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INDUCED MAPS ON n–FOLD HYPERSPACE SUSPENSIONS

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Dedicated to Professor Janusz J. Charatonik

ABSTRACT. For a given map between continua, we begin the study of induced maps on the n–fold hyperspace suspensions.

1. INTRODUCTION

In [11], the hyperspace suspension of a continuum was introduced, and it was shown that it has the fixed point property for the class of chainable continua. In [4], a more detailed study of the hyperspace suspension of continua was done, and in [7], n–fold hyperspace suspensions were introduced and studied. As was done with hyperspaces (see, for example, [10], [5], and [2]), it is natural to study the induced maps on the n–fold hyperspace suspensions.

In section 2, we give the necessary definitions for the paper. In section 3, we present the main theorems of the paper and pose some questions.

2. DEFINITIONS

If (Z, d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$, the interior of A is

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denoted by $Int_Z(A)$, and the closure of A is denoted by $Cl_Z(A)$. A *map* means a continuous function.

A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* is a continuum contained in a space Z. An *arc* is any space homeomorphic to $[0,1]$.

An onto map $f\colon X \to Y$ between continua is said to be:

ε–map if diameter $(f^{-1}(y)) < \varepsilon$ for each $y \in Y$;

monotone if $f^{-1}(y)$ is connected for every $y \in Y$;

weakly confluent provided that for each subcontinuum Q of Y, there exists a subcontinuum K of X such that $f(K) = Q$;

refinable provided that for each $\varepsilon > 0$, there exists an ε–map $g\colon X \to Y$ such that $d(f(x), g(x)) < \varepsilon$ for every $x \in X$;

monotonely refinable if for each $\varepsilon > 0$, there exists a monotone ε–map $g\colon X \to Y$ such that $d(f(x), g(x)) < \varepsilon$ for every $x \in X$.

Given a continuum X and a positive integer n, $\mathcal{C}_n(X)$ denotes the *n–fold hyperspace* of X; that is:
$\mathcal{C}_n(X) = \{A \subset X \mid A$ nonempty, closed, with $\leq n$ components$\}$ topologized with the Hausdorff metric, which is defined as follows:

$$\mathcal{H}_X(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}^d_\varepsilon(B) \text{ and } B \subset \mathcal{V}^d_\varepsilon(A)\}.$$

$\mathcal{H}_X$ always denotes the Hausdorff metric on the n–fold hyperspace of a continuum X.

The symbol $\mathcal{F}_n(X)$ denotes the *n–fold symmetric product* of X; that is:

$$\mathcal{F}_n(X) = \{A \subset X \mid A \text{ has at most } n \text{ points}\}.$$

Note that $\mathcal{F}_n(X) \subset \mathcal{C}_n(X)$. It is known that $\mathcal{C}_n(X)$ is an arcwise connected continuum (for $n = 1$, see (1.12) of [10]; for $n \geq 2$, see 3.1 of [6]).

An *order arc* in $\mathcal{C}_n(X)$ is an arc $\alpha\colon [0,1] \to \mathcal{C}_n(X)$ such that if $0 \leq s < t \leq 1$, then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$.

By the *n–fold hyperspace suspension* of a continuum X, denoted by $HS_n(X)$, we mean the quotient space:

$$HS_n(X) = \mathcal{C}_n(X)/\mathcal{F}_n(X)$$

with the quotient topology. The fact that $HS_n(X)$ is a continuum follows from 3.10 of [13]. Notice that $HS_1(X)$ corresponds to the hyperspace suspension $HS(X)$ defined by S. B. Nadler, Jr. in [11].

Notation 2.1. Given a continuum X, $q_X^n\colon \mathcal{C}_n(X) \to HS_n(X)$ denotes the quotient map. Also, let F_X^n denote the point $q_X^n(\mathcal{F}_n(X))$.

Remark 2.2. Note that $HS_n(X)\setminus\{F_X^n\}$ and $HS_n(X)\setminus\{q_X^n(X), F_X^n\}$ are homeomorphic to $\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$ and $\mathcal{C}_n(X) \setminus (\{X\} \cup \mathcal{F}_n(X))$, respectively, using the appropriate restriction of q_X^n.

Given a map $f\colon X \to Y$ between continua and a positive integer n, the function $\mathcal{C}_n(f)\colon \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ given by $\mathcal{C}_n(f)(A) = f(A)$ is the *induced map by f between the n–fold hyperspaces of X and Y*. Note that $\mathcal{C}_n(f)$ is continuous (this follows easily from 13.3 of [5]). Also, we have an induced map $HS_n(f)\colon HS_n(X) \to HS_n(Y)$, given by

$$HS_n(f)(A) = \begin{cases} q_Y^n(f((q_X^n)^{-1}(A))), & \text{if } A \neq F_X^n; \\ F_Y^n, & \text{if } A = F_X^n; \end{cases}$$

called the *induced map by f between the n–fold hyperspace suspensions of X and Y*. Note that, by 4.3 on page 126 of [3], $HS_n(f)$ is continuous. In addition, the following diagram

$$\begin{array}{ccc} \mathcal{C}_n(X) & \xrightarrow{\mathcal{C}_n(f)} & \mathcal{C}_n(Y) \\ {\scriptstyle q_X^n}\downarrow & & \downarrow{\scriptstyle q_Y^n} \\ HS_n(X) & \xrightarrow[HS_n(f)]{} & HS_n(Y) \end{array} \qquad (*)$$

is commutative.

3. Main Results

We begin with a proposition.

Proposition 3.1. *If $f\colon X \to Y$ is a surjective map between continua, and n is a positive integer, then both $HS_n(f)^{-1}(F_Y^n)$ and $HS_n(f)^{-1}(q_Y^n(Y))$ are arcwise connected subcontinua of $HS_n(X)$.*

Proof: Let $\chi \in HS_n(f)^{-1}(F_Y^n) \setminus \{F_X^n\}$. Then, by the commutativity of $(*)$, $\mathcal{C}_n(f)((q_X^n)^{-1}(\chi)) \in \mathcal{F}_n(Y)$. For each component C of $(q_X^n)^{-1}(\chi)$, let $x_C \in C$. Note that the set $\{x_C \mid C \text{ is a component of } (q_X^n)^{-1}(\chi)\}$ belongs to $\mathcal{F}_n(X)$ and, by (1.8) of [10], there exists an order arc $\gamma\colon [0,1] \to \mathcal{C}_n(X)$ such that $\gamma(0) = \{x_C \mid C \text{ is a component of } (q_X^n)^{-1}(\chi)\}$ and $\gamma(1) = (q_X^n)^{-1}(\chi)$. Hence, $\mathcal{C}_n(f)(\gamma(t)) \in \mathcal{F}_n(Y)$ for every $t \in [0,1]$. Let $\eta\colon [0,1] \to HS_n(X)$ be given by $\eta(t) = q_X^n(\gamma(t))$. Then η is an arc such that $\eta(0) = F_X^n$, $\eta(1) = \chi$ and

$HS_n(f)(\eta(t)) = F_Y^n$ for each $t \in [0,1]$. Therefore, $HS_n(f)^{-1}(F_Y^n)$ is arcwise connected.

Let $\chi \in HS_n(f)^{-1}(q_Y^n(Y)) \setminus \{q_X^n(X)\}$. Then, by the commutativity of $(*)$, $\mathcal{C}_n(f)((q_X^n)^{-1}(\chi)) = Y$. As we did in the previous paragraph, we can use the image of an order arc in $\mathcal{C}_n(X)$ from $(q_X^n)^{-1}(\chi)$ to X, under q_X^n, to see that there is an arc from χ to $q_X^n(X)$ contained in $HS_n(f)^{-1}(q_Y^n(Y))$. Thus, $HS_n(f)^{-1}(q_Y^n(Y))$ is arcwise connected. □

Proposition 3.2. *Let $f\colon X \to Y$ be a map between continua, and let n be a positive integer. Then $\mathcal{C}_n(f)$ is surjective if and only if $HS_n(f)$ is surjective.*

Proof: If $\mathcal{C}_n(f)$ is surjective, then it follows easily, by the commutativity of $(*)$, that $HS_n(f)$ is surjective.

Suppose $HS_n(f)$ is surjective. Then there exists a point $\chi \in HS_n(X)$ such that $HS_n(f)(\chi) = q_Y^n(Y)$. This implies, by $(*)$, that $\mathcal{C}_n(f)((q_X^n)^{-1}(\chi)) = Y$. Hence, $\mathcal{C}_n(f)(X) = Y$, i.e., $f(X) = Y$. Therefore, f is surjective.

Let $B \in \mathcal{C}_n(Y)$. If $B \in \mathcal{F}_n(Y)$, then, since f is surjective, there exists $A \in \mathcal{F}_n(X)$ such that $f(A) = B$. Thus, $\mathcal{C}_n(f)(A) = B$. Suppose now that $B \in \mathcal{C}_n(Y) \setminus \mathcal{F}_n(Y)$. Then $q_Y^n(B) \in HS_n(Y) \setminus \{F_Y^n\}$. Since $HS_n(f)$ is surjective, there exists $\chi \in HS_n(X) \setminus \{F_X^n\}$ such that $HS_n(f)(\chi) = q_Y^n(B)$. This implies, by $(*)$, that $\mathcal{C}_n(f)((q_X^n)^{-1}(\chi)) = B$. Therefore, $\mathcal{C}_n(f)$ is surjective. □

As a consequence of Proposition 3.2 and Proposition 1 of [2], we have the following:

Corollary 3.3. *Let $f\colon X \to Y$ be a map between continua, and let n be a positive integer. Then the following are equivalent:*

(1) f is weakly confluent;

(2) $\mathcal{C}_n(f)$ is surjective;

(3) $HS_n(f)$ is surjective.

Theorem 3.4. *Let $f\colon X \to Y$ be a map between continua, and let n be a positive integer. Then $\mathcal{C}_n(f)$ is a homeomorphism if and only if $HS_n(f)$ is a homeomorphism.*

Proof: If $\mathcal{C}_n(f)$ is a homeomorphism, then, by 3.2 and $(*)$, $HS_n(f)$ is a homeomorphism.

Suppose $HS_n(f)$ is a homeomorphism. By 3.2, $\mathcal{C}_n(f)$ is surjective. Note that $\mathcal{C}_n(f)|_{\mathcal{C}_n(X)\setminus\mathcal{F}_n(X)}$ is one–to–one and

$$\mathcal{C}_n(f)\left(\mathcal{C}_n(X)\setminus\mathcal{F}_n(X)\right)\subset\mathcal{C}_n(Y)\setminus\mathcal{F}_n(Y).$$

Let $B\in\mathcal{C}(Y)\setminus\mathcal{F}_1(Y)$. Since f is weakly confluent (by 3.3), there exists $A\in\mathcal{C}(X)$ such that $f(A)=B$. Hence, $\mathcal{C}_n(f)(A)=B$ and $\mathcal{C}_n(f)^{-1}(B)=\{A\}$. Thus, $\mathcal{C}_n(f)^{-1}(B)\cap\mathcal{C}(X)\neq\emptyset$. Then, by (1.49) of [10], $\bigcup\mathcal{C}_n(f)^{-1}(B)\in\mathcal{C}(X)$. Since $\bigcup\mathcal{C}_n(f)^{-1}(B)=f^{-1}(B)$, we have that $f^{-1}(B)$ is connected.

Let $y\in Y$, and let $\{K_m\}_{m=1}^{\infty}$ be a decreasing sequence of nondegenerate subcontinua of Y such that $\bigcap_{m=1}^{\infty}K_m=\{y\}$. This implies that $\bigcap_{m=1}^{\infty}f^{-1}(K_m)=f^{-1}(\{y\})$. Hence, $f^{-1}(\{y\})$ is connected. If $f^{-1}(\{y\})\in\mathcal{C}(X)\setminus\mathcal{F}_1(X)$, then $\mathcal{C}_n(f)(f^{-1}(\{y\}))\in\mathcal{C}(Y)\setminus\mathcal{F}_1(Y)$. But, $\mathcal{C}_n(f)(f^{-1}(\{y\}))=f(f^{-1}(\{y\}))=\{y\}$, a contradiction. Therefore, $f^{-1}(\{y\})\in\mathcal{F}_n(X)$. Since $f^{-1}(\{y\})\in\mathcal{C}(X)$, $f^{-1}(\{y\})\in\mathcal{F}_n(X)\cap\mathcal{C}(X)=\mathcal{F}_1(X)$. Therefore, f is a homeomorphism. Hence, $\mathcal{C}_n(f)$ is a homeomorphism by Theorem 46 of [2]. □

Theorem 3.5. *Let $f\colon X\to Y$ be a map between continua and let n be a positive integer. Consider the following statements:*

(1) f is monotone;

(2) $\mathcal{C}_n(f)$ is monotone;

(3) $HS_n(f)$ is monotone.

Then, for $n\geq 2$, (1), (2) and (3) are equivalent. For $n=1$, (1) and (2) are equivalent and (2) implies (3).

Proof: The equivalence of (1) and (2) is in Theorem 4 of [2].

Since q_Y^n is monotone by $(*)$ and by (5.15) of [8], we have that if $\mathcal{C}_n(f)$ is monotone, then $HS_n(f)$ is monotone.

Next, suppose $n\geq 2$ and $HS_n(f)$ is monotone. Let $B\in\mathcal{C}(Y)\setminus\mathcal{F}_1(Y)$. Then, by $(*)$ and our hypothesis, $\mathcal{C}_n(f)^{-1}(B)$ is connected. Since f is weakly confluent (by 3.3), there exists $A\in\mathcal{C}(X)$ such that $f(A)=B$. Hence, $A\in\mathcal{C}_n(f)^{-1}(B)$. This implies that $\mathcal{C}_n(f)^{-1}(B)\cap\mathcal{C}(X)\neq\emptyset$. Thus, $\bigcup\mathcal{C}_n(f)^{-1}(B)\in\mathcal{C}(X)$ by (1.49) of [10]. Note that $\bigcup\mathcal{C}_n(f)^{-1}(B)\subset f^{-1}(B)$. Let $x\in f^{-1}(B)$. Then $A\cup\{x\}\in\mathcal{C}_n(f)^{-1}(B)$ and $x\in\bigcup\mathcal{C}_n(f)^{-1}(B)$. Therefore, $\bigcup\mathcal{C}_n(f)^{-1}(B)=f^{-1}(B)$. In particular, $f^{-1}(B)$ is connected. Repeating the argument given in the last paragraph of the proof of

3.4, we obtain that for each $y \in Y$, $f^{-1}(y)$ is connected. Therefore, f is monotone. $\square$

Question 3.6. Let $f\colon X \to Y$ be a map between continua. If $HS(f)$ is monotone, then is f monotone?

An onto map $f\colon X \to Y$ between continua is a *CE–map* provided that for each $y \in Y$, $f^{-1}(y)$ has trivial shape in the sense of K. Borsuk [1].

Nadler proved that a map f between continua is monotone if and only if the induced map $\mathcal{C}(f)$ is a CE–map (see Lemma 2.1 of [12]). This result was generalized to any induced map $\mathcal{C}_n(f)$ in Proposition 6 of [2]. It is natural to ask if the same is true for induced maps on n–fold hyperspace suspensions. This is the content of Question 3.8. The following example says that a positive answer to this question is possible.

Example 3.7. Let $\mathcal{S}^1$ denote the unit circle in the Euclidean plane $\mathbb{R}^2$. Let $r\colon \mathcal{S}^1 \times [0,1] \to \mathcal{S}^1$ be given by $r((z,t)) = z$. Note that for each $z \in \mathcal{S}^1$, $r^{-1}(z) = \{z\} \times [0,1]$. Hence, r is a monotone map. It is easy to see that, in this case, $\mathcal{C}(r)^{-1}(\{z\}) = \mathcal{C}(\{z\} \times [0,1])$. Hence, $\mathcal{C}(r)^{-1}(\mathcal{F}_1(\mathcal{S}^1)) = \bigcup\{\mathcal{C}(r)^{-1}(\{z\}) \mid z \in \mathcal{S}^1\} = \bigcup\{\mathcal{C}(\{z\} \times [0,1]) \mid z \in \mathcal{S}^1\}$. Thus, $\mathcal{C}(r)^{-1}(\mathcal{F}_1(\mathcal{S}^1))$ is homeomorphic to a solid torus. Observe that $\mathcal{F}_1(\mathcal{S}^1 \times [0,1])$ is in the manifold boundary of $\mathcal{C}(r)^{-1}(\mathcal{F}_1(\mathcal{S}^1))$. This implies that $q^1_{\mathcal{S}^1\times[0,1]}(\mathcal{C}(r)^{-1}(\mathcal{F}_1(\mathcal{S}^1))) = HS(r)^{-1}(F^1_{\mathcal{S}^1})$ is homeomorphic to the solid of revolution obtained by rotating the set $\{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1\}$ about the y axis. Note that this space is contractible. In particular, it has trivial shape in the sense of Borsuk [1, 5.5, p. 28]. Therefore, by Proposition 6 of [2], $HS(r)$ is a CE–map.

Question 3.8. Let $f\colon X \to Y$ be a map between continua and let n be a positive integer. If f is monotone, then is $HS_n(f)$ a CE–map?

Theorem 3.9. *Let $f\colon X \to Y$ be a map between continua and let n be a positive integer. If $\mathcal{C}_n(f)$ is open, then $HS_n(f)$ is open.*

Proof: Let Γ be an open subset of $HS_n(X)$. Then $(q^n_X)^{-1}(\Gamma)$ is an open subset of $\mathcal{C}_n(X)$. Hence, since $\mathcal{C}_n(f)$ is open, $\mathcal{C}_n(f)((q^n_X)^{-1}(\Gamma))$ is open in $\mathcal{C}_n(Y)$. Observe that either $\mathcal{F}_n(Y) \subset \mathcal{C}_n(f)((q^n_X)^{-1}(\Gamma))$

or $\mathcal{F}_n(y) \cap \mathcal{C}_n(f)((q_X^n)^{-1}(\Gamma)) = \emptyset$. This implies that

$$q_Y^n(\mathcal{C}_n(f)((q_X^n)^{-1}(\Gamma)))$$

is open in $HS_n(Y)$. Since $q_Y^n(\mathcal{C}_n(f)((q_X^n)^{-1}(\Gamma))) = HS_n(f)(\Gamma)$ (by $(*)$), $HS_n(f)(\Gamma)$ is open in $HS_n(Y)$. Therefore, $HS_n(f)$ is open. □

Question 3.10. Let $f\colon X \to Y$ be a map between continua and let n be a positive integer. If $HS_n(f)$ open, then is $\mathcal{C}_n(f)$ open?

Remark 3.11. Note that the example proposed in Remark 9 of [2] also shows that the openness of $HS_n(f)$ does not imply the openness of f.

Nadler, in (2.3) of [11], defines a metric for a quotient space obtained by shrinking a subcontinuum to a point. We follow him to define a metric on the n–fold hyperspace suspensions. Let X be a continuum and let n be a positive integer. Let

$$\Im_n = \{\mathcal{F}_n(X) \cup \{A\} \mid A \in \mathcal{C}_n(X)\}.$$

Note that $\Im_n(X) \subset \mathcal{C}_2(\mathcal{C}_n(X))$. Define $G_n\colon HS_n(X) \to \Im_n(X)$ by

$$G_n(\chi) = \mathcal{F}_n(X) \cup (q_X^n)^{-1}(\chi).$$

Then G_n is a homeomorphism. Next, define

$$\rho_X^n\colon HS_n(X) \times HS_n(X) \to [0, \infty)$$

by

$$\rho_X^n(\chi_1, \chi_2) = \mathcal{H}_X^2(G_n(\chi_1), G_n(\chi_2)), \qquad (**)$$

where $\mathcal{H}_X^2$ is the Hausdorff metric induced by the Hausdorff metric, $\mathcal{H}_X$, on $\mathcal{C}_n(X)$. Then ρ_X^n is a metric.

Rewriting (2.3) of [11] with our terminology, we obtain:

Theorem 3.12. *Let $f\colon X \to Y$ be a map between continua, let n be a positive integer and let $\varepsilon > 0$. If $\mathcal{C}_n(f)$ is an ε–map, then $HS_n(f)$ is an ε–map.*

Lemma 3.13. *Let $g, f\colon X \to Y$ be maps between continua, let n be a positive integer and let $\varepsilon > 0$. If $d(f(x), g(x)) < \varepsilon$ for every $x \in X$, then*

$$\rho_Y^n(HS_n(f)(\chi), HS_n(g)(\chi)) < \varepsilon$$

for each $\chi \in HS_n(X)$.

Proof: First, note that, under our hypothesis,

$$\mathcal{H}_Y(\mathcal{C}_n(f)(A), \mathcal{C}_n(g)(A)) < \varepsilon$$

for each $A \in \mathcal{C}_n(X)$ by Lemma 37 of [2].

Second, observe that, since $HS_n(f)(F_X^n) = HS_n(g)(F_X^n) = F_Y^n$, we have that $\rho_Y^n(HS_n(f)(F_X^n), HS_n(g)(F_X^n)) = 0 < \varepsilon$.

Finally, let $\chi \in HS_n(X) \setminus \{F_X^n\}$. Then there exists $A \in \mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$ such that $q_X^n(A) = \chi$. Hence,

$$\begin{aligned}\rho_Y^n(HS_n(f)(\chi), HS_n(g)(\chi)) &= \mathcal{H}_Y^2(\mathcal{F}_n(Y) \cup \{f(A)\}, \mathcal{F}_n(Y) \cup \{g(A)\}) \\ &= \mathcal{H}_Y(f(A), g(A)) = \mathcal{H}_Y(\mathcal{C}_n(f)(A), \mathcal{C}_n(g)(A)) < \varepsilon.\end{aligned}$$

□

A continuum Y is in *Class*(W) provided that for each continuum X, each onto map $f \colon X \to Y$ is weakly confluent.

Theorem 3.14. *Let X and Y be continua and let n be a positive integer. If Y is Class(W) and $f \colon X \to Y$ is a refinable map, then $HS_n(f)$ is refinable.*

Proof: Let $\varepsilon > 0$. Since f is refinable, there exists an ε–map $g \colon X \to Y$ such that $d(f(x)), g(x)) < \varepsilon$ for each $x \in X$. Since g is an ε–map and Y is in Class(W), by Lemma 36 of [2] and 3.3, $\mathcal{C}_n(g)$ is an ε–map. Hence, by 3.12, $HS_n(g)$ is an ε–map. Now, by 3.13, $\rho_Y^n(HS_n(f)(\chi), HS_n(g)(\chi)) < \varepsilon$ for each $\chi \in HS_n(X)$. Therefore, $HS_n(f)$ is refinable. □

Theorem 3.15. *Let X and Y be continua and let n be a positive integer. If $f \colon X \to Y$ is monotonely refinable, then $HS_n(f)$ is monotonely refinable.*

Proof: Let $\varepsilon > 0$. Since f is monotonely refinable, there exists a monotone ε–map $g \colon X \to Y$ such that $d(f(x)), g(x)) < \varepsilon$ for each $x \in X$. By 3.5 and the proof of 3.14, $HS_n(g)$ is a monotone ε–map such that $\rho_Y^n(HS_n(f)(\chi), HS_n(g)(\chi)) < \varepsilon$ for each $\chi \in HS_n(X)$. Therefore, $HS_n(f)$ is monotonely refinable. □

Recall that a subcontinuum R of a continuum X is *terminal in* X provided that if T is a subcontinuum of X such that $T \cap R \neq \emptyset$, then either $T \subset R$ or $R \subset T$.

Our next theorem characterizes atomic maps between continua (see (1.2) of [9]):

Theorem 3.16. *A map $f\colon X \to Y$ between continua is atomic if and only if $f^{-1}(y)$ is a terminal subcontinuum in X for each $y \in Y$.*

Theorem 3.17. *Let X and Y be continua and let n be a positive integer. If $f\colon X \to Y$ is a map such that $HS_n(f)$ is atomic, then f is a homeomorphism.*

Proof: Note that, by 3.16, $HS_n(f)^{-1}(\chi)$ is a terminal subcontinuum in $HS_n(X)$ for each $\chi \in HS_n(Y)$. Since $HS_n(X)$ is arcwise connected, $HS_n(X)$ does not contain nondegenerate, proper, terminal subcontinua. Hence, $HS_n(f)^{-1}(\chi)$ consists of just one point for every $\chi \in HS_n(Y)$. Thus, $HS_n(f)$ is one–to–one. Hence, $HS_n(f)$ is a homeomorphism. Therefore, f is a homeomorphism by 3.4 and Theorem 46 of [2]. □

We finish the paper with a question. Nadler proved that the hyperspace suspensions of arc–like continua have the fixed point property (see (3.1) of [11]). It is natural to ask:

Question 3.18. Do n–fold hyperspace suspensions of arc–like continua have the fixed point property?

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