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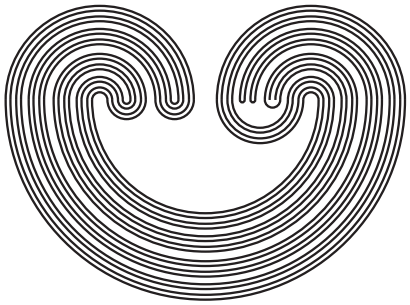

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A NATURAL CLASSIFYING SPACE FOR COHOMOLOGY WITH COEFFICIENTS IN A FINITE CHAIN COMPLEX

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Abstract. We provide a natural classifying space for cohomology with coefficients in a finite chain complex.

1. Introduction

The notion of cohomology with coefficients in a chain complex extends the notion of ordinary cohomology with coefficients in a group. It was first introduced in [1] where it was proved that if X is a space and B is a chain complex of abelian groups, there is an isomorphism:

$$H^n(X;B) \cong \prod_m H^m(X, H_{m-n}(B)).$$

This proves that $\prod_n K(H_{n-m}(B), n)$ is a classifying space for this cohomology theory. However, the isomorphism above is not natural in the coefficient complex. We shall provide a *natural* classifying space for cohomology with coefficients in a *finite* chain complex.

In the case of a finite chain complex of the form $A \xrightarrow{h} B$ we define $K(A \xrightarrow{h} B, n)$ to be the homotopy fiber of the map

$$K(h,n) : K(A,n) \to K(B,n).$$

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In general we define $K(B_0 \to \ldots \to B_k, n)$ to be the homotopy fiber of the map

$$K(B_0 \to \ldots \to B_{k-1}, n) \to K(0 \to \ldots \to 0 \to B_k, n)$$

defined inductively by functoriality of the definition of $K(B_0 \to \ldots \to B_{k-1}, n)$. We shall prove that this defines a natural classifying space. This inductive construction gives the restriction to finite chain complexes. We will, however, be able to deal with chains of the form $B_0 \to B_1 \to \ldots$.

In the next three sections, we recapitulate cohomology with coefficients in a chain complex and define the notion of a natural classifying space. In section 5, we describe a long exact sequence that provides the natural connection between cohomology with coefficients in a finite chain and cohomology with coefficients in shorter chains. This provides means for induction.

In section 6, we prove equivalence of cellular and singular cohomology, which is important, since the isomorphism $H^n(X, A) \cong [X, K(A, n)]$ in ordinary cohomology is defined using cellular cohomology.

In section 7, we extend the Eilenberg-MacLane functor to the case of finite chains, and in the last three sections, we extend the definition of the natural transformation $[-, K(-, n)] \to H^n(-, -)$ and prove that it defines an equivalence of functors.

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2. Cohomology with coefficients in a chain complex

Definition 2.1. Let (A, ∂_A) and (B, ∂_B) be chain complexes of abelian groups. We define the cochain complex $\mathrm{Hom}(A, B)$ by setting

$$\mathrm{Hom}(A, B)_p = \prod_n \mathrm{Hom}(A_n, B_{n-p})$$

with differential $\delta : \mathrm{Hom}(A, B)_p \to \mathrm{Hom}(A, B)_{p+1}$ given by $\delta f = \partial_B f - (-1)^p f \partial_A$.

It is easily verified that this does in fact define a cochain complex. The definition of this differential was first given in [4].

Definition 2.2. A finite chain complex is a finite sequence of abelian groups and homomorphisms

$$B_0 \overset{h_0}{\to} B_1 \to \ldots \overset{h_{r-1}}{\to} B_r$$

such that $h_i \circ h_{i-1} = 0$.

The class of finite chain complexes becomes a subcategory of the category of chain complexes under the inclusion that maps the finite complex

$$B_0 \overset{h_0}{\to} B_1 \to \ldots \overset{h_{r-1}}{\to} B_r$$

to the infinite chain complex which has B_0 to B_r in dimensions 0 to r and 0 in all other dimensions.

We define the length of a finite complex to be the number of nontrivial groups in it. That is, the length of $B_0 \to \ldots \to B_r$ is $r + 1$.

The following definition is due to R. Brown [1].

Definition 2.3. Let (A, ∂_A) and (B, ∂_B) be chain complexes. We define the cohomology of A with coefficients in B to be

$$H^n(A; B) = H^n(\mathrm{Hom}(A, B)).$$

Remark 2.4. If B_0 is an abelian group, we can view B_0 as a finite chain complex. This way Definition 2.3 extends ordinary cohomology.

One may easily prove:

Proposition 2.5. There exists an isomorphism

$$H^n(A; B_0 \to \ldots \to B_k) \cong H^{n+1}(A; 0 \to B_0 \to \ldots \to B_k)$$

that is natural in both variables.

In [1], the following theorem is proved:

Theorem 2.6. *For all chain complexes A, B the groups $H^n(A; B)$ and $\prod_m H^n(A; H_{m-n}(B))$ are isomorphic. The isomorphism is natural in A.*

It is also shown that any such isomorphism cannot be natural in the chain complex B, so that cohomology with coefficients in a chain complex cannot be reduced to ordinary cohomology.

3. Cohomology of spaces

We are now ready to define cohomology of a space.

Definition 3.1. If X is a topological space, let $\Delta_*(X)$ denote the singular chain complex. If B is a chain complex, we define (singular) cohomology of X with coefficients in B to be

$$H^n(X;B) = H^n(\Delta_*(X);B).$$

If (X, A) is a pair of spaces, we define the relative cohomology with coefficients in B to be

$$H^n(X, A; B) = H^n(\Delta_*(X, A); B)$$

where $\Delta_*(X, A) = \Delta_*(X)/\Delta_*(A)$ is the singular chain complex of the pair.

We define reduced cohomology of a space using the augmented singular chain complex.

One may now prove the following theorem:

Theorem 3.2. *Singular cohomology with coefficients in a chain complex is a generalized cohomology theory on the category of topological spaces. Reduced cohomology with coefficients in a chain complex is a reduced cohomology theory on the category of CW-complexes.*

The proof of this is standard. For details, see [8].

Remark 3.3. Theorem 2.6 shows that this cohomology theory cannot satisfy the dimension axiom since

$$H^n(\{*\};B) \cong H_{-n}(B).$$

4. Natural classifying spaces

In this article we let $[X, Y]$ denote the homotopy classes of basepoint-preserving maps $X \to Y$.

It is known from [5] that there is a functorial construction of Eilenberg-MacLane spaces. That is, for each $n \geq 0$, we have a functor $K(-, n)$ from the category of abelian groups to the category of pointed CW-complexes. For any group A, $K(A, n)$ is a CW-complex with $(n-1)$-skeleton consisting only of the basepoint $*$. In case $A = 0$, we have $K(0, n) = *$. In case $n = 0$, we take $K(-, 0)$ to be the functor, that to A associates A as a discrete group.

The functor $K(-,n)$ is a right inverse to the functor π_n. This provides us with a natural isomorphism $\pi_n K(A,n) \cong A$.

We also have another natural equivalence,

$$S : [\Sigma -, -] \to [-, \Omega -],$$

where Σ denotes the reduced suspension.

This natural equivalence induces a natural equivalence

$$S^l : \pi_n \to \pi_{n-l}\Omega^l$$

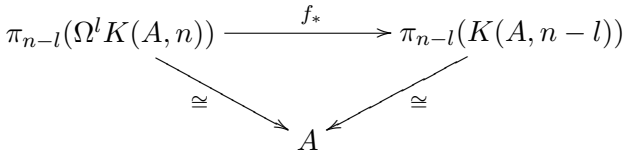
for all $l \geq 0$. Using this equivalence, we get an identification

$$\pi_{n-l}\Omega^l K(A,n) \cong A.$$

Lemma 4.1. *Suppose $n \geq l$. We can choose a homotopy equivalence,*

$$f : \Omega^l K(A,n) \to K(A, n-l),$$

such that the following diagram commutes:



Proof: This follows from the fact that

$$[K(A,n), K(B,n)] \cong \mathrm{Hom}(A,B). \qquad \square$$

In the rest of this article, whenever a homotopy equivalence $K(A, n-l)$ to $\Omega^l K(A,n)$ is mentioned, we mean a homotopy equivalence satisfying the requirements of Lemma 4.1

Definition 4.2. Suppose $h^n(-,-)$ is a cohomology theory, with coefficients in a chain complex. A natural classifying space for $h^n(-,-)$ is a functor $K_n(-)$ such that there exists a natural equivalence from the functor $h^n(-,-)$ to the functor $[-, K_n(-)]$.

Example 4.3. The functor $K(-,n)$ is a natural classifying space for ordinary reduced cohomology with coefficients in the category of groups.

The aim of this article is to construct a natural classifying space for cohomology with coefficients in the category of finite chain complexes.

5. The long exact sequences

Theorem 5.1. *Suppose*

$$B_0 \stackrel{h_0}{\to} B_1 \to \ldots \stackrel{h_{k-1}}{\to} B_k$$

is a finite chain complex. For every chain complex A of abelian groups, and $r \in \{0, \ldots, k-1\}$ there is a long exact sequence on the form:

$$\cdots \longrightarrow H^n(A, B_0 \to \ldots B_k) \xrightarrow{\psi} H^n(A, B_0 \to \ldots B_r) \xrightarrow{(h_r)_*}$$
$$H^{n-r}(A, B_{r+1} \to \ldots B_k) \xrightarrow{\phi} H^{n+1}(A, B_0 \to \ldots B_k) \longrightarrow \cdots .$$

Here, the map ψ is induced by projection of chains and the map ϕ is induced by inclusion of chains. This sequence is natural in both A and the coefficient complex.

Remark 5.2. The map $(h_r)_*$ is the map induced by the chain map:

$$\begin{array}{ccccccccccc}
B_0 & \longrightarrow & \cdots & \longrightarrow & B_{r-1} & \longrightarrow & B_r & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow{\scriptstyle h_r} & & \downarrow & & & & \downarrow \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & B_{r+1} & \longrightarrow & B_{r+2} & \longrightarrow & \cdots & \longrightarrow & B_k
\end{array}$$

Proof: The chain map i

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & B_{r+1} & \longrightarrow & \cdots & \longrightarrow & B_k \\
\downarrow & & & & \downarrow & & \downarrow{\scriptstyle id} & & & & \downarrow{\scriptstyle id} \\
B_0 & \longrightarrow & \cdots & \longrightarrow & B_r & \longrightarrow & B_{r+1} & \longrightarrow & \cdots & \longrightarrow & B_k
\end{array}$$

induces the map

$$\begin{aligned} \phi : \quad & H^n(A; 0 \to \ldots \to 0 \to B_{r+1} \to \ldots \to B_k) \to \\ & H^n(A; B_0 \to \ldots \to B_k), \end{aligned}$$

and the chain map p

$$\begin{array}{ccccccccc}
B_0 & \longrightarrow & \cdots & \longrightarrow & B_r & \longrightarrow & B_{r+1} & \longrightarrow & \cdots & \longrightarrow & B_k \\
\downarrow{\scriptstyle id} & & & & \downarrow{\scriptstyle id} & & \downarrow & & & & \downarrow \\
B_0 & \longrightarrow & \cdots & \longrightarrow & B_r & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0
\end{array}$$

induces the map

$$\psi : H^n(A; B_0 \to \ldots \to B_k) \to H^n(A; B_0 \to \ldots \to B_r).$$

Since clearly the sequence

$$0 \longrightarrow \mathrm{Hom}(A; 0 \to \ldots \to 0 \to B_{r+1} \to \ldots \to B_k)_n \xrightarrow{i_*}$$
$$\mathrm{Hom}(A; B_0 \to \ldots \to B_k)_n \xrightarrow{p_*} \mathrm{Hom}(A; B_0 \to \ldots B_r)_n \longrightarrow 0$$

is short exact for every n, the long exact sequence follows from the Snake Lemma and Proposition 2.5.

The only thing left to prove is that the connecting homomorphism is in fact the map $(h_r)_*$. This is done by tracing through the definition of the connecting map in the Snake Lemma. □

The version of the long exact sequence that we will use most is the case $r = k - 1$.

Corollary 5.3. *Suppose $k > n + 1$ and X is a space. Then the map*

$$\psi : \tilde{H}^n(X, B_0 \to \ldots \to B_k) \to \tilde{H}^n(X, B_0 \to \ldots \to B_{n+1})$$

is an isomorphism. This map is natural in both variables.

Proof: This follows easily from Theorem 5.1 and Theorem 2.6. □

Thus, the only interesting cases of cohomology with coefficients in a finite complex are the ones with $k \leq n+1$, and we can restrict our attention to those.

Had we done the analysis for Theorem 5.1 in the case of infinite chains, the arguments from the corollary above would have shown, that for a general chain complex B, $\tilde{H}^n(X, B)$ is only affected by the groups in B in dimension less than or equal to $n + 1$.

6. Cellular cohomology

Definition 6.1. Let X be a CW-complex, and let $C_*(X)$ denote the cellular complex of X. We define the cellular cohomology of X with coefficients in the chain complex B as

$$H^n_{CW}(X; B) = H^n(C_*(X); B).$$

Likewise, we define cellular cohomology of a pair and reduced cohomology using the cellular complex of the pair and the augmented cellular complex, respectively.

Theorem 6.2. *Let $\tilde{H}^n(-,-)$ and $\tilde{H}^n_{CW}(-,-)$ denote reduced singular and cellular cohomology with coefficients in a chain complex. The functors $\tilde{H}^n(-,-)$ and $\tilde{H}^n_{CW}(-,-)$ are naturally equivalent on the category of CW-complexes.*

Proof: We will first prove that the two functors are equivalent on the category of simplicial complexes.

Suppose X is a simplicial complex. Each cell of X has an attaching map that is a homeomorphism from Δ_n to the cell. We may view this attaching map as an element of $\Delta_n(X)$. This gives a chain map

$$C_*(X) \to \Delta_*(X)$$

that induces an isomorphism on homology, and therefore also on cohomology. This way we obtain an isomorphism

$$H^n_{CW}(X;B) \to H^n(X;B)$$

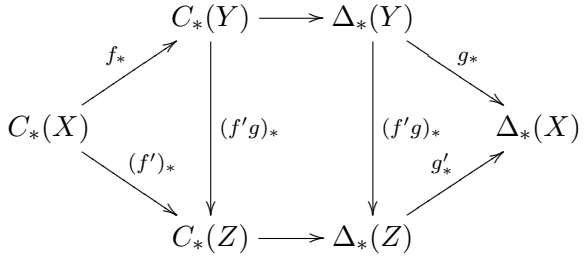
that is natural in X and B as desired. This is an equivalence of functors on the category of simplicial complexes.

Now suppose X is a CW-complex. Choose a simplicial complex Y homotopic to X and choose cellular homotopy equivalences $f : X \to Y$ and $g : Y \to X$, that are inverses of each other. Let T_X be the composition

$$C_*(X) \xrightarrow{f_*} C_*(Y) \longrightarrow \Delta_*(Y) \xrightarrow{g_*} \Delta_*(X)$$

where the middle map is the map defined above. We define the natural equivalence of functors to be the map induced by T_X. There is a choice of simplicial approximations involved here, but we claim that this cannot be seen at the level of cohomology.

Consider a different choice of simplicial space Z with maps $f' : X \to Z$ and $g' : Z \to X$, and consider the following diagram:



Since the maps f' and $f'gf$ are homotopic, the left triangle commutes up to chain homotopy. The middle square clearly commutes, and the right triangle commutes up to chain homotopy as before. This proves well-definedness.

Naturality is clear. This proves the theorem and the proof can easily be generalized to relative cohomology (cohomology of pairs) which takes care of the case of reduced cohomology. □

7. The functor $K(-, n)$

The aim of this section is to extend the functor $K(-, n) : Ab \to Top$, where Ab is the category of abelian groups, to the category of finite chain complexes. We will begin with the case of a complex $A \xrightarrow{h} B$. Suppose we have a natural classifying space $K(A \xrightarrow{h} B, n)$. As a consequence of Theorem 5.1, we then have an exact sequence

$$\cdots \longrightarrow [X, \Omega K(A,n)] \xrightarrow{\Omega K(h,n)_*} [X, \Omega K(B,n)] \longrightarrow$$
$$[X, K(A \xrightarrow{h} B, n)] \longrightarrow [X, K(A,n)] \xrightarrow{K(h,n)_*}$$
$$[X, K(B,n)] \longrightarrow \cdots .$$

This long exact sequence corresponds to a right exact sequence of spaces that resembles the Puppe sequence for the map $K(h, n)$. This motivates the definition of $K(A \xrightarrow{h} B, n)$ to be the homotopy fiber of $K(h, n)$.

To see that this defines a functor, note that a morphism of complexes

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle \phi_0}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ A' & \xrightarrow{h'} & B' \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} K(A,n) & \xrightarrow{K(h,n)} & K(B,n) \\ {\scriptstyle K(\phi_0,n)}\downarrow & & \downarrow{\scriptstyle K(\phi_1,n)} \\ K(A',n) & \xrightarrow{K(h',n)} & K(B',n) \end{array}$$

which induces a map between the homotopy fibers.

We note two important obvious consequences of this definition: $K(A \to 0, n) = K(A, n)$ and $K(0 \to B, n) = \Omega K(B, n)$.

To define $K(-, n)$ in general, we proceed by induction over the length of the chain complex. So suppose we have defined $K(-, n)$ on all chains of length k or shorter, such that

$$K(0 \to B_1 \to \ldots \to B_{k-1}, n) = \Omega K(B_1 \to \ldots \to B_{k-1}, n).$$

Suppose we are given a chain $B_0 \to \ldots \to B_k$. Since we have a commutative diagram of chains of length $k-1$

$$\begin{array}{ccc} (B_0 \to \ldots \to B_{k-2}) & \xrightarrow{(0,\ldots,0,h_{k-2})} & (0 \to \ldots \to 0 \to B_{k-1}) \\ \downarrow & & \downarrow{\scriptstyle (0,\ldots,0,h_{k-1})} \\ (0 \to \ldots \to 0) & \longrightarrow & (0 \to \ldots \to 0 \to B_k) \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} K(B_0 \to \ldots \to B_{k-2}, n) & \xrightarrow{K((0,\ldots,0,h_{k-2}),n)} & \Omega^{k-2} K(B_{k-1}, n) \\ \downarrow & & \downarrow{\scriptstyle \Omega^{k-2}(h_{k-1},n)} \\ \star & \longrightarrow & \Omega^{k-2} K(B_k, n) \end{array}$$

which induces a map

$$K((0, \ldots, 0, h_{k-1}), n) : K(B_0 \to \ldots \to B_{k-1}, n) \longrightarrow \Omega^{k-1} K(B_k, n).$$

We can now define:

Definition 7.1. $K(B_0 \to \ldots \to B_k, n)$ is the homotopy fiber of the map

$$K((0, \ldots, 0, h_{k-1}), n) : K(B_0 \to \ldots \to B_{k-1}, n) \longrightarrow \Omega^{k-1} K(B_k, n).$$

Clearly, we still get $K(0 \to B_1 \to \ldots \to B_k, n) = \Omega K(B_1 \to \ldots \to B_k, n)$ and $K(A \to 0 \to \ldots \to 0, n) = K(A, n)$.

The next lemma should be considered in relation to Corollary 5.3.

Lemma 7.2. *Suppose $k > n + 1$. Then there exists a natural homotopy equivalence*

$$K(B_0 \to \ldots \to B_k, n) \to K(B_0 \to \ldots \to B_{n+1}, n).$$

Proof: It suffices to show that there exists a natural homotopy equivalence:

$$K(B_0 \to \ldots \to B_k, n) \to K(B_0 \to \ldots \to B_{k-1}, n).$$

If $k > n+1$, then $\Omega^{k-1}K(B_k, n)$ is null homotopic. So we can construct a commutative diagram:

$$\begin{array}{ccc} K(B_0 \to \ldots \to B_{k-1}, n) & \longrightarrow & \Omega^{k-1}K(B_k, n) \\ \| & & \downarrow \simeq \\ K(B_0 \to \ldots \to B_{k-1}, n) & \longrightarrow & \star \end{array}$$

Since both vertical maps here are homotopy equivalences, this map induces a weak homotopy equivalence between the fibers. That it is in fact a homotopy equivalence is proved in [2]. Naturality is clear. □

8. A CW-structure on $K(B_0 \to \ldots \to B_k, n)$

We will prove that $K(B_0 \to \ldots \to B_k, n)$ is a natural classifying space using cellular cohomology, so we need a CW-structure on this space.

Lemma 8.1. *The space $K(B_0 \to \ldots \to B_k, n)$ is homotopy equivalent to a CW-complex if $n \geq k$.*

Proof: This lemma can be proven by induction on k using the facts that the homotopy fiber of a map between CW-complexes is homotopy equivalent to a CW-complex [7], and the fact that a pair of homotopy equivalences in a commutative diagram induces a homotopy equivalence on the homotopy fibers [2]. □

For our purpose, however, we need a special CW-structure on $K(B_0 \to \ldots \to B_k, n)$.

Lemma 8.2. *$K(B_0 \to \ldots \to B_k, n)$ is homotopy equivalent to a CW-complex X that is obtained from $K(B_k, n-k)$ by attaching cells of dimension $n-k+1$ and higher. Furthermore, the homotopy equivalence is a homotopy equivalence of pairs:*

$$(X, K(B_k, n-k)) \to (K(B_0 \to \ldots \to B_k, n), \Omega^k K(B_k, n)).$$

To prove this we need the following lemma.

Lemma 8.3. *The pair* $(K(B_0 \to \ldots \to B_k, n), \Omega^k K(B_k, n))$ *is* $(n-k)$*-connected.*

Proof: This is an easy consequence of the long exact sequence of a fibration. □

Now, Lemma 8.2 follows from the Cellular Approximation Theorem, CW-models, and [6, page 76].

By the definition of $K(B_0 \to \ldots \to B_k, n)$, we have a commutative diagram of maps.

Diagram 8.1

$$\begin{array}{ccccc} \Omega^k K(B_k,n) & \xrightarrow{i} & K(B_0 \to \ldots B_k, n) & \xrightarrow{p_k} & K(B_0 \to \ldots B_{k-1}, n) \\ & & \downarrow{\scriptstyle q_k} & & \downarrow{\scriptstyle K((0,\ldots,0,h_{k-1}),n)} \\ & & P\Omega^{k-1}K(B_k,n) & \xrightarrow{\eta} & \Omega^{k-1}K(B_k,n) \end{array}$$

Here, p_k and q_k are the projections, i is the inclusion, and η is the map $\gamma \mapsto \gamma(1)$. Notice that $p_k i$ is the constant map to $*$.

If we replace the spaces in the upper line of this diagram with CW approximations, as given in Lemma 8.2, we get a diagram that only commutes up to homotopy, but still has $p_k i = *$. According to the Cellular Approximation Theorem, we may also replace p_k with a cellular map that is homotopic to p_k through a homotopy that maps $\Omega^k K(B_k, n)$ to $*$ at all times. Having done this replacement, we know that $K((0, \ldots, 0, h_{k-1}), n) \circ p_k$ is homotopic to $\eta \circ q_k$ through a homotopy that maps $\Omega^k K(B_k, n)$ to $*$ at all times.

9. The natural transformation

In the following we shall always assume that $n \geq k$.

To define the natural transformation

$$T : [-, K(B_0 \to \ldots \to B_k, n)] \to \tilde{H}^n(-, B_0 \to \ldots \to B_k)$$

we will construct an element

$$\xi \in H^n(K(B_0 \to \ldots \to B_k, n), B_0 \to \ldots \to B_k)$$

and define $T([f]) = f^*(\xi)$. An element in $\tilde{H}^n(X, B_0 \to \ldots \to B_k)$ is represented by a set of maps $(\xi_0, \ldots, \xi_k)$ such that the following

diagram commutes up to sign:

$$\begin{array}{ccccccccc} C_{n+1} & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \cdots & \longrightarrow & C_{n-k}(X) \\ \downarrow & & \downarrow{\scriptstyle \xi_0} & & \downarrow{\scriptstyle \xi_1} & & & & \downarrow{\scriptstyle \xi_k} \\ 0 & \longrightarrow & B_0 & \xrightarrow{h_0} & B_1 & \longrightarrow & \cdots & \longrightarrow & B_k \end{array}$$

To be precise, we must have the equations

$$\xi_0 \circ \partial = 0 \tag{9.1}$$

$$\forall l : \xi_l \circ \partial = (-1)^n h_{l-1} \xi_{l-1} \ . \tag{9.2}$$

In the case $k = 0$, we know that ξ_0 is the map that to an n-cell associates its attaching map, considered as an element of $\pi_n(K(B, n))$. Notice that this works for all $n \geq 0$.

We define the set $(\xi_0, \ldots, \xi_k)$ inductively. Suppose the maps $(\xi'_0, \ldots, \xi'_{k-1})$ are defined. We define

$$(\xi_0, \ldots, \xi_k) = (p_k^*(\xi'_0), \ldots, p_k^*(\xi'_{k-1}), \xi_k)$$

where ξ_k is the map known from ordinary cohomology using the fact that

$$C_{n-k}(K(B_0 \to \ldots \to B_k, n)) = C_{n-k}(K(B_k, n-k)).$$

Another way to desribe $\xi_i : C_{n-i}(K(B_0 \to \ldots \to B_k, n)) \to B_i$ is that if σ is the attaching map of an $n-i$ cell, then ξ_i associates to this cell the element

$$p_{i+1} \circ \ldots p_k \circ \sigma \in \pi_{n-i}(K(B_i, n-i)).$$

Notice that since the CW-approximation used for

$$K(\underbrace{0 \to \ldots \to 0}_{l} \to B, n) = \Omega^l K(B, n)$$

is in fact $K(B, n-l)$ in this case, the transformation

$$T : [-, \Omega^l K(B, n)] \to H^n(-, \underbrace{0 \to \ldots \to 0}_{l} \to B) = H^{n-l}(-, B)$$

reduces to the transformation known from ordinary cohomology. Thus, we have:

Proposition 9.1. For $n \geq l$ the restriction of T to CW-complexes and abelian groups:

$$T : [-, \Omega^l K(-, n)] \to \tilde{H}^{n-l}(-, -)$$

is an equivalence of functors.

In what follows, we will prove several things about the definition of T. First of all, we need to prove that $\xi = (\xi_0, \ldots, \xi_k)$ is a cycle, so that T will be well-defined. This is done by proving that it satisfies (9.1) and (9.2).

The second thing we need to prove is that we have defined a transformation that is natural in the coefficient variable. This is the requirement that will prove that $K(B_0 \to \ldots \to B_k, n)$ is a *natural* classifying space.

The last thing we need to prove is that the transformation T is in fact an equivalence.

Lemma 9.2. *The set $(\xi_0, (-1)^n\xi_1, \ldots, (-1)^{nk}\xi_k)$ with $\xi_0, \ldots, \xi_k$ as defined above is a cycle.*

Proof: The proof will be by induction on k. The case $k = 0$ is known from ordinary cohomology.

For the induction step, suppose $(\xi'_0, (-1)^n\xi'_1, \ldots, (-1)^{(k-1)n}\xi'_{k-1})$ is a cycle. We need to prove that the element

$$(p_k^*(\xi'_0), (-1)^n p_k^*(\xi'_1), \ldots, (-1)^{(k-1)n} p_k^*(\xi'_{k-1}), (-1)^{nk}\xi_k)$$

satisfies (9.1) and (9.2). Most of the equations follow trivially from the induction hypothesis. The only one left to prove is

$$\xi_k \partial = h_{k-1} p_k^*(\xi'_{k-1}).$$

Suppose σ is an $(n-k+1)$-cycle of $K(B_0 \to \ldots \to B_k, n)$ represented by its characteristic map $\sigma : D^{n-k+1} \to K(B_0 \to \ldots \to B_k, n)$. Then $p_k^*(\xi'_{k-1})(\sigma)$ is by definition the element

$$p_k \circ \sigma \in \pi_{n-k+1}(\Omega^{k-1} K(B_{k-1}, n)) \cong B_{k-1} \ .$$

Since the identification $\pi_{n-k+1}(\Omega^{k-1}K(B_{k-1}, n)) \cong B_{k-1}$ comes from the natural transformation from $\pi_{n-k+1}(\Omega^{k-1}K(-, n))$ to the identity functor, $h_{k-1} \circ p_k^*(\xi'_{k-1})(\sigma)$ is the element

$$\Omega^{k-1}K(h_{k-1}, n) \circ p_k \circ \sigma \in \pi_{n-k+1}(\Omega^{k-1}K(B_k, n)) \cong B_k \ .$$

It is clear from definitions that the map

$$K((0, \ldots, 0, h_{k-1}), n) : K(B_0 \to \ldots \to B_{k-1}, n) \to \Omega^{k-1}K(B_k, n)$$

restricted to $\Omega^{k-1}K(B_{k-1}, n)$ is $\Omega^{k-1}K(h_{k-1}, n)$, and thus we know that $h_{k-1} \circ p_k^*(\xi'_{k-1})(\sigma)$ is represented by the map $K(h_{k-1}, n) \circ p_k \circ \sigma$.

Since Diagram 8.1 commutes up to a homotopy, which is the constant map to $*$ on $K(B_0 \to \ldots \to B_k, n)^{(n-k)} \times I$, we have that the same element is represented by the map

$$\eta \circ q_k \circ \sigma \in \pi_{n-k+1}(\Omega^{k-1} K(B_k, n)) \cong B_k \ .$$

On the other side of the equation we have $\xi_k \partial\sigma$ which is the element represented by the map $\partial\sigma = \sigma|_{S^{n-k}} : S^{n-k} \to \Omega^k K(B_k, n)$. Using the natural transformation

$$S^{-1} : \pi_{n-k}\Omega \to \pi_{n-k+1}$$

we get that the same element is represented by

$$S^{-1}(\partial\sigma) \in \pi_{n-k+1}(\Omega^{k-1} K(B_k, n)).$$

Now, all we need to complete the proof is a homotopy from $S^{-1}(\partial\sigma)$ to $\eta \circ q_k \circ \sigma$ inside $\Omega^{k-1} K(B_k, n)$. Let

$$F : D^{n-k+1} \times I / D^{n-k+1} \times \{0\} \to \Omega^{k-1} K(B_k, n)$$

be defined as $F(x,t) = (q_k \circ \sigma(x))(t)$. Then

$$F|_{D^{n-k+1} \times \{1\}} = \eta \circ q_k \circ \sigma$$

and

$$F|_{S^{n-k} \times I} = S^{-1}(\partial\sigma).$$

So F provides the desired homotopy. □

Thus, if $\xi = (\xi_0, \ldots, (-1)^{nk}\xi_k)$, we can define a natural transformation

$$T : [-, K(B_0 \to \ldots \to B_k, n)] \to \tilde{H}^n(-, B_0 \to \ldots \to B_k)$$

by setting $T([f]) = f^*(\xi)$. Since our goal was to define a natural classifying space, the next step is to prove that this transformation is natural in the coefficient variable.

We have to prove that if $\phi : (B_0 \to \ldots \to B_k) \longrightarrow (B'_0 \to \ldots \to B'_k)$ is a chain map then the following diagram commutes for all X:

$$\begin{array}{ccc} [X, K(B_0 \to \ldots \to B_k, n)] & \xrightarrow{K(\phi,n)_*} & [X, K(B'_0 \to \ldots \to B'_k, n)] \\ \downarrow T & & \downarrow T \\ \tilde{H}^n(X, B_0 \to \ldots \to B_k) & \xrightarrow{\phi_*} & \tilde{H}^n(X, B'_0 \to \ldots \to B'_k) \end{array}$$

Since all maps in this diagram are natural transformations, it suffices to prove that

$$\phi_* T([id]) = TK(\phi, n)_*([id]).$$

Lemma 9.3. *The transformation T is natural in the coefficient variable.*

Proof: The proof will be by induction on k. The case $k = 0$ is known from ordinary cohomology. In this case we can actually prove that if

$$\xi : C_n(K(A, n)) \to A$$

and

$$\eta : C_n(K(B, n)) \to B$$

are the maps that give the natural transformation, then

$$h \circ \xi = \eta \circ K(h, n)_* \ .$$

We shall prove that this identity holds in general by induction. For the induction, suppose

$$\xi' = (\xi'_0, \ldots, (-1)^{n(k-1)} \xi'_{k-1})$$

and

$$\eta' = (\eta'_0, \ldots, (-1)^{n(k-1)} \eta'_{k-1})$$

are the elements giving the transformation in the cases of the chains $B_0 \to \ldots \to B_k$ and $B'_0 \to \ldots \to B'_k$, respectively. We shall use the names $\phi = (\phi_0, \ldots, \phi_{k-1})$ and $\bar{\phi} = (\phi_0, \ldots, \phi_k)$. The induction hypothesis will be that

$$\phi \circ \xi' = \eta' \circ K(\phi, n)_* \ .$$

We define

$$\xi = (p_k^*(\xi'_0), \ldots, (-1)^{n(k-1)} p_k^*(\xi'_{k-1}), (-1)^{nk} \xi_k)$$

and

$$\eta = (p_k^*(\eta'_0), \ldots, (-1)^{n(k-1)} p_k^*(\eta'_{k-1}), (-1)^{nk} \eta_k)$$

as usual. We need to prove that

$$\bar{\phi} \circ \xi = \eta \circ K(\bar{\phi}, n)_* \ .$$

Let us first concentrate on the first $k-1$ places above. Suppose $l \in \{1, \ldots, k-1\}$. We need to prove that

$$\phi_l \circ \xi'_l \circ (p_k)_* = \eta_l \circ (p_k)_* K(\bar{\phi}, n)_* \ .$$

If we can prove that

$$p_k \circ K(\bar{\phi}, n) = K(\phi, n) \circ p_k,$$

the equation will follow from the induction hypothesis, but the latter equation follows from the definitions.

The last thing we need to prove is that

$$\phi_k \circ \xi_k = \eta_k \circ K(\bar{\phi}, n)_* \ .$$

Notice that on $K(B_0 \to \ldots \to B_k, n)^{(n-k)} = \Omega^k K(B_k, n)$ we have that $K(\bar{\phi}, n)$ is just $\Omega^k K(\phi_k, n)$. This reduces the equation to the one known from ordinary cohomology. □

10. The main theorem

By now we have defined a natural transformation, that we would like to be an equivalence. Consider the following:

Diagram 10.1

$$\begin{array}{ccc}
[X, \Omega K(B_0 \to \ldots \to B_{k-1}, n)] & \xrightarrow{T} & \tilde{H}^{n-1}(X, B_0 \to \ldots \to B_{k-1}) \\
\downarrow{\scriptstyle \Omega K(h_{k-1}, n)} & & \downarrow{\scriptstyle (h_{k-1})_*} \\
[X, \Omega^k K(B_k, n)] & \xrightarrow{T} & \tilde{H}^{n-k}(X, B_k) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \phi} \\
[X, K(B_0 \to \ldots \to B_k, n)] & \xrightarrow{T} & \tilde{H}^{n}(X, B_0 \to \ldots \to B_k) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \psi} \\
[X, K(B_0 \to \ldots \to B_{k-1}, n)] & \xrightarrow{T} & \tilde{H}^{n}(X, B_0 \to \ldots \to B_{k-1}) \\
\downarrow{\scriptstyle K((0,\ldots,0,h_{k-1}), n)} & & \downarrow{\scriptstyle (h_{k-1})_*} \\
[X, \Omega^{k-1} K(B_k, n)] & \xrightarrow{T} & \tilde{H}^{n-k+1}(X, B_k)
\end{array}$$

The diagram should be interpreted using identifications of the form:

$$\Omega^k K(B_k, n) = K(\underbrace{0 \to \ldots \to 0}_{k} \to B_k, n)$$

and

$$\tilde{H}^{n-k}(X, B_k) = \tilde{H}^n(X, \underbrace{0 \to \ldots \to 0}_{k} \to B_k).$$

Lemma 10.1. *Diagram 10.1 commutes.*

Proof: This is a consequence of the fact that T is a natural transformation. □

If we replace all instances of X in Diagram 10.1 by ΣX, we know that all maps involved are homomorphisms; therefore, we may apply the Five Lemma to this diagram and prove that T induces an isomorphism by induction.

Theorem 10.2. *The functors $[-, \Omega K(-, n+1)]$ and $\tilde{H}^n(-,-)$, taking the first variable in the category of pointed CW-complexes and the second in the category of finite chain complexes of abelian groups, are naturally equivalent.*

Proof: By induction on Diagram 10.1 we know that the maps

$$T : [\Sigma X, \Omega^l K(B_0 \to \ldots \to B_k, n)] \to \tilde{H}^{n-l}(\Sigma X, B_0 \to \ldots \to B_k)$$

are isomorphisms as long as $n - k - l \geq 0$ (l can be zero).

This proves that the functors $[\Sigma -, K(-, n)]$ and $\tilde{H}^n(\Sigma -, -)$ are naturally equivalent functors defined on chains of length at most $n + 1$.

It follows easily from the Meyer-Vietoris sequence that the functors $\tilde{H}^{n+1}(\Sigma -, -)$ and $\tilde{H}^n(-,-)$ are equivalent. Using this equivalence and the equivalence

$$[-, \Omega K(-, n+1)] \cong [\Sigma -, K(-, n+1)],$$

we get a natural equivalence of functors

$$[-, \Omega K(-, n+1)] \cong \tilde{H}^n(-,-)$$

defined on chains of length at most $n + 2$. Now suppose $k > n + 2$. Then we have natural homotopy equivalences

$$\Omega K(B_0 \to \ldots \to B_k, n+1) \simeq K(0 \to B_0 \to \ldots \to B_k, n+1) \simeq$$
$$K(0 \to B_0 \to \ldots \to B_{n+1}, n+1) \simeq \Omega K(B_0 \to \ldots \to B_{n+1}, n+1)$$

by Lemma 7.2, and since Corollary 5.3 gives a natural equivalence

$$\tilde{H}^n(-, B_0 \to \ldots \to B_k) \cong \tilde{H}^n(-, B_0 \to \ldots \to B_{n+1}),$$

the equivalence of functors holds in the case of arbitrary finite chains. This concludes the proof. □

If we allow the chains to be infinite in one direction, that is, allow chains on the form

$$B_0 \to B_1 \to \dots,$$

then the fact that the groups B_k for $k > n+1$ do not affect $\tilde{H}^n$ shows that $K(B_0 \to \dots \to B_{n+1}, n)$ is a natural classifying space for cohomology with coefficients in $B_0 \to B_1 \to \dots$.

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