Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY
PROCEEDINGS
Volume 28, No. 1, 2004
Pages 277-281

A REFLECTION THEOREM FOR I-WEIGHT

ALEJANDRO RAMÍREZ-PÁRAMO

ABSTRACT. We will show that, under GCH, for every cardinal $\kappa > \omega$ and an arbitrary compact Hausdorff space X, if we have $iw(Y) < \kappa$ whenever $Y \subset X$ and $|Y| \leq \kappa$, then $iw(X) < \kappa$.

1. Introduction

The following remarkable theorem is due to A. Hajnal and I. Juhász [5]: The weight function w reflects every infinite cardinal κ (that is, if $w(X) \geq \kappa$, then $w(Y) \geq \kappa$ for some subspace Y of cardinality $\leq \kappa$). The main aim of this paper is to prove an analogous statement for the i-weight: Assume GCH (Generalized Continuum Hypothesis); for the class of compact Hausdorff spaces, iw strongly reflects all infinite cardinals. The study of reflection and the increasing union property was initiated by M. G. Tkacenko in [9] and continued by Juhász in [8]. R. E. Hodel and J. E. Vaughan in [7] made a systematic study of reflection theorems for cardinal functions.

Let w, χ and ψ denote the following standard cardinal functions: weight, character, and pseudo-character, respectively. If ϕ is a cardinal function, then $h\phi$ is the hereditary version of ϕ; i.e., $h\phi(X) = sup\,\{\phi(Y) : Y \subseteq X\}$. As well, it is known that ϕ is monotone if and only if $\phi = h\phi$. (For definitions, see [6].)

2000 *Mathematics Subject Classification.* Primary 54X10, 58Y30; Secondary 55Z10.

Key words and phrases. cardinal function, character, compact space, i-weight, pseudo-character reflection, weight.

A map $f : X \to Y$ is called a map onto if $f(X) = Y$. A condensation is a bijective continuous map onto.

For a topological space X, $iw(X)$ denotes the minimal weight of all (Tychonoff) spaces onto which X can be condensed. The cardinal invariant $iw(X)$ is called the $i-weight$ of X. For example, a space X has $i-$weight ω iff X has a weaker separable metric topology.

The next theorem summarizes some properties of the cardinal invariant iw. (See [1] and [2].)

Theorem 1.1. *(a) The function iw is monotone. (b) $\psi(X) \leq iw(X)$. (c) For each Tychonoff space X, $iw(X) \leq w(X)$. If, in addition, X be a compact Hausdorff space, then $iw(X) = w(X)$.*

Finally, a compact Hausdorff space X is called a dyadic space if X is a continuous image of the Cantor cube D^κ for some cardinal number κ.

2. Main Results

For the sake of the reader's comfort the formulations of the necessary results of Reflection Theory are given here. (See [7].)

Definition 2.1. Let ϕ be a cardinal function and $\kappa \geq \omega$ a cardinal number.

(a) ϕ reflects κ means: if $\phi(X) \geq \kappa$, then there exists $Y \subseteq X$ with $|Y| \leq \kappa$ and $\phi(Y) \geq \kappa$.

(b) ϕ strongly reflects κ means: if $\phi(X) \geq \kappa$, then there exists $Y \subseteq X$ such that:

(1) $|Y| \leq \kappa$;

(2) if $Y \subseteq Z \subseteq X$, then $\phi(Z) \geq \kappa$.

Note. For some cardinal functions, it is necessary to restrict the class of spaces under consideration in order to obtain a reflection theorem. The appropriate definition in this case is as follows: ϕ reflects κ for the class $\mathcal{C}$ if given $X \in \mathcal{C}$ with $\phi(X) \geq \kappa$, there exist $Y \subseteq X$ with $|Y| \leq \kappa$ and $\phi(X) \geq \kappa$.

The next two lemmas play a very important role in streamlining reflection proofs. (See [7].)

Lemma 2.2. *If ϕ reflects κ^+, then $h\phi$ strongly reflects κ^+.*

Lemma 2.3. *If ϕ strongly reflects all successor cardinals, then ϕ strongly reflects all infinite cardinals. In particular, if ϕ is monotone and reflects all successor cardinals, then ϕ strongly reflects all infinite cardinals.*

To present our main result, we need the next theorem. (See [7].)

Theorem 2.4. *Assume GCH: For the class of compact Hausdorff spaces, ψ strongly reflects all infinite cardinals.*

Theorem 2.5. *Assume GCH: For the class of compact Hausdorff spaces, iw strongly reflects all infinite cardinals.*

Proof: Since iw is monotone, it suffices to prove that iw reflects every successor cardinal κ^+. (See lemmas 2.2, 2.3.)

Let X be a compact Hausdorff space such that $iw(X) \geq \kappa^+$. We will show that there is a subset Z of X with $|Z| \leq \kappa^+$ and $iw(Z) \geq \kappa^+$.

Since $iw(X) = w(X)$, then $w(X) \geq \kappa^+$; hence, by the Hajnal-Juhász reflection theorem, there is a subspace Y of X with $|Y| \leq \kappa^+$ and $w(Y) \geq \kappa^+$. Let $Y_0 = cl_X(Y)$, then Y_0 is a compact Hausdorff space.

We now consider two cases:

(a) $\psi(Y_0) \geq \kappa^+$. By Theorem 2.4, there is a subspace Z of Y_0 such that $|Z| \leq \kappa^+$ and $\psi(Z) \geq \kappa^+$. It follows that $iw(Z) \geq \kappa^+$. (See Theorem 1.1 (a).)

(b) $\psi(Y_0) < \kappa^+$; i.e., $\psi(Y_0) \leq \kappa$. Since Y_0 is a compact Hausdorff space, $\psi(Y_0) = \chi(Y_0)$, and $|Y_0| \leq 2^{\chi(Y_0)}$. Hence, $|Y_0| \leq 2^\kappa = \kappa^+$, (by GCH). On the other hand, it is clear that at $iw(Y_0) = w(Y_0) \geq w(Y) \geq \kappa^+$. Thus, $Z = Y_0$ witnesses the fact that iw reflects κ^+. The proof is complete. $\square$

The next assertion is an obvious corollary of the previous theorem.

Proposition 2.6. *Assume GCH. Let X be a compact Hausdorff space. Suppose that $X = \bigcup\{X_\alpha : \alpha \in \lambda\}$, where $\{X_\alpha : \alpha \in \lambda\}$ is an increasing family ($\alpha < \beta$ implies $X_\alpha \subseteq X_\beta$) and λ is regular. If $iw(X_\alpha) < \kappa$ for all $\alpha \in \lambda$ and $\kappa < \lambda$, then $iw(X) < \kappa$.*

At the moment the author does not know the answer to the following questions:

Question 2.7. For the class of compact Hausdorff spaces, does iw strongly reflect all infinite cardinals? More generally, does iw strongly reflect all infinite cardinals?

Question 2.8. Let X be a compact Hausdorff space. Suppose that $X = \bigcup\{X_\alpha : \alpha \in \lambda\}$, where $\{X_\alpha : \alpha \in \lambda\}$ is an increasing family ($\alpha < \beta$ implies $X_\alpha \subseteq X_\beta$) and λ is regular. If $iw(X_\alpha) \leq \kappa$ for all $\alpha \in \lambda$ and $\lambda \leq \kappa$, then is $iw(X) < \kappa$?

In connection with Question 2.7, we now show that iw strongly reflects all infinite cardinals for the class of dyadic spaces.

Theorem 2.9. *For the class of dyadic spaces, iw strongly reflects all infinite cardinals.*

Proof: It suffices to prove that iw reflects every successor cardinal κ^+. Let X be a dyadic space such that $iw(X) \geq \kappa^+$. By Theorem 1.1 (c), $w(X) \geq \kappa^+$; hence, by the Efimov-Gerlitz-Hagler theorem [4], X contains a topological copy of D^{κ^+}. Let $p \in D^{\kappa^+}$; since $\chi(p, D^{\kappa^+}) = \kappa^+$, by Efimov's theorem [3], there exists $M \subseteq D^{\kappa^+}$ such that M is discrete, $|M| = \kappa^+$, and $Y = M \cap \{p\}$ is homeomorphic to $A(\kappa^+)$, where $A(\kappa^+)$ denotes the one-point compactification of a discrete space of cardinality κ^+. Therefore, $\kappa^+ \leq iw(Y)$. □

Acknowledgments. The author would like to thank the referee for many valuable suggestions and very careful corrections. I also want to thank my wife Homaira Athenea Ramírez Gutiérrez.

References

1. A. V. Arahangel'skii, *Topological Function Spaces.* Dordrecht: Kluwer Academic Publishers, 1991.
2. F. Casarrubias-Segura, *Invariantes cardinales supertopolgicos*, Manuscrito, IV Taller de Investigación en Topología General, Oaxaca, Oax. M'exico, noviembre 1996. Unpublished.
3. B. A. Efimov, *Dyadic compacta*, (Russian), Trudy Moskov. Mat. Obšč., **14** (1965), 211–247.
4. B. A. Efimov, *Mappings and embeddings of dyadic spaces*, (Russian), Mat. Sb. (N.S.), **103(145)** (1977), 52–68.
5. A. Hajnal and I. Juhász, *Having a small weight is determined by the small subspaces*, Proc. Amer. Math. Soc., **79** (1980), 657–658.
6. R. Hodel, *Cardinal functions I*, in Handbook of Set-Theoretic Topology, Ed. K. Kunen and J. E. Vaughan. Amsterdam: North-Holland, 1984. pp. 1–61.

7. R. E. Hodel and J. E. Vaughan, *Reflection theorems for cardinal functions*, Topology Appl. **100** (2000), 47–66
8. I. Juhász, *Cardinal Functions in Topology—Ten Years Later*. 2nd ed. Mathematical Centre Tracts, 123. Amsterdam: Mathematisch Centrum, 1980.
9. M. G. Tkacenko, *Chains and cardinals*, Soviet Math. Dokl. **9** (1978), 382–385.

Facultad de Ciencias Fsico-Matemáticas; Universidad Autónoma de Puebla; Puebla, México
E-mail address: aparamo@fcfm.buap.mx