n : \overline{U} \cap U_n \neq \emptyset\}$, $B = \{n : \overline{V} \cap U_n \neq \emptyset\}$ are both finite and $U \cap \mathrm{int}(U_n) = \emptyset = V \cap \mathrm{int}(U_n)$ for all n. Since both A and B are closed subsets of (ω, ρ) it follows that $\mathrm{cl}_\sigma(U) = \mathrm{cl}_\tau(U)$ and $\mathrm{cl}_\sigma(V) = \mathrm{cl}_\tau(V)$ are the required disjoint closed neighbourhoods of x and y respectively.

For each $q \in \mathbb{Q}$, consider the set $Y_q = \{h_n^{-1}[q] : n \in \omega\}$. It is clear that $(Y_q, \sigma|Y_q)$ is homeomorphic to (ω, ρ) and hence is connected. Since the space (ω, ρ) is countable, connected and Urysohn, it follows from [11; Lemma 1] that it is not Urysohn-closed and hence we can find an infinite Urysohn family $\mathcal{W} = \{W_n : n \in \omega\}$ in (ω, ρ). It is then clear that the infinite family

$$\mathcal{V} = \{V_n = \mathrm{cl}_\sigma(O(\mathbb{Q}, \mathrm{int}_\rho(W_n))) : n \in \omega\}$$

is a Urysohn family in (X, σ).

Enumerate a countable dense subset of $X \setminus Z$ as $\{x_n : n \in \omega, n > 0\}$ and let $\{K_n : n \in \omega\}$ be an infinite family of mutually disjoint infinite subsets of ω. For each n, $\{V_j : j \in K_n\}$ is a Urysohn family in (X, σ) and let $\mathcal{G}_j$ be the free Urysohn filter with base $\{\cup\{\mathrm{int}_\sigma(V_n) : n \in K_j \text{ and } n \geq m\} : m \in \omega\}$. Define a topology ν on X by requiring that $\mathcal{G}_j$ converge to x_j.

We omit the, by now familiar, proof that (X, ν) is a Urysohn space and we claim that (X, ν) is connected. To see this, note that for each $q \in \mathbb{Q}$, $x_1 \in \mathrm{cl}_\nu(Y_q)$ and hence $S = \cup\{Y_q : q \in \mathbb{Q}\} \cup \{x_1\}$ is connected. However, S is dense in (X, ν) and so this latter space is connected. Finally note that $\{V_j : j \in K_0\}$ is a Urysohn family in (X, ν) and hence this latter space is a Urysohn space with a countable network which is not Urysohn-closed. Applying Theorem 2.3, (X, ν) can be condensed onto a second countable connected Urysohn space. □

Combining Theorem 2.10 and Corollary 2.6 we have:

Theorem 2.11. *A disconnected Urysohn space with a countable network can be condensed onto a connected second countable Urysohn space if and only if it is not Urysohn-closed.*

Proof. It remains only to show that a disconnected Urysohn-closed space (X, τ) cannot be condensed onto a connected Urysohn space. However, if U is an open and closed subset of (X, τ), then by Remark 1.2, U and $X \setminus U$ are Urysohn-closed and hence each is closed in any weaker Urysohn topology. □

The space (X, τ) of Example 2.2 is first countable but neither feebly compact nor Urysohn-closed; however, we do not know whether it can be condensed onto a connected Urysohn space. However, we have:

Example 2.12. The space ω_1 with the order topology can not be condensed onto a connected (or even a dense-in-itself) Urysohn space.

Proof. Let μ denote the order topology on ω_1; if $\sigma \subset \mu$ is a Urysohn topology on ω_1 strictly weaker than μ, then there is some $\alpha \in \omega_1$ such that the open filter $\mathcal{U}$ of all open σ-neighbourhoods of α does not contain the open filter of all μ-neighbourhoods of α. We will show that (ω_1, σ) is Urysohn-closed and to this end, without loss of generality, we assume that μ and σ coincide on $\omega_1 \setminus \{\alpha\}$. Since (ω_1, σ) is a Urysohn space, $\mathcal{U}$ is a Urysohn filter in (ω_1, σ) and hence is a Urysohn filter base in (ω_1, μ). Since α is the unique cluster point of $\mathcal{U}$ in (ω_1, σ), the only possible cluster point of $\mathcal{U}$ in (ω_1, μ) is again α. However, since every initial closed interval of (ω_1, μ) is compact, it follows that every open filter with a unique cluster point in (ω_1, μ) which contains a bounded set, must converge. Thus every element of $\mathcal{U}$ must be unbounded. Let $\mathcal{G}$ be any free open filter on (ω_1, σ). Again, since initial closed intervals of (ω_1, σ) are compact, $\mathcal{G}$ can contain no bounded set and hence all elements of $\mathcal{G}$ are unbounded; but then, if $G \in \mathcal{G}$ and $U \in \mathcal{U}$ it follows that $\mathrm{cl}_\sigma(G) \cap \mathrm{cl}_\sigma(U) \supseteq \mathrm{cl}_\mu(G) \cap \mathrm{cl}_\mu(U) \neq \emptyset$, since closed unbounded subsets of (ω_1, μ) are not disjoint and so $\mathcal{G}$ is not a Urysohn filter. Thus (ω_1, σ) is Urysohn-closed and since each successor ordinal $\beta \in \omega_1$ distinct from α is isolated, it follows that (ω_1, σ) is disconnected. The result now follows by an argument identical to that used in the proof of Theorem 2.11. □

We note in passing that in the above example, if there is some $U \in \mathcal{U}$ which contains no final interval, then (ω_1, σ) is an example of a Urysohn-closed space which is neither regular nor H-closed. By way of a contrast, it was shown in [3] that (ω_1, μ) can be condensed onto a connected Hausdorff space.

The techniques used in this section to condense a second countable space (X, τ) onto a connected Urysohn space depend on the existence of a countably infinite Urysohn family in (X, τ). If $w(X) = \kappa > \omega$, then to apply the same methods, a Urysohn family of size κ would appear to be needed. This leads to the following problem which we state rather informally:

Problem 2.13. If X is a Urysohn space of weight κ, find conditions under which X possesses a Urysohn family of size κ.

3. The countable case

Recall that a (Hausdorff) space is *H-closed* if it is closed in every Hausdorff space in which it is embedded. It is well-known (and may easily be deduced from the results in [4; 3.12.5]) that Lindelöf (hence countable and second countable) Hausdorff spaces are H-closed if and only if they are feebly compact.

In general, a second countable Urysohn-closed space is not H-closed (the first example was given in [5]), but for countable spaces the following result was proved in [11]:

Theorem 3.1. *A countable Urysohn space is Urysohn-closed if and only if it is H-closed.*

Corollary 3.2. *A countable Urysohn space is Urysohn-closed if and only if it is feebly compact.*

Furthermore, it follows from [11; Lemma 1] that a countable Urysohn-closed space must have a dense set of isolated points and hence cannot be dense-in-itself. Thus a countable Urysohn space can be condensed onto a connected Urysohn space only if it is not Urysohn-closed and we have proved:

Theorem 3.3. *A countable Urysohn space can be condensed onto a connected (first countable) Urysohn space if and only if it is not feebly compact.*

Phrased another way,

Corollary 3.4. *A Urysohn topology on a countable set contains a (first countable) connected Urysohn subtopology if and only if it is not feebly compact.*

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