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CONTINUA ON WHICH ALL REAL–VALUED CONNECTED FUNCTIONS ARE CONNECTIVITY FUNCTIONS

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Dedicated to B. D. Garrett

ABSTRACT. It is shown that the continua on which every connected real–valued function is a connectivity function, as well as the continua on which every connected real–valued function is a Darboux function, are precisely the dendrites X each of whose arcs contains only finitely many branch points of X.

1. Introduction

A *continuum* is a nonempty compact connected metric space. (For basic terminology in connection with continua, see [8].) If X and Y are topological spaces and f is a function from X to Y, then $\Gamma(f)$ denotes the graph of f in the Cartesian product space $X \times Y$; that is,

$$\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in X\}.$$

We denote the restriction of a function f to a subset A of its domain by $f|A$.

We are concerned with three types of functions that are defined (below) in terms of properties of continuous functions. There is a vast literature concerning these types of functions – we refer the

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reader to [2] and [9] for excellent surveys of results and substantial bibliographies.

Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Then

- f is a *connected function* provided that $\Gamma(f)$ is connected.
- f is a *connectivity function* provided that $\Gamma(f|C)$ is connected for all connected subsets C of X.
- f is a *Darboux function* provided that $f(C)$ is connected for all connected subsets C of X.

Let X and Y be spaces such that X is connected. It is obvious that connectivity functions from X to Y are connected functions; also, using the projection map of $X \times Y$ to Y, we see that connectivity functions from X to Y are Darboux functions. These are the only implications that hold in general, even for real–valued functions defined on continua; in particular, there are real–valued Darboux functions defined on $[0, 1]$ that are not connected (such functions are not of Baire class 1 [7]); we will have occasion to discuss one such function near the end of the paper.

Thus, we became interested in determining those continua on which every connected real–valued function is a connectivity function and those continua on which every connected real–valued function is a Darboux function. We completely characterize such continua as being, in both cases, the dendrites X each of whose arcs contains only finitely many branch points of X (Theorem 8).

We note that such dendrites can contain an infinite number of branch points; for example, let X be the union of a null sequence of simple triods with the same end point p and otherwise disjoint. (We thank Alejandro Illanes for pointing out this example, which led us to correct the original version of Theorem 8.)

After this paper was written, Francis Jordan made us aware of B. D. Garrett's paper [4]. Garrett shows that a real–valued connected function on a dendrite with only finitely many branch points is a connectivity function. This result is a special case of (3) implies (1) in our Theorem 8; thus, our paper completes the study in [4].

2. Additional Terminology and Notation

In addition to terminology and notation in the introduction, we note the following (see [8] for terminology not given).

A *Peano continuum* is a locally connected continuum. A *dendrite* is a Peano continuum that contains no simple closed curve. A *branch point of a dendrite* is a point of the dendrite that is of order > 2 [8, p. 174].

Let A be an arc in a space X, and let p and q be the end points of A. If $A - \{p, q\}$ is open in X, then A is called a *free arc in* X.

A *retraction* is a continuous function that is the identity on its range. (Since our results are about functions that are not necessarily continuous, it is important to remember that retractions *are* continuous.)

When referring to a space as *nondegenerate*, we mean that the space contains at least two points.

We denote the real line by $\mathbb{R}^1$ and the interval $[0, 1]$ by I. We signify the closure of a subset A of a space X by $\overline{A}$ and the interior of A in X by $int_X(A)$.

3. The Characterization

Our characterization is Theorem 8. We break the proof of Theorem 8 down into several lemmas. At two stages of the development, Lemma 4 and Lemma 7, we prove the two main parts of the characterization.

The proof of the following lemma consists of simple observations about the proof of a theorem due to J. L. Cornette [3].

Lemma 1 (Cornette). *If S and T are nondegenerate connected separable metric spaces, then there is a connected function $f : S \overset{onto}{\rightarrow} T$ such that $\Gamma(f)$ is dense in $S \times T$.*

Proof: The proof of Theorem 1 of [3, p. 185] begins by observing that a connected function defined on I to a connected separable metric space is a connectivity function. The proof then goes on to construct a connected function from I onto a given connected separable metric space Y. It follows from the construction that Γ, used in [3] to denote the graph of the resulting function, is dense in $I \times Y$. (The closure of any nonempty open set in $I \times Y$ is a member of the collection denoted by H in the proof in [3], and Γ intersects each member of H.) It is easy to see that the construction applies to the situation when I is replaced by S in our lemma. □

Lemma 2. *Let X be a continuum such that every connected real–valued function on X is a Darboux function. Then X does not contain a nowhere dense nondegenerate continuum.*

Proof: Let X be a continuum such that X contains a nowhere dense nondegenerate continuum A. Let $\mathcal{K}$ denote the collection of all components of $X - A$. We note from [8, 5.6, p. 74] that

(1) $\overline{K} \cap A \neq \emptyset$ for each $K \in \mathcal{K}$.

By (1), each $K \in \mathcal{K}$ is nondegenerate. Hence, we can apply Lemma 1 to each $K \in \mathcal{K}$ to obtain a connected function $f_K : K \overset{\text{onto}}{\to} I$ such that $\Gamma(f_K)$ is dense in $K \times I$.

By [8, 6.6, p. 89] there is a point $a \in A$ such that $A - \{a\}$ is connected. Let $A_0 = A - \{a\}$, and define a function $f : X \overset{\text{onto}}{\to} I$ as follows:

$$f(x) = \begin{cases} f_K(x) & \text{if } x \in K \in \mathcal{K} \\ 1 & \text{if } x = a \\ 0 & \text{if } x \in A_0. \end{cases}$$

Since $f(A)$ is not connected, f is not a Darboux function. We complete the proof of the lemma by proving that f is a connected function.

Let

$$\mathcal{F} = \{K \in \mathcal{K} : \overline{K} \cap A_0 \neq \emptyset\}.$$

Let $K \in \mathcal{F}$. Since $\Gamma(f_K) = \Gamma(f|K)$ is connected and dense in $K \times I$, it follows that $\Gamma(f|\overline{K})$ is connected and dense in $\overline{K} \times I$. Also, since A_0 is connected and $f|A_0$ is constant, $\Gamma(f|A_0)$ is connected. Thus, since $\overline{K} \cap A_0 \neq \emptyset$, we have that $\Gamma(f|(K \cup A_0))$ is connected. Therefore, letting

$$Y = (\cup\mathcal{F}) \cup A_0,$$

it follows that

(2) $\Gamma(f|Y)$ is connected.

Since $\Gamma(f_K) = \Gamma(f|K)$ is dense in $K \times I$ for each $K \in \mathcal{K}$, it follows that $\Gamma(f| \cup \mathcal{F})$ is dense in $(\cup\mathcal{F}) \times I$. Thus, $\overline{\Gamma(f| \cup \mathcal{F})} \supset \overline{(\cup\mathcal{F}) \times I} = \overline{(\cup\mathcal{F})} \times I$. Also, $\overline{\cup\mathcal{F}} \supset A$ (since A is nowhere dense in X). Hence,

$$\overline{\Gamma(f| \cup \mathcal{F})} \supset A \times I.$$

Thus, $(a,1) \in \overline{\Gamma(f|\cup\mathcal{F})}$. Therefore, since $\cup\mathcal{F} \subset Y$, it is clear that

(3) $(a,1) \in \overline{\Gamma(f|Y)}$.

Now, let $\mathcal{M} = \mathcal{K} - \mathcal{F}$. We note that $\mathcal{M}$ may be empty; nevertheless, by (1),

$$\mathcal{M} = \Big\{K \in \mathcal{K} : \overline{K} \cap A = \{a\}\Big\}.$$

We also see from (1) that

(4) $Y \cup (\cup\mathcal{M}) \cup \{a\} = X$.

Let $K \in \mathcal{M}$. Then $\overline{K} = K \cup \{a\}$. Thus, since $\Gamma(f_K)$ is dense in $K \times I$, $(a,1) \in \Gamma(f|\overline{K})$. Hence, by (3), the sets $\Gamma(f|\overline{K})$ and $\Gamma(f|Y)$ are not mutually separated. Also, $\Gamma(f|\overline{K})$ is connected since $\Gamma(f_K)$ is connected and dense in $K \times I$, and $\Gamma(f|Y)$ is connected by (2). Therefore, $\Gamma(f|(Y \cup \overline{K}))$ is connected. This proves that $\Gamma(f|(Y \cup \overline{K}))$ is connected for each $K \in \mathcal{M}$. Therefore, it now follows from (2) and (4) that $\Gamma(f)$ is connected. □

Lemma 3. *Let X be a continuum such that every connected real–valued function on X is a Darboux function. Then X does not contain a simple closed curve.*

Proof: Assume that X satisfies the assumptions in our lemma but that X contains a simple closed curve S. We obtain a contradiction by constructing a connected function $f : X \to I$ that is not a Darboux function.

By Lemma 2, there is an arc $A = \widehat{pq} \subset S$ such that $A - \{p,q\}$ is open in X. Let $U = A - \{p,q\}$, and fix a point $a_0 \in U$.

By the Tietze Extension Theorem [6, p. 127], there is a retraction from $X - U$ onto $S - U$. Since U is open in X, it is clear that we can extend the given retraction to a retraction $r : X \overset{\text{onto}}{\to} S$ by letting $r(s) = s$ for all $s \in U$. We note for use later that

(1) $r(X - U) = S - U$.

Let φ be a one-to-one continuous function from $[0,1)$ onto S such that $\varphi(0) = a_0$. Now, define a function $f : X \overset{\text{onto}}{\to} [0,1)$ by letting

$$f = \varphi^{-1} \circ r.$$

We let A_p and A_q denote the subarcs of A from a_0 to p and from a_0 to q, respectively. We assume without loss of generality that

(2) $\varphi^{-1}(p) < \varphi^{-1}(q)$ and, hence, $\varphi^{-1}|A_p$ is continuous.

We see that f is not a Darboux function since, by (2),

$$f(A_q) = [\varphi^{-1}(q), 1) \cup \{0\}$$

and, therefore, $f(A_q)$ is not connected.

We now prove that f is a connected function. This will complete the proof (as indicated at the beginning of the proof).

Since r is the identity on S, it follows from (1) that $r(X-\{a_0\}) = S - \{a_0\}$. Thus, since $\varphi^{-1}|(S - \{a_0\})$ is continuous, we see that $f|(X - \{a_0\})$ is continuous. Furthermore, $X - \{a_0\}$ is connected (as is seen from [8, 5.6, p. 74] by using that $S - \{a_0\}$ is connected and that $a_0 \notin \overline{X - S}$). Therefore, since continuous functions on connected spaces have connected graphs, we have that

(3) $\Gamma(f|(X - \{a_0\}))$ is connected.

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in $A_p - \{a_0\}$ such that

$$\lim_{n\to\infty} p_n = a_0.$$

Then, since $\varphi^{-1}|A_p$ is continuous by (2), $\lim_{n\to\infty} \varphi^{-1}(p_n) = \varphi^{-1}(a_0)$; thus, since r is the identity on S and $f = \varphi^{-1} \circ r$,

$$\lim_{n\to\infty} f(p_n) = f(a_0).$$

Thus, since $(p_n, f(p_n)) \in \Gamma(f|(X-\{a_0\}))$ for each n and $\lim_{n\to\infty} p_n = a_0$, we see that

$$(a_0, f(a_0)) \in \overline{\Gamma(f|(X - \{a_0\}))}.$$

Therefore, by (3), $\Gamma(f)$ is connected. □

Our next lemma proves one of the implications in Theorem 8. First, note the following terminology.

A *null comb* is a continuum that is homeomorphic to the continuum N in the plane consisting of the line segment L from $(0,0)$ to $(1,0)$ together with the line segments L_n from $(\frac{1}{n+1}, 0)$ to $(\frac{1}{n+1}, \frac{1}{n+1})$, $n = 1, 2, \ldots$.

Lemma 4. *Let X be a continuum such that every connected real–valued function on X is a Darboux function. Then X is a dendrite that does not contain a null comb; hence, each arc in X contains only finitely many branch points of X.*

Proof: By Lemma 2 and [8, 5.12, p. 76], X is a locally connected continuum. Therefore, by Lemma 3, X is a dendrite.

Now, assume that X contains a null comb Y. Let

$$Y = B \cup (\cup_{n=1}^{\infty} A_n),$$

where, for a given homeomorphism h of the null comb $N = L \cup (\cup_{n=1}^{\infty} L_n)$ defined above onto Y,

$$B = h(L), \;\; A_n = h(L_n), \;\; b_n = h(L \cap L_n), \;\; p = h((0,0)).$$

We define a connected function $f : X \to I$ such that f is not a Darboux function. This will prove our lemma.

For each $n = 1, 2, ...,$ let $h_n : A_n \overset{\text{onto}}{\to} I$ be a homeomorphism such that $h_n(b_n) = 0$. Define a function $g : Y \to I$ by

$$g(y) = \begin{cases} h_n(y) & \text{if } y \in A_n \\ 0 & \text{if } y \in B - \{p\} \\ 1 & \text{if } y = p. \end{cases}$$

By [8, 10.25, p. 176], there is a monotone retraction $r : X \overset{\text{onto}}{\to} Y$. (The fact that r is monotone is not explicitly stated in [8, 10.25], but is easy to verify.) Finally, let $f : X \to I$ be given by

$$f = g \circ r.$$

Since $f(B) = \{0, \frac{1}{2}\}$, f is not a Darboux function. We prove that f is a connected function.

Since $Y - \{p\}$ is connected and r is monotone, $r^{-1}(Y - \{p\}) = X - r^{-1}(p)$ is connected [8, 8.46, p. 137]. Also, since r and $g|(Y - \{p\})$ are continuous, $f|(X - r^{-1}(p))$ is continuous; hence,

(1) $\Gamma(f|(X - r^{-1}(p)))$ is connected.

Since $f|r^{-1}(p)$ is constantly 1 and $r^{-1}(p)$ is connected,

(2) $\Gamma(f|r^{-1}(p))$ is connected.

Let a_n denote the end point of A_n different from b_n for each n; then $f(a_n) = h_n(a_n) = 1$ for each n and $\lim_{n\to\infty} a_n = p$. Thus, $(a_n, 1) \in \Gamma(f|(X - r^{-1}(p)))$ for each n and $\lim_{n\to\infty}(a_n, 1) = (p, 1) \in \Gamma(f|r^{-1}(p))$. Hence, the sets in (1) and (2) are not mutually separated. Therefore, by (1) and (2), $\Gamma(f)$ is connected; in other words, f is a connected function. $\square$

We now head towards proving Lemma 7, which is the remaining major implication in Theorem 8.

Lemma 5. *Let X be a dendrite, and let f be a connected real–valued function defined on X. Then $f|A$ is a connectivity function for each free arc A in X.*

Proof: Assume that X is a dendrite and that $f : X \to \mathbb{R}^1$ is a function for which there is a free arc $A = \widehat{pq}$ in X such that $f|A$ is not a connectivity function. Then, as noted at the beginning of the proof of Lemma 1, $f|A$ is not a connected function. Hence, there are nonempty mutually separated sets G and H such that $\Gamma(f|A) = G \cup H$. Note that

(1) $\Gamma(f) = \Gamma(f|(X - A)) \cup G \cup H$.

We prove that f is not a connected function (which proves our lemma). We take two cases which, by symmetry, exhaust all possibilities.

Case 1: $(p, f(p)) \in G$ and $(q, f(q)) \in G$. Then the two sets $\Gamma(f|X - A) \cup G$ and H are mutually separated. Hence, by (1), f is not a connected function.

Case 2: $(p, f(p)) \in G$ and $(q, f(q)) \in H$. By [8, 10.2, p. 166], p and q are separated in X by a third point c; clearly, $c \in A - \{p, q\}$. Hence, letting $U = A - \{p, q\}$, p and q are separated in X by U. Thus, there are mutually separated sets P and Q such that

$$p \in P, \quad q \in Q, \text{ and } \quad X - U = P \cup Q.$$

Then the two sets $(P \times \mathbb{R}^1) \cup G$ and $(Q \times \mathbb{R}^1) \cup H$ are mutually separated and, by (1), their union contains $\Gamma(f)$ (since $\Gamma(f|A) = G \cup H$). Hence, f is not a connected function. □

Lemma 6. *Let X be a nondegenerate dendrite such that each arc in X contains only finitely many branch points of X. Then $X = \cup_{n=1}^{\infty} A_n$, where each A_n is a free arc in X.*

Proof: Let $\mathcal{M}$ denote the collection of all maximal arcs in X. By [8, 10.47, p. 188], $X = \cup\mathcal{M}$.

Let $M \in \mathcal{M}$. By assumption in the lemma, M contains only finitely many branch points of X. Thus, $M = \cup_{i=1}^{k} A_{i,M}$, $k < \infty$, where each $A_{i,M}$ is a maximal free arc in X. Let

$$\mathcal{A} = \{A_{i,M} : M \in \mathcal{M}\}.$$

Now, note that if $A_{i,M}$ and $A_{j,M'}$ are different arcs in $\mathcal{A}$, then

$$int_X(A_{i,M}) \cap int_X(A_{j,M'}) = \emptyset.$$

Thus, since X is a separable metric space, we see that $\mathcal{A}$ is countable. Therefore, since $X = \cup\mathcal{M} = \cup\mathcal{A}$, we have proved our lemma. $\square$

Lemma 7. *Let X be a dendrite such that each arc in X contains only finitely many branch points of X. Then every connected real-valued function on X is a connectivity function.*

Proof: Let $f : X \to \mathbb{R}^1$ be a connected function. To prove that f is a connectivity function, let C be a nonempty connected subset of X. By Lemma 6, $X = \cup_{n=1}^{\infty} A_n$ where each A_n is a free arc in X. Let

$$\mathcal{A} = \{A_n : A_n \cap C \neq \emptyset\}.$$

We show that we can index the members of A so that

$$\mathcal{A} = \{A_{n_1}, A_{n_2}, ... : A_{n_i} \cap A_{n_{i+1}} \neq \emptyset \text{ for each } i\}$$

as follows: Let $n_1 = 1$. Let n_2 be the first $n > n_1$ such that $A_n \cap A_{n_1} \neq \emptyset$. Let n_3 be the first $n > n_2$ such that $A_n \cap A_{n_2} \neq \emptyset$ if such an n exists; otherwise, let $n_3 = n_1$. Let n_4 be the first integer $n > \max\{n_1, n_2, n_3\}$ such that $A_n \cap A_{n_3} \neq \emptyset$. Continue the indicated pattern.

For each positive integer k, let $B_k = \cup_{i=1}^{k} A_{n_i}$ and let $C_k = B_k \cap C$. Applying Lemma 5 k times, it follows that $f|B_k$ is a connectivity function for each k. Thus, by [8, 10.10, p. 169], since C_k is connected for each k, we have that

$$\Gamma(f|(C_k) \text{ is connected for each } k.$$

Therefore, since $C_k \subset C_{k+1}$ for each k and $\cup_{k=1}^{\infty} C_k = C$, we see that $\Gamma(f|(C)$ is connected. $\square$

We now have our characterization theorem:

Theorem 8. *For a continuum X, the following three statements are equivalent:*

(1) every connected real–valued function on X is a connectivity function;

(2) every connected real–valued function on X is a Darboux function;

(3) X is a dendrite such that each arc in X contains only finitely many branch points of X.

Proof: The fact that (1) implies (2) is simply because connectivity functions are Darboux functions (as we noted in section 1). By Lemma 4, (2) implies (3). By Lemma 7, (3) implies (1). □

We will give a corollary concerning almost continuous functions. A function $f : X \to Y$ is *almost continuous* provided that each neighborhood of $\Gamma(f)$ in $X \times Y$ contains the graph of a continuous function from X to Y [10, p. 252].

By Proposition 3 of J. Stallings [10, p. 260], almost continuous real–valued functions on continua are connected functions. Hence, by Theorem 8, we have the following result:

Corollary 9. *If X is a dendrite such that each arc in X contains only finitely many branch points of X, then every almost continuous real–valued function on X is a connectivity function.*

We note that the converse of the condition (2) in Theorem 8 is false even for real–valued functions on I. This is well known, and an example is as follows: Simply extend the function $f : (0, 1) \to I$ in [5, pp. 383 - 384] to a function $g : I \to I$ by letting $g(0) = g(1) = \frac{1}{2}$; then g is a Darboux function that is not a connected function (since $\Gamma(g)$ is separated by the line $y = x$).

A more natural way to extend f in the example we just presented would be to define $g(0) = 0$ and $g(1) = 1$; however, then the function g would be a connected function – in fact, g would then be almost continuous [1] and, therefore, a connectivity function by Corollary 9.

We remark that the converse of the conclusion of Corollary 9 is false; in fact, there is even a connectivity function from I to I that is not almost continuous [3, p. 189].

We have recently obtained a result that adds to the equivalences in Theorem 8. Namely, we have shown that the continua for which every connected real–valued function is a local connectivity function are the types of continua in (3) of Theorem 8. This result, along with others, is in a paper entitled "Local connectivity functions on arcwise connected spaces and certain continua," which we have submitted for publication.

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