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**ALMOST CONTINUOUS RETRACTS AND THE
FIXED POINT PROPERTY FOR CERTAIN
NON-SEPARATING PLANE CONTINUA**

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ABSTRACT. Suppose M is a non-separating plane continuum. There is a ray S lying in the complement of M and converging to the boundary of M , such that the continuum $M \cup S$ has the fixed point property, and there is a continuum N such that N is an almost continuous retract of a disk and M is an almost continuous retract of N .

1. INTRODUCTION

In [8], J. Stallings described, though he did not name, almost continuous retracts (also called quasi-retracts in [1]) and suggested the possibility of solving the problem of whether or not every non-separating plane continuum has the fixed point property by finding that every such continuum is an almost continuous retract of a disk. In 1984, V. N. Akis [1] showed that solution to be impossible by proving that a circular disk with a spiral limiting to its boundary is not an almost continuous retract of a disk. Still, almost continuous retracts have shown to be useful tools in the study of the fixed point property [1], [2], [6], [7] and investigations of the properties of almost continuity have themselves been interesting [4], [3], [2]. The results below are to offer two principal types of results: first, a fixed point result and second, an interesting property of almost continuous retracts and non-separating plane continua. First, though Akis

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showed that a spiral to the boundary of a disk is not an almost continuous retract, we show that, if M is any non-separating plane continuum, there is a ray, S , in $\mathbb{R}^2 \setminus M$, whose limit is the boundary of M , and the continuum $M \cup S$ is an almost continuous retract of a disk; thereby, $M \cup S$ has the fixed point property. Second, if M is any non-separating plane continuum, then there is a non-separating plane continuum N such that M is an almost continuous retract of N and N is an almost continuous retract of a disk (perhaps regenerating the possibility of using almost continuous retracts to attack the problem of whether or not every non-separating plane continuum has the fixed point property).

2. NOTATION AND TERMINOLOGY

Generally, unless otherwise stated or inferred or made clear from context, notation and terminology are of standard usage. Thus, \mathbb{R} denotes the real line; I is the closed unit interval, $[0, 1]$; its product is $I^2 = I \times I$; and the plane is \mathbb{R}^2 . We confine our interest herein only to subsets of the plane and, while the results and properties claimed herein may be stated only for that case, their generalization to other spaces may be known or may be obvious. A continuum is a bounded, closed, and connected subset of the plane and a continuum with exactly two non-cut points is an arc. That D is a disk means D is homeomorphic to I^2 . A space, X , has the fixed point property if, for every continuous function $f : X \rightarrow X$, there is an x such that $f(x) = x$. The distance from A to B is $d(A, B)$ and $\text{Cl}(X)$, $\text{Bd}(X)$, and $\text{Int}(X)$ are, respectively, the closure, the boundary, and the interior of the set X . A function and its graph are identical, so that $f : X \rightarrow Y$ is a subset of $X \times Y$, no two ordered pairs of which have the same first term. The subset Y of X is a retract of X if there is a continuous function $f : X \rightarrow Y$ such that $f(y) = y$, for each y in Y .

3. DEFINITIONS AND PROPOSITIONS

The definitions and propositions below are stated to hold for subsets of the plane, which is where we will be using them, but may not be stated exactly or as generally as they are in the referenced sources.

Definition 1. A function $f : X \rightarrow Y$ is said to be almost continuous if, whenever U is an open set of $X \times Y$ and $f \subset U$, then there is a continuous $g : X \rightarrow Y$, with $g \subset U$.

Definition 2. The subset Y of X is an almost continuous retract of X provided that there is an almost continuous function $f : X \rightarrow X$ such that $f(X) = Y$ and for every y in Y , $f(y) = y$.

Proposition 1 ([8]). If X has the fixed point property and $f : X \rightarrow X$ is almost continuous, then for some x , $f(x) = x$.

Proposition 2 ([8]). A continuum that is an almost continuous retract of a disk has the fixed point property.

Proposition 3 ([1], [7]). If $X = X_0$ is a continuum and X_0, X_1, X_2, \dots is a sequence of subsets of X such that, for each positive integer i , $f_i : X_{i-1} \rightarrow X_i$ is a retraction onto X_i and, for any point x of X_{i-1} such that $f_i(x)$ is not x , the point $f_i(x)$ lies in every X_j , $j = 1, 2, 3, \dots$, then the continuum that is the common part of X_0, X_1, X_2, \dots is an almost continuous retract of X .

4. ARGUMENTS

Theorem 1. *Suppose M is a non-separating plane continuum. There is a ray S lying in $\mathbb{R}^2 \setminus M$ so that $\text{Cl}(S) \setminus S = \text{Bd}(M)$ and such that the continuum $M \cup S$ is an almost continuous retract of a disk.*

Proof: There is no loss of generality in considering the non-separating plane continua we are concerned with to be lying in the unit square. Our proof involves constructing a sequence of continua, X_0, X_1, X_2, \dots , and a sequence of retractions, f_1, f_2, f_3, \dots , such that, for each positive integer i , $f_i : X_{i-1} \rightarrow X_i$ is a retraction onto X_i and, for any point x of X_{i-1} such that $f_i(x)$ is not x , the point $f_i(x)$ lies in every X_j . Then, provided X_0 is a disk, by Proposition 3, the continuum that is left after all of the retractions, $X_0 \cap X_1 \cap X_2 \cap \dots$, will be an almost continuous retract of a disk.

We begin by imbedding the non-separating plane continuum M in the unit square so that, except for its point $P = (1/2, 1)$, the continuum M is a subset of $\text{Int}(I^2)$.

For use in the construction of these continua, we will first describe a sequence of arcs, $A_0, A_1, A_2, A_3, \dots$, in $I^2 = X_0$, converging to the

boundary of the non-separating plane continuum M . In addition to the point P , we will name several other objects in I^2 , beginning as follows.

$$D_0 = I^2 = X_0.$$

The point $(0, 1)$ is L_0 ; for each positive integer i , $l_i = (1/2 - 1/2^{i+1})$ and L_i is the point $(l_i, 1)$.

The point $(1, 1)$ is R_0 ; for each positive integer i , $r_i = (1/2 + 1/2^{i+1})$ and R_i is the point $(r_i, 1)$.

For $i = 1, 2, 3, \dots$, the horizontal interval from L_i to R_i is denoted by H_i ; the horizontal interval from L_{i-1} to L_i is E_i ; and the horizontal interval from R_{i-1} to R_i is F_i .

The arc A_0 is the arc in $\text{Bd}(I^2)$ with endpoints L_0 and R_0 and containing the origin. Note that the disk with boundary consists of the arcs A_0 and H_0 is $D_0 = I^2$.

Using the fact that a non-separating plane continuum is the intersection of a monotonic sequence of disks containing it, along with the fact that a disk is arcwise connected, we will construct arcs A_1, A_2, A_3, \dots as follows.

The arc A_1 has end points L_1 and R_1 , lies otherwise in $\text{Int}(D_0) \setminus M$, and $d(x, M) \leq 1/4$, for x in A_1 . Denote by D_1 the disk with boundary $A_1 \cup H_1$.

The arc A_2 has end points L_2 and R_2 , lies otherwise in $\text{Int}(D_1) \setminus M$, and $d(x, M) \leq 1/8$, for x in A_2 . The disk with boundary $A_2 \cup H_2$ is D_2 .

Continuing in this manner, for each positive integer i , we define an arc A_i with end points L_i and R_i ; the arc A_i is a subset of $\text{Int}(D_{i-1}) \setminus M$, and $d(x, M) \leq (1/2^{i+1})$, for x in A_i . The disk with boundary $A_i \cup H_i$ is D_i .

For each integer $1, 2, 3, \dots$, the disk G_i is the disk with boundary $A_{i-1} \cup A_i \cup E_i \cup F_i$.

A retraction of X_0 that leaves every point of X_0 fixed that is not a point of $\text{Int}(G_1)$ or a non-endpoint of F_1 and moves those points to $A_0 \cup A_1 \cup E_1$ is the function $f_1 : X_0 \rightarrow X_1$. We see that the continuum X_1 is the disk D_1 , bounded by $A_1 \cup H_1$, plus the arc β_1 , beginning at $(1, 1)$, moving down along A_0 through $(1, 0)$ and $(0, 0)$ and up to $(0, 1)$, then to the right along the horizontal arc E_1 to the point L_1 of the disk D_1 ; ($\beta_1 = A_0 \cup E_1$).

The retraction $f_2 : X_1 \rightarrow X_2$ moves any non-endpoint of the arc E_2 and any point of $\text{Int}(G_2)$ to a point of $A_1 \cup A_2 \cup F_2$, and

otherwise leaves points of X_1 fixed. The continuum X_2 is the disk D_2 , bounded by $A_2 \cup H_2$, plus the arc β_2 , beginning at $(1, 1)$, then moving along A_0 to where it meets E_1 , along E_1 to its common end with the arc A_1 , along A_1 to R_1 , and along F_2 to where it meets A_2 ; ($\beta_2 = A_0 \cup E_1 \cup A_1 \cup F_2$).

The retraction $f_3 : X_2 \rightarrow X_3$ pushes all points of $\text{Int}(G_3)$ and non-endpoints of F_3 onto $A_2 \cup A_3 \cup E_3$ but leaves all other points of X_2 fixed. The continuum X_3 is the disk D_3 , bounded by $A_3 \cup H_3$, plus the arc beginning at $(1, 1)$, moving along A_0 , then along E_1 to the arc A_1 , along A_1 to F_2 , along F_2 to A_2 to E_3 , and along E_3 to D_3 ; ($\beta_3 = A_0 \cup E_1 \cup A_1 \cup F_2 \cup A_2 \cup E_3$).

Inductively then, we construct a sequence of subcontinua X_1, X_2, X_3, \dots , and a sequence of retractions $f_i : X_{i-1} \rightarrow X_i$, for $i = 1, 2, 3, \dots$; each f_i is a retraction of X_{i-1} onto X_i so that,

- (1) for each odd positive integer i , the function f_i moves each point of $\text{Int}(G_i)$ and each non-endpoint of F_i onto the set $A_{i-1} \cup A_i \cup E_i$, but moves no other point and,
- (2) for an even positive integer i , the function $f_i : X_{i-1} \rightarrow X_i$ moves any non-endpoint of the arc E_i and the points of $\text{Int}(G_i)$ to a point of $A_{i-1} \cup A_i \cup F_i$ and otherwise leaves points of X_2 fixed.

Each of sets X_0, X_1, X_2, \dots is a continuum; their intersection is a continuum N , which can be seen to be the union of the non-separating continuum M and a ray $S = \beta_0 \cup \beta_1 \cup \beta_2 \cup \dots$, which converges to the boundary of M .

By Proposition 3, the continuum $M \cup S$ is an almost continuous retract of I^2 . \square

Corollary 1. *If M is a non-separating plane continuum, there is a ray S converging to the boundary of M such that the continuum $M \cup S$ has the fixed point property.*

Proof: With M and S as in Theorem 1, apply Proposition 2. \square

Theorem 2. *If M is a non-separating plane continuum, there is a sequence A_0, A_1, A_2, \dots of arcs, with one common endpoint P in M , such that*

- (1) for $i \neq j$ in $0, 1, 2, \dots$, except for the point P , the arcs A_i and A_j are mutually exclusive and neither contains a point of M other than P ,

- (2) the sequence A_0, A_1, A_2, \dots converges to the boundary of M , and
- (3) the continuum $M \cup A_0 \cup A_1 \cup A_2 \cup \dots$ is an almost continuous retract of a disk.

Proof: With certain variations, we will begin this argument by performing the first part of the construction from the proof of Theorem 1. The definitions and constructions of the proof of Theorem 1, through the construction of the sequence A_1, A_2, A_3, \dots of arcs, will hold here except that, for each integer $i = 0, 1, 2, \dots$, the point $L_i = (1/2, 1)$. Thus, where, in the proof of Theorem 1, for each $i = 0, 1, 2, \dots$, E_i was a horizontal interval, in the current proof, E_i is the point $(1/2, 1)$. Now, after completing the definitions of A_0, A_1, A_2, \dots , we have a sequence of arcs such that, if i and j are in $1, 2, 3, \dots$,

- (1) except for its endpoints, A_i lies in $\text{Int}(I^2)$,
- (2) except for their common endpoint P , which lies in M , the arcs A_i and A_j do not intersect and neither contains a point of M ,
- (3) the other endpoint of A_i , $i = 1, 2, 3, \dots$, is the point $(1/2 + 1/2^{i+1}, 1)$, and
- (4) if x is in A_i , then $d(x, M) \leq (1/2^{i+1})$.

We will construct the sequences of continua X_0, X_1, X_2, \dots and retractions f_1, f_2, f_3, \dots we need in order to apply Proposition 3. In this construction we begin each retraction with one of the horizontal intervals F_1, F_2, F_3, \dots of $[1/2, 1] \times \{1\}$, between endpoints of successive pairs of arcs in A_0, A_1, A_2, \dots . Again, $I^2 = X_0$ and, now, A_0 is the arc in $\text{BD}(I^2)$ from $(1, 1)$ to $(1/2, 1)$ that contains $(0, 0)$.

The retraction $f_1 : X_0 \rightarrow X_1$ moves all the points of I^2 that are between A_0 and A_1 (in I^2) onto those two arcs, but leaves all other points of $I^2 = X_0$ in place.

The retraction $f_2 : X_1 \rightarrow X_2$ moves all the points of I^2 that are between A_1 and A_2 (in I^2) onto those two arcs, but leaves all other points of X_1 in place.

Generally, the retraction $f_i : X_{i-1} \rightarrow X_i$ moves all the points of I^2 that are between A_{i-1} and A_i onto those two arcs, but leaves all other points of X_{i-1} in place.

The result of this construction is a continuum N , which is the sum of the non-separating continuum M we started with and the arcs of the sequence A_0, A_1, A_2, \dots . By Proposition 3, the continuum N is an almost continuous retract of I^2 . \square

Corollary 2. *The continuum $M \cup A_0 \cup A_1 \cup A_2 \cup \dots$ of Theorem 2 has the fixed point property.*

Proof: Via Proposition 2 and Proposition 3. \square

Theorem 3. *If M is a non-separating plane continuum, then M is an almost continuous retract of a continuum that is an almost continuous retract of a disk.*

Proof: Suppose M and the arcs A_0, A_1, A_2, \dots are as was constructed in Theorem 2 and suppose we denote the continuum that is their sum by X_0 . The retraction $f_1 : X_0 \rightarrow X_1$ leaves all of X_0 fixed, except for points of A_0 that are not the end point P in M , and moves those points to P . The retraction $f_2 : X_1 \rightarrow X_2$ leaves all of X_1 fixed, except for points of A_1 that are not the end point P in M , and moves those points to P . The retraction $f_3 : X_2 \rightarrow X_3$ leaves all of X_2 fixed, except for points of A_2 that are not the end point P in M , and moves those points to P . Continuing this process, for each $i = 1, 2, 3, \dots$, the retraction $f_i : X_{i-1} \rightarrow X_i$ leaves all of X_{i-1} fixed, except for points of A_i that are not the end point P in M , and moves those points to P . After this process, the remaining continuum is the non-separating continuum M , which is, by Proposition 3, an almost continuous retract of the continuum X_0 , which is by Theorem 2, an almost continuous retract of a disk. \square

5. SOME OBSERVATIONS AND PROBLEMS

It is known that, for a sufficiently “nice” space X (such as an n -cell), any continuum that is an almost continuous retract of X will have the fixed point property. But the behavior of the almost continuous retracts isn’t so predictable or friendly as that of the retracts, where continua will inherit the fixed point property from a continuum from which they were retracted. The continuum N , constructed in Theorem 2, is a retract of a disk, so we know it has the fixed point property. But the continuum N need not be “nice” enough that we know it to pass the fixed point property to its subcontinua (such as the non-separating plane continuum M)

that are almost continuous retracts of N . We don't know what conditions would guarantee that the fixed point property would be inherited by almost continuous retracts or what conditions might allow a continuum without the fixed point property to be an almost continuous retract of a continuum with the fixed point property. Indeed, it is known that there is a plane continuum X with the fixed point property, having an almost continuous retract Y which is a continuum without the fixed point property [2]. But we point out that both that continuum X and that subcontinuum Y separate the plane. When a plane continuum with the fixed point property does not separate the plane, it is not known if a continuum that is its almost continuous retract must have the fixed point property (and, certainly, no example is known of a non-separating plane continuum with a subcontinuum that is an almost continuous retract but without the fixed point property). It is obvious, by the construction, that the continuum N , constructed in Theorem 2 does not separate the plane. (H. Rosen [7] has shown more generally that any almost continuous retract of an n -cell cannot separate the n -cell.)

Question 1. Is it possible that a continuum which is an almost continuous retract of the continuum N (constructed in Theorem 2) must have the fixed point property?

If the answer is yes, that would, via Corollary 2, show that every non-separating plane continuum has the fixed point property. On the other hand, if there is a non-separating plane continuum without the fixed point property, the answer to Question 1 is clearly negative. So, what we have is a problem equivalent to the problem of whether or not non-separating plane continua have the fixed point property.

The next two questions are generalizations of Question 1 and a positive answer to either clearly gives a positive answer to Question 1.

Question 2. If N is a continuum that is an almost continuous retract of a disk and the continuum M is an almost continuous retract of N , does M have the fixed point property?

Question 3. If M is a continuum which is an almost continuous retract of a non-separating plane continuum N with the fixed point property, does M have the fixed point property?

Some years ago, I thought I had extended Rosen's result from [7], so that a continuum that is an almost continuous retract of a continuum that is an almost continuous retract of an n -cell could not separate that n -cell. I can not find the notes from those years and can not now produce an argument to settle the following questions.

Question 4. If the continuum Y is an almost continuous retract of a continuum X , which is an almost continuous retract of an n -cell C can Y separate C ?

Question 5. If N is a plane continuum that is an almost continuous retract of a disk and M is a continuum that is an almost continuous retract of N , is M a non-separating plane continuum?

Question 6. In order that the continuum M be a non-separating plane continuum, is it necessary and sufficient that M be an almost continuous retract of a continuum that is an almost continuous retract of a disk?

Remark. The referee, via the editor, supplied a copy of an unpublished manuscript [5] by S. D. Iliadis, University of Patras, Department of Mathematics, which reaches results analogous to those of Theorem 2, herein. However, that manuscript does not make use of almost continuity.

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