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WEAK AND SMALL WHITNEY PROPERTIES

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Abstract. Notions of weak, small, and small weak Whitney properties are introduced, and examples are constructed showing that all these notions are different. Several results on non-Whitney properties are strengthened to be not small or even not small weak Whitney ones.

In a larger part of research concerning hyperspace of subcontinua the most frequently used tools are Whitney maps and related notions. Among them we have Whitney properties, that is, properties carried from a continuum X onto the hyperspace $C(X)$ of its subcontinua, and Whitney reversible properties, that is, the ones carried from $C(X)$ onto X itself. Large parts of the books [9] and [13] are devoted to such properties. In particular, several variations of Whitney reversible properties are studied: for example, a strong Whitney reversible property in [13] and a sequential strong Whitney reversible property in [9]. In this paper we define and study similar variants for Whitney properties, and we show their relations to Whitney reversible properties. The introduced concepts give new tools with which to investigate attributes of the hyperspace of subcontinua of a given continuum.

A *continuum* means a compact, connected metric space, and a *mapping* means a continuous transformation. The symbol $\mathbb{Q}$ denotes the Hilbert cube. Given a continuum X with a metric d, we

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denote by $C(X)$ the hyperspace of all subcontinua of X equipped with the Hausdorff metric (see, e.g., [13, (0.1), p. 1 and (0.12), p. 10]); for a point $p \in X$ we put $C(p, X) = \{P \in C(X) : p \in P\}$; and $F_1(X)$ means the hyperspace of singletons of X. A *Whitney map* for $C(X)$ is a mapping $\mu : C(X) \to [0, \infty)$ such that:

(0.1) $\mu(A) < \mu(B)$ for every two $A, B \in C(X)$ such that $A \subset B$ and $A \neq B$;

(0.2) $\mu(A) = 0$ if and only if $A \in F_1(X)$.

For the concept and existence of a Whitney map see [9, Section 13, pp. 105–110]. For each $t \in [0, \mu(X)]$, the preimage $\mu^{-1}(t)$ is called a *Whitney level*. It is known that each Whitney level is a continuum; see [9, p. 159].

Let $\mathcal{P}$ be a topological property. We write $X \in \mathcal{P}$ to denote that a space X has the property $\mathcal{P}$. A property $\mathcal{P}$ is said to be:

— a *Whitney property* (*Wp*), provided that for each continuum X

(0.3) if $X \in \mathcal{P}$, then for all Whitney maps $\mu : C(X) \to [0, \infty)$ and for all $t \in [0, \mu(X))$, we have $\mu^{-1}(t) \in \mathcal{P}$ (see [9, Definition 27.1 (a), p. 232]);

— a *Whitney-reversible property* (*Wrp*), provided that for each continuum X the following implication holds:

(0.4) if for all Whitney maps $\mu : C(X) \to [0, \infty)$ and for all $t \in [0, \mu(X))$ we have $\mu^{-1}(t) \in \mathcal{P}$, then $X \in \mathcal{P}$ (see [9, Definition 27.1 (b), p. 232]);

— a *strong Whitney-reversible property* (*sWrp*), provided that for each continuum X

(0.5) if there is a Whitney map $\mu : C(X) \to [0, \infty)$ such that for all $t \in (0, \mu(X))$ we have $\mu^{-1}(t) \in \mathcal{P}$, then $X \in \mathcal{P}$ (see [9, Definition 27.1 (c), p. 232]);

— a *sequential strong Whitney-reversible property* (*ssWrp*), provided that for each continuum X

(0.6) if there are a Whitney map $\mu : C(X) \to [0, \infty)$ and a sequence $\{t_n : n \in \mathbb{N}\}$ in $(0, \mu(X))$ such that $\lim t_n = 0$ and for each $n \in \mathbb{N}$ we have $\mu^{-1}(t_n) \in \mathcal{P}$, then $X \in \mathcal{P}$ (see [9, Definition 27.1 (d), p. 233]).

In this paper we introduce three more associated concepts, and we investigate relations between them. Namely, a property $\mathcal{P}$ is said to be:

— a *weak Whitney property* (*wWp*), provided that for each continuum X there is a Whitney map $\mu : C(X) \to [0,\infty)$ such that the following implication holds:

(0.7) if $X \in \mathcal{P}$, then for each $t \in [0,\mu(X))$ we have $\mu^{-1}(t) \in \mathcal{P}$;

— a *small Whitney property* (*sWp*), provided that for each continuum X and for each Whitney map $\mu : C(X) \to [0,\infty)$ there is a number $s \in (0,\mu(X))$ such that ,

(0.8) if $X \in \mathcal{P}$, then for each $t \in [0,s)$ we have $\mu^{-1}(t) \in \mathcal{P}$;

— a *small weak Whitney property* (*swWp*), provided that for each continuum X there are a Whitney map $\mu : C(X) \to [0,\infty)$ and a number $s \in (0,\mu(X))$ such that ,

(0.9) if $X \in \mathcal{P}$, then for each $t \in [0,s)$ we have $\mu^{-1}(t) \in \mathcal{P}$.

A continuum X is said to be *Whitney stable* [9, p. 428], (*weak Whitney stable* [9, p. 454], *small Whitney stable, small weak Whitney stable*) provided that the property "to be homeomorphic to X" is a (Wp) (a (wWp), (sWp), (swWp), respectively).

Proposition 1. *A property $\mathcal{P}$ is a (ssWrp) if and only if non-$\mathcal{P}$ is a (sWp).*

Proof: First, assume that $\mathcal{P}$ is a (ssWrp), let X be a continuum with the property non-$\mathcal{P}$, and let $\mu : C(X) \to [0,\infty)$ be an arbitrary Whitney map. Suppose on the contrary that non-$\mathcal{P}$ is not (sWp). Then there is a sequence $t_n \in (0,\mu(X))$ such that $\lim t_n = 0$ and $\mu^{-1}(t_n) \in \mathcal{P}$. This contradicts the fact that $\mathcal{P}$ is a (ssWrp).

Second, assume that non-$\mathcal{P}$ is a (sWp), take a continuum X and an arbitrary Whitney map $\mu : C(X) \to [0,\infty)$. To show that $\mathcal{P}$ is a (ssWrp) suppose on the contrary that there is a sequence $t_n \in (0,\mu(X))$ such that $\lim t_n = 0$ and for each $n \in \mathbb{N}$ we have $\mu^{-1}(t_n) \in \mathcal{P}$ while $X \notin \mathcal{P}$. This contradicts that non-$\mathcal{P}$ is (sWp). □

As a consequence of Proposition 1 and of the fact that unicoherence is a (ssWrp) (see [9, 64.3, p. 292]), we get the following.

Corollary 2. *Non-unicoherence is a (sWp).*

In the next results we will use the following construction.

Construction 3. Consider a sequence of pointed continua $\{(P_n, p_n)\}$ such that

$$p_n \in P_n \quad \text{for each} \quad n \in \mathbb{N} \quad \text{and} \quad \lim \operatorname{diam} P_n = 0.$$

On a straight line segment S take a sequence of points s_n tending to an end point of S, and locate each P_n so that, for each $n \in \mathbb{N}$,

$$P_m \cap P_n = \emptyset \text{ for } m \neq n \text{ and } P_n \cap S = \{p_n\} = \{s_n\}.$$

Then the continuum

$$P = S \cup \bigcup \{P_n : n \in \mathbb{N}\} \tag{3.1}$$

is named a *bunch of continua* P_n.

To prove the next theorem recall the following concept (see [9, Definition 25.1, p. 216]). For a continuum X and its hyperspace $C(X)$, a Whitney map $\mu : C(X) \to [0, \infty)$ is called an *admissible Whitney map for* $C(X)$ provided that there is a homotopy $h : C(X) \times [0, 1] \to C(X)$ satisfying the following conditions:

(a) $h(A, 0) \in F_1(X)$ and $h(A, 1) = A$ for all $A \in C(X)$;
(b) if $\mu(h(A, t)) > 0$ for some $A \in C(X)$ and $t \in [0, 1]$, then $\mu(h(A, s)) < \mu(h(A, t))$ whenever $0 \leq s < t$.

Theorem 4. *The following implications are the only ones that hold between the conditions* $(Wp), (wWp), (sWp), (swWp)$*:*

$$\begin{array}{ccc} (Wp) & \Rightarrow & (wWp) \\ \Downarrow & & \Downarrow \\ (sWp) & \Rightarrow & (swWp) \end{array} \tag{4.1}$$

Proof: The implications follow from their definitions. To see that they are the only ones, consider the examples below.

1) The Hilbert cube $\mathbb{Q}$ is weak Whitney stable, while neither Whitney stable nor small Whitney stable. Indeed, since the cone over $\mathbb{Q}$ is homeomorphic to $\mathbb{Q}$, there is an admissible Whitney map for $C(\mathbb{Q})$; see [9, Example 25.10, (a), p. 218]. Thus, $\mathbb{Q}$ is weak Whitney stable according to [9, Theorem 25.3 (b), p. 217]. Further, $\mathbb{Q}$ is not Whitney stable; see [13, Theorem 14.42.1, p. 436]. To see that it is not small Whitney stable let us accept the following

notation.

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z \in [-1, \tfrac{1}{2}]\};$$
$$B = D \cap \{(x, y, z) \in \mathbb{R}^3 : z = \tfrac{1}{2}\} \quad \text{and} \quad p = (0, 0, -1) \in D.$$

Thus, D is a disk having the simple closed curve B as its relative boundary.

For each $n \in \mathbb{N}$ let D_n be a homeomorphic copy of D with $p_n \in D_n$ being the copy of p, and with $\operatorname{diam} D_n < \frac{1}{2^n}$. Then the bunch X of continua D_n as defined in (3.1) of Construction 3 is a compact absolute retract, so it is a Hilbert cube factor; i.e., $X \times \mathbb{Q}$ is homeomorphic to $\mathbb{Q}$; see [12, 6.8, p. 304].

Let d be the metric for X derived from $\mathbb{R}^3$, and let $\mu : C(X) \to [0, \infty)$ be the Whitney map for $C(X)$ constructed as in [9, 13.5, p. 108] from the metric d. For each $n \in \mathbb{N}$ let B_n be a copy of B in each D_n. Arguing as in [13, Theorem 14.42.1, pp. 436–440], one can show that for each $n \in \mathbb{N}$, the Whitney level $\mu^{-1}(\mu(B_n))$ is not contractible. Using [13, Lemma (*) in 14.42.1, p. 436], one can construct a Whitney map $\mu^* : C(X \times \mathbb{Q}) \to [0, \infty)$ such that for each point $q \in \mathbb{Q}$ the continuum $(\mu^*)^{-1}(\mu^*(B_n \times \{q\}))$ is not contractible, either. Thus, we have constructed a Whitney map μ^* for the hyperspace of subcontinua of the Hilbert cube $X \times \mathbb{Q}$ having non-contractible Whitney levels $(\mu^*)^{-1}(t_n)$ for a sequence of positive numbers t_n tending to zero. This shows that the Hilbert cube is not small Whitney stable.

Consequently, (wWp) implies neither (Wp) nor (sWp).

2) The $\sin(1/x)$-curve $\mathbb{S}$ is small Whitney stable, while not weak Whitney stable, and consequently, not Whitney stable. Really, let $\mu : C(\mathbb{S}) \to [0, \infty)$ be an arbitrary Whitney map for $C(\mathbb{S})$. If L is the limit segment of $\mathbb{S}$, and $s = \mu(L)$, then one can observe that for each $t \in [0, s)$ the Whitney level $\mu^{-1}(t)$ is homeomorphic to $\mathbb{S}$, while for $t \in [s, \mu(\mathbb{S}))$ it is an arc; compare [9, 27.5, p. 233]. Thereby, $\mathbb{S}$ is not weak Whitney stable and thus not Whitney stable; (see also [13, Theorem 14.41, p. 435]).

Therefore, (sWp) implies neither (Wp) nor (wWp).

3) In the Cartesian coordinates $(x_0, x_1, x_2, \dots)$ put

$$p = (0, 0, 0, \dots), \quad A = \{(x_0, 0, 0, \dots) : x_0 \in [0, 1]\},$$
$$\mathbb{Q} = \{0, x_1, x_2, \dots) : x_i \in [0, 1] \text{ for each } i \in \mathbb{N}\},$$

and let

$$X = A \cup \mathbb{Q}.$$

Thus, A is an arc, $\mathbb{Q}$ is a Hilbert cube, and X is the one-point union of A and $\mathbb{Q}$ such that $A \cap \mathbb{Q} = \{p\}$ and p is an endpoint of A.

Define homotopies $H : X \times [0,1] \to X$ by $H(x,t) = x \cdot t$ for $x \in X$ and $t \in [0,1]$ and $h : C(X) \times [0,1] \to C(X)$ by $h(P,t) = H(P,t)$ for $P \in C(X)$ and $t \in [0,1]$. Thus,

$$\text{(4.2)} \qquad \text{if } \ p \in A, \ \text{ then } \ p \in h(A,t) \ \text{ for each } \ t \in [0,1].$$

Let $\mu : C(X) \to [0,\infty)$ be the Whitney map as constructed in the proof of [9, Theorem 13.4, p. 107]. Observe that the homotopy H diminishes the distances of points, so it is the homotopy that is needed in the definition of an admissible Whitney map. Thus, μ is admissible. Then for each $t \in (0, \mu(\mathbb{Q}))$, the Whitney level $\mu^{-1}(t) \cap C(\mathbb{Q})$ is homeomorphic to $\mathbb{Q}$ [9, Theorem 25.3 (b), p. 217]. Further, for each $t \in [0, \mu(A))$ define

$$\mathcal{K}_1 = \mu^{-1}(t) \cap C(\mathbb{Q}),$$
$$\mathcal{K}_2 = \mu^{-1}(t) \cap C(p,X),$$
$$\mathcal{K}_3 = \mu^{-1}(t) \cap C(A),$$

and note that

$$\mu^{-1}(t) = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3.$$

We have already observed that $\mathcal{K}_1$ is homeomorphic to a Hilbert cube. We will show that $\mathcal{K}_2$ is also a Hilbert cube, the intersection $\mathcal{K}_1 \cap \mathcal{K}_2$ is a Hilbert cube, and $\mathcal{K}_1 \cap \mathcal{K}_2$ is a Z-set in $\mathcal{K}_1$, whence it will follow from [7, Theorem 1, p. 21] that the union $\mathcal{K}_1 \cup \mathcal{K}_2$ is homeomorphic to a Hilbert cube.

Note that $\mathcal{K}_3$ is an arc as a Whitney level of an arc, that the intersection $\mathcal{K}_3 \cap (\mathcal{K}_1 \cup \mathcal{K}_2)$ is a singleton being an end point of the arc $\mathcal{K}_3$, and the union $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$ is homeomorphic to X.

To show that $\mathcal{K}_2$ and $\mathcal{K}_1 \cap \mathcal{K}_2$ are Hilbert cubes we will use Toruńczyk's characterization theorem; see [9, Theorem 9.3, p. 79]. First, we will show that they are absolute retracts. Note that $C(p,X)$ is a Hilbert cube by [9, Exercise 15.15 (3), p. 126]. We will show that $\mathcal{K}_2$ is a retract of $C(p,X)$. To this aim accept the following notation.

For a given $P \in C(X)$ and $s > 0$, denote by $B(P, s)$ the closed ball of radius s around P in X; i.e., $B(P, s) = \{x \in X :$ there is $y \in P$ such that $d(x, y) \leq s\}$, where d stands for the metric in X.

Define $r : C(p, X) \to \mathcal{K}_2$ as follows. For a given $P \in C(p, X)$, if $\mu(P) \geq t$, then $r(P)$ is the only continuum of the form $h(P, s)$ such that $\mu(h(P, s)) = t$. If $\mu(P) < t$, choose $s > 0$ so that $\mu(B(P, s)) = t$ and put $r(P) = B(P, s)$. Note that r is a retraction, whence it follows that $\mathcal{K}_2$ is an absolute retract. Defining analogously $r' : C(p, \mathbb{Q}) \to \mathcal{K}_1 \cap \mathcal{K}_2$, we see that $\mathcal{K}_1 \cap \mathcal{K}_2$ is a retract of $C(p, \mathbb{Q})$; since $C(p, \mathbb{Q})$ is a Hilbert cube, again by [9, Exercise 15.15 (3), p. 126], we conclude that $\mathcal{K}_1 \cap \mathcal{K}_2$ is an absolute retract.

Now we will show that the identity mappings on $\mathcal{K}_2$ and on $\mathcal{K}_1 \cap \mathcal{K}_2$ are the uniform limits of Z-mappings. To this aim define a sequence of points $p_n \in \mathbb{Q}$ putting

$$p_n = (0, 0, \dots, 0, 1, 0, \dots),$$

where 1 is the nth coordinate of p_n, and observe that $Z_n = \{P \in C(p, X) : p_n \in P\}$ is a Z-set in $C(p, X)$. Similarly, $\mu^{-1}(t) \cap Z_n$ is a Z-set in $\mu^{-1}(t) \cap C(p, X) = \mathcal{K}_2$. For each $n \in \mathbb{N}$ define $f_n : \mathcal{K}_2 \to \mu^{-1}(t) \cap Z_n$ by $f_n(P) = r(P \cup pp_n)$, where pp_n is the straight line segment in $\mathbb{Q}$ from p to p_n. Observe that the mappings f_n are Z-mappings and that they converge uniformly to the identity on $\mathcal{K}_2$. Thus, $\mathcal{K}_2$ is a Hilbert cube.

Analogously, for each $n \in \mathbb{N}$, the intersections $Z_n \cap C(\mathbb{Q})$ are Z-sets in $C(p, \mathbb{Q})$ and $\mu^{-1}(t) \cap Z_n \cap C(\mathbb{Q})$ are Z-sets in $\mu^{-1}(t) \cap C(p, \mathbb{Q}) = \mathcal{K}_1 \cap \mathcal{K}_2$. Define $f'_n : \mathcal{K}_1 \cap \mathcal{K}_2 \to \mu^{-1}(t) \cap Z_n \cap C(\mathbb{Q})$ by $f'_n(P) = r'(P \cup pp_n)$. Again the identity on $\mathcal{K}_1 \cap \mathcal{K}_2$ is the uniform limit of the mappings f'_n which are Z-mappings in $\mathcal{K}_1 \cap \mathcal{K}_2$, and thus $\mathcal{K}_1 \cap \mathcal{K}_2$ is the Hilbert cube.

The last thing to observe is that $\mathcal{K}_1 \cap \mathcal{K}_2$ is a Z-set in $\mathcal{K}_1$. Thus all the assumptions of [7, Theorem 1, p. 21] are fulfilled, so $\mathcal{K}_1 \cup \mathcal{K}_2$ is a Hilbert cube.

To see that X is not small Whitney stable let us recall that in part 1) above there was shown that there are a Whitney map μ^* for $C(\mathbb{Q})$ and a sequence of positive numbers t_n tending to zero such that $(\mu^*)^{-1}(t_n)$ was not contractible. Extending this Whitney map μ^* to a Whitney map $\mu : C(X) \to [0, \infty)$ we get a Whitney map for $C(X)$ having noncontractible Whitney levels $\mu^{-1}(t_n)$ for a sequence of positive numbers t_n tending to zero.

To see that X is not weak Whitney stable, take an arbitrary Whitney map $\mu : C(X) \to [0, \infty)$ and let $t \in (\mu(A), \mu(X))$. Then each element P of $\mu^{-1}(t)$ is the center of an ∞-od in X, and thus $\mu^{-1}(t)$ is infinite dimensional at P by [13, Theorem 14.33, p. 430]. Therefore, $\mu^{-1}(t)$ is not homeomorphic to X, so X is not weak Whitney stable, as needed.

Thus, (swWp) implies neither (sWp) nor (wWp).

Consequently, the implications shown in diagram (4.1) are the only ones that hold between the four conditions. □

Some properties of continua are known not to be Whitney properties; see, e.g., [9, Table Summarizing Chapter VIII, pp. 294-298]. For these properties it is interesting to know if they are small, weak, or small weak Whitney properties. In some cases an answer can be deduced just from known proofs (or examples) that concern (Wp). The next theorem collects information about this. The reader is referred to [9] for the needed definitions.

Theorem 5. *The cited proofs showing that the following are not Whitney properties show even more, namely that the properties are not* ($swWp$)*:*

(5.1) *being an arc-continuum* [3, Example 1, p. 636]*;*
(5.2) *atriodicity* [16, Example 5.6, p. 583]*; see also* [9, 34.1, p. 251]*;*
(5.3) *being* C^**-smooth* [1, Remark 2.10, p. 309]*;*
(5.4) *circle-likeness* [9, 39.4, p. 260, and Fig. 40, p. 261]*;*
(5.5) δ*-connectedness* [2, Example 5, p. 387]*; compare* [9, 51.2, p. 279]*;*
(5.6) *covering property = being in Class* (W) *= being absolute* C^**-smooth* [6, Example 4.5, p. 383]*;* [2, Example 5, p. 387]*; compare* [9, 51.2, p. 279, and 67.1, p. 320]*;*
(5.7) *indecomposability* [9, 44.14, p. 268]*;*
(5.8) *irreducibility* [4, Example 3.2, p. 362]*; compare* [9, 49.8, p. 276]*;*
(5.9) *planarity* [9, 54.4, p. 284] *(consider a simple triod);*
(5.10) *unicoherence* [10, Example 5.4, p. 178]*;* [16, Example 5.6, p. 583]*; compare* [13, Example 14.12, p. 413] *and* [9, 64.1, p. 292].

Theorem 6. *Having a cut point is not a* ($swWp$).

Proof: Take an arbitrary dendrite X containing no free arc (equivalently, having the set $E(X)$ of its end points dense in X. You may take as X the Ważewski universal dendrite D_ω; see, e.g., [14, 10.37, p. 181]). Thus, each point of $X \setminus E(X)$ is a cut point of X; see [14, Theorem 10.7, p. 168]. On the other hand, for each Whitney map small Whitney levels are homeomorphic to the Hilbert cube; see [9, 25.3 (b), p. 217] and [5, Theorem 2.17, p. 680]. □

Theorem 7. *Contractibility is not a* $(swWp)$.

Proof: In [8, Section 2, Theorem 2.1, p. 1072], a contractible continuum X is constructed with a property that for each Whitney map there is a positive Whitney level which is not contractible. Take a bunch P of continua (see Construction 3) such that each P_n is homeomorphic to X. Then P is a continuum having the needed properties. Evidently, P is contractible. To see that for an arbitrary Whitney map $\mu : C(P) \to [0, \infty)$ there are arbitrarily small numbers $t > 0$ such that $\mu^{-1}(t)$ is not contractible, take, for each $n \in \mathbb{N}$, a retraction $r_n : P \to P_n$ such that $r_n(P \setminus P_n) = \{p_n\}$ (where $p_n \in P_n$ is the distinguished point, as in Construction 3), and note that r_n does not increase the values of μ; i.e.,

(7.1) $\mu(r_n(K)) \le \mu(K)$ for each continuum $K \in C(P)$.

Since P_n is homeomorphic to X, there is $t_n \in (0, \mu(P))$ such that the continuum $\mu^{-1}(t_n) \cap C(P_n)$ is noncontractible. Obviously the sequence t_n tends to 0 as $n \to \infty$. Let $\mathcal{A}$ be an order arc from $\{p_n\}$ to $\{P_n\}$ in $C(P)$. For each $n \in \mathbb{N}$ define a retraction $\rho_n : \mu^{-1}(t_n) \to \mu^{-1}(t_n) \cap C(P_n)$ by the condition $\rho(A) = r_n(A) \cup B$, where $B \in \mathcal{A}$ is determined by $\mu(r_n(A) \cup B) = t_n$. It follows from (7.1) that ρ_n is well defined. Then $\mu^{-1}(t_n)$ is not contractible since contractibility is an invariant property under any retraction; see, e.g., [11, §54, V, Theorem 3, p. 371]. The proof is complete. □

Let us consider again the bunch D of continua D_n as in part 1) of the proof of Theorem 4. Note that D is an absolute retract, while the continua $\mu^{-1}(t_n)$ have essential mappings onto the 2-sphere. This example shows the following.

Theorem 8. *The following properties are not* (sWp)*:*

(8.1) *being an absolute retract;*

(8.2) *acyclicity;*

(8.3) *arc-smoothness;*
(8.4) *n-connectedness for each integer $n > 1$;*
(8.5) *having a trivial shape.*

Theorem 9. *The following properties are not (sWp):*

(9.1) *being an absolute neighborhood retract;*
(9.2) *local contractibility;*
(9.3) *local n-connectedness for each integer $n > 1$.*

Proof: In [15, Example 5, p. 278], a 2-cell X and a Whitney map $\widehat{\mu} : C(X) \to [0, \infty)$ are constructed such that there is a number $t \in (0, \widehat{\mu}(X))$ for which $\widehat{\mu}^{-1}(t)$ is not locally contractible. Take a bunch P of continua P_n such that each P_n is homeomorphic to X and the distinguished point p_n lies in the circle A_0 (in the definition of X). Let $\mu : C(P) \to [0, \infty)$ be a Whitney map such that $\mu | C(P_n) = \frac{1}{n}\hat{\mu}$. Then P is an absolute retract while there is a sequence of positive numbers t_n tending to zero such that $\mu^{-1}(t_n)$ is not locally 2-connected. □

Question 10. Consider the following properties of continua:

(10.1) being an absolute retract;
(10.2) being an absolute neighborhood retract;
(10.3) having a trivial shape.

Which one of them is (wWp)?

Remark 11. It has been observed in [9, Exercise 34.8, p. 253] that being an n-od is a (sWp).

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