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COUNTABLE MIGRANT COVERS

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Abstract. This paper contains results from my dissertation done at the University of South Carolina under Dr. Peter Nyikos. It expands upon work done by Zoltan Balogh, Joe Mashburn, and Peter Nyikos concerning when a countable closed (or open) migrant cover can be obtained.

1. Introduction

The paper by Z. Balogh, J. Mashburn, and P. Nyikos [1] used the following notation. Let G be a group of bijections from a set X to itself and let e denote the identity element. In this situation they made the following definition.

Definition 1.1. A subset A of X is *G–migrant* (or just *migrant*) if $A \cap g(A) = \varnothing$ for all $g \in G - \{e\}$.

Balogh et al. pointed out that A is migrant if and only if for $g, h \in G$

$$g \neq h \implies g(A) \cap h(A) = \varnothing.$$

They also used the terminology of P. E. Conner and E. E. Floyd [2] as defined below.

Definition 1.2. *G acts freely* on X if every element of $G - \{e\}$ is fixed point free. That is, for all $g \in G - \{e\}$ and for all $x \in X$ we have $g(x) \neq x$.

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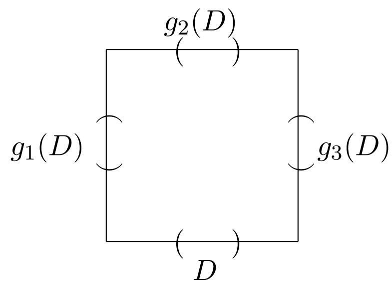

FIGURE 1. Space X with Migrant Set D

We will be dealing with a space X with a group G of autohomeomorphisms acting freely on X. As an example, consider Figure 1. X is the boundary of a square and $G = \{e, g_1, g_2, g_3\}$ is a group of autohomeomorphisms where e is the identity map and g_1, g_2, and g_3 are $90°$, $180°$, and $270°$ rotations, respectively. The open set D is migrant since it does not meet $g_1(D)$, $g_2(D)$, or $g_3(D)$.

We will use covering maps to show that if a space X is Hausdorff and subparacompact then X has a countable closed migrant cover.

2. Main Results

The first question considered by Balogh et al. [1] was for what type of space is every closed migrant set contained in some open migrant set. They answered this with the following lemma.

Lemma 2.1. *Let X be a normal space and let G be a finite group of autohomeomorphisms of X. If F is a closed migrant subset of X then there exists an open migrant subset U of X such that $F \subset U$.*

They point out that the following can also be shown.

Lemma 2.2. *Let X be a Hausdorff space and let G be a finite group of autohomeomorphisms acting freely on X. Then every element of X has a migrant open neighborhood.*

Balogh et al. [1] next considered the question of when can a countable closed or open migrant cover be obtained. We will now give their basic result.

Theorem 2.3. *Let X be a Hausdorff paracompact space. Let G be a finite group of autohomeomorphisms acting freely on X. Then X has a countable closed (or open) migrant cover.*

In this paper, we will show that "paracompact" can be replaced with "subparacompact" in this result to get the countable closed migrant cover. The space must also be normal to obtain the countable open migrant cover. This term "subparacompact" is defined by J. Nagata [4] as follows.

Definition 2.4. A space X is *subparacompact* if every open cover has a σ-discrete closed refinement.

In proving this result, we will utilize covering maps and will use the following notation from J. R. Munkres [3].

Definition 2.5. Let $\phi : X \to Y$ be a continuous surjective map. The open (or closed) set U of Y is said to be *evenly covered* by ϕ if there exists a collection $\{ V_\varepsilon : \varepsilon < \lambda \}$ of disjoint open (closed) subsets of X such that

$$\phi^{-1}(U) = \bigcup_{\varepsilon<\lambda} V_\varepsilon$$

and so that for each $\varepsilon < \lambda$ the restriction of ϕ to V_ε is a homeomorphism of V_ε onto U. The collection $\{ V_\varepsilon : \varepsilon < \lambda \}$ will be called a partition of $\phi^{-1}(U)$ into *slices.* When we choose such a collection, we will always assume it to be faithfully indexed. If every element of Y has an open neighborhood U that is evenly covered by ϕ then ϕ is called a *covering map.*

See Figure 2 for an example of a covering map ϕ. Here ϕ wraps each side of the square X around the circle Y. The open set U is evenly covered. The partition of $\phi^{-1}(U)$ is $\{V_1, V_2, V_3, V_4\}$.

Our basic approach will be as follows. We will obtain a covering map $\phi : X \to Y$ such that $\phi^{-1}(\{\phi(x)\}) = \{g(x) : g \in G\}$ for all $x \in X$. Then, we will show that Y inherits subparacompactness from X. Finally, we will construct a countable closed cover of Y consisting of evenly covered sets and "lift" it to obtain the desired cover of X.

Now for a simple result dealing with homeomorphisms and discrete closed collections. The proof is omitted.

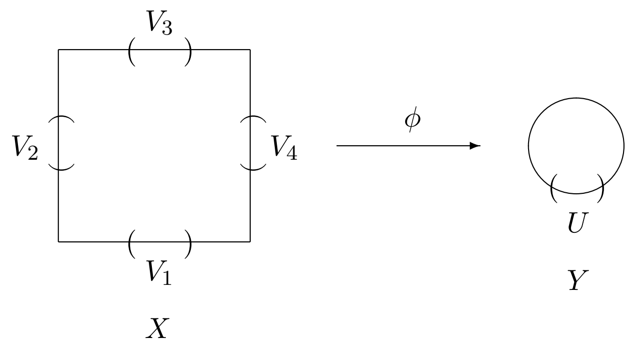

FIGURE 2. Covering Map

Lemma 2.6. *Let $\mathcal{K} = \{ K_\alpha : \alpha < \beta \}$ be a faithfully indexed discrete collection of closed subsets of X. Let $\mathcal{L} = \{ L_\alpha : \alpha < \beta \}$ be a faithfully indexed discrete collection of closed subsets of Y. Let $E = \bigcup \mathcal{K}$ and $F = \bigcup \mathcal{L}$. Let $f : E \to F$ be a function such that, for all $\alpha < \beta$, $f(K_\alpha) = L_\alpha$ and the restricted mapping $f \restriction K_\alpha : K_\alpha \to L_\alpha$ is a homeomorphism. Then f is a homeomorphism.*

Next, we will give two results on covering maps.

Lemma 2.7. *Let $\phi : X \to Y$ be a covering map. Let L be closed (or open) in Y. Let U be evenly covered and open in Y with $L \subset U$. Then L is evenly covered. Additionally, if $\{ V_\varepsilon : \varepsilon < \lambda \}$ is a partition of $\phi^{-1}(U)$ into slices, then we can choose $\{ K_\varepsilon : \varepsilon < \lambda \}$, a partition of $\phi^{-1}(L)$ into slices, such that $K_\varepsilon \subset V_\varepsilon$ for all $\varepsilon < \lambda$.*

Proof: Let $\{ V_\varepsilon : \varepsilon < \lambda \}$ be a partition of $\phi^{-1}(U)$ into slices. Let $\psi_\varepsilon : V_\varepsilon \to U$ be the restriction of ϕ to V_ε. Let $K_\varepsilon = \psi_\varepsilon^{-1}(L)$. Note that $\psi_\varepsilon(K_\varepsilon) = \psi_\varepsilon(\psi_\varepsilon^{-1}(L)) = L$ since ψ_ε is a bijection. Since the V_ε's are disjoint and $K_\varepsilon \subset V_\varepsilon$, the K_ε's are also disjoint.

Clearly,

$$\phi^{-1}(L) = \bigcup_{\varepsilon < \lambda} K_\varepsilon.$$

Fix $\varepsilon < \lambda$.

Case 1: If L is open then clearly K_ε is open.

Case 2: If L is closed, then we need to show that K_ε is closed in X. Let $x \in \overline{K}_\varepsilon$. Since ϕ is continuous, $\phi^{-1}(L)$ is closed. So

$$x \in \overline{K}_\varepsilon \subset \overline{\phi^{-1}(L)} = \phi^{-1}(L) = \bigcup_{\delta<\lambda} K_\delta.$$

Suppose that $x \notin K_\varepsilon$. Then, $x \in K_\delta \subset V_\delta$ for some $\delta \neq \varepsilon$. Since $x \in \overline{K}_\varepsilon$ and V_δ is open,

$$V_\delta \cap V_\varepsilon \supset V_\delta \cap K_\varepsilon \neq \varnothing.$$

But this is a contradiction since the V_δ's are disjoint. So $x \in K_\varepsilon$. Therefore, $\overline{K}_\varepsilon \subset K_\varepsilon$, and thus K_ε is closed in X.

Now let $\eta_\varepsilon : K_\varepsilon \to L$ be the restriction of ϕ to K_ε, or equivalently, the restriction of ψ_ε to K_ε. Since $\psi_\varepsilon : V_\varepsilon \to U$ is a homeomorphism and $K_\varepsilon \subset V_\varepsilon$ with $L = \psi_\varepsilon(K_\varepsilon)$, we see that η_ε is a homeomorphism. So $\{ K_\varepsilon : \varepsilon < \lambda \}$ is a partition of $\phi^{-1}(L)$ into slices such that $K_\varepsilon \subset V_\varepsilon$ for all $\varepsilon < \lambda$. So L is evenly covered. □

Lemma 2.8. *Let $\phi : X \to Y$ be a covering map with $|\phi^{-1}(\{y\})| = \lambda$ for all $y \in Y$. Let U be an open (or closed) subset of Y that is evenly covered by ϕ. Then $\phi^{-1}(U)$ has a partition into slices indexed by λ.*

Proof: Let $y \in U$. Let $\mathcal{V}$ be a partition of $\phi^{-1}(U)$ into slices. Each $V \in \mathcal{V}$ contains exactly one element of $\phi^{-1}(\{y\})$. This is true since, for each $V \in \mathcal{V}$, $\psi : V \to U$, the restriction of ϕ to V, is a homeomorphism, and so V contains $\psi^{-1}(y)$ and no other element of $\phi^{-1}(\{y\})$. So $|\mathcal{V}| = |\phi^{-1}(\{y\})| = \lambda$. Therefore, $\mathcal{V}$ can be indexed by λ. □

The next result uses lemmas 2.7 and 2.8 to show that, given a covering map $\phi : X \to Y$, a discrete closed collection of subsets of Y can be "lifted" to a discrete closed collection of subsets of X.

Lemma 2.9. *Let $\phi : X \to Y$ be a covering map with $|\phi^{-1}(\{y\})| = \lambda$ for all $y \in Y$. Let $\mathcal{L} = \{ L_\alpha : \alpha < \beta \}$ be a faithfully indexed discrete collection of closed subsets of Y. Let $\mathcal{U} = \{ U_\alpha : \alpha < \beta \}$ be a collection of evenly covered open subsets of Y with $L_\alpha \subset U_\alpha$ for all $\alpha < \beta$. Let $\{ V_{\varepsilon\alpha} : \varepsilon < \lambda \}$ be a faithfully indexed partition of $\phi^{-1}(U_\alpha)$ into slices and let $\{ K_{\varepsilon\alpha} : \varepsilon < \lambda \}$ be a faithfully indexed partition of $\phi^{-1}(L_\alpha)$ into slices such that $K_{\varepsilon\alpha} \subset V_{\varepsilon\alpha}$ for all $\alpha < \beta$ and $\varepsilon < \lambda$. Then $\mathcal{K} = \{ K_{\varepsilon\alpha} : \varepsilon < \lambda \text{ and } \alpha < \beta \}$ is a faithfully indexed discrete collection of closed subsets of X.*

Proof: Let $\psi_{\varepsilon\alpha} : K_{\varepsilon\alpha} \to L_\alpha$ be the restriction of ϕ to $K_{\varepsilon\alpha}$. To show that $\mathcal{K}$ is faithfully indexed, suppose that $K_{\varepsilon_1\alpha_1} = K_{\varepsilon_2\alpha_2}$. So

$$L_{\alpha_1} = \phi(K_{\varepsilon_1\alpha_1}) = \phi(K_{\varepsilon_2\alpha_2}) = L_{\alpha_2}.$$

So $\alpha_1 = \alpha_2$ since $\{ L_\alpha : \alpha < \beta \}$ is faithfully indexed. But

$$K_{\varepsilon_1\alpha_1} = K_{\varepsilon_2\alpha_2} = K_{\varepsilon_2\alpha_1} \implies \varepsilon_1 = \varepsilon_2$$

since $\{ K_{\varepsilon\alpha_1} : \varepsilon < \lambda \}$ is faithfully indexed. Therefore, $\mathcal{K}$ is faithfully indexed. We already know that each $K_{\varepsilon\alpha}$ is closed. To show that $\mathcal{K}$ is discrete, let $x \in X$. We must find an open set B containing x which meets at most one element of $\mathcal{K}$.

Case 1: Suppose $\phi(x) \notin \bigcup \mathcal{L}$.

Let $A = Y - \bigcup \mathcal{L}$. So $\phi(x) \in A$ and A is open in Y (since $\bigcup \mathcal{L}$ is closed). Let $B = \phi^{-1}(A)$. Clearly, $x \in B$ and B is open in X. Suppose $B \cap K_{\varepsilon\alpha} \neq \varnothing$ for some $\varepsilon < \lambda$ and $\alpha < \beta$. Let $w \in B \cap K_{\varepsilon\alpha}$. Then

$$\phi(w) \in A \cap \phi(K_{\varepsilon\alpha}) = A \cap L_\alpha$$

and so $A \cap L_\alpha \neq \varnothing$. This contradicts the definition of A, and so B meets nothing in $\mathcal{K}$.

Case 2: Suppose $\phi(x) \in \bigcup \mathcal{L}$.

Since $\mathcal{L}$ is discrete, we can choose A open in Y with $\phi(x) \in A$ such that A meets nothing in $\mathcal{L} - \{L_\alpha\}$ for some α. So we must have $\phi(x) \in L_\alpha \subset U_\alpha$ and so $x \in \phi^{-1}(U_\alpha)$. Choose $\varepsilon_0 < \lambda$ such that $x \in V_{\varepsilon_0\alpha}$. Let

$$B = \phi^{-1}(A) \cap V_{\varepsilon_0\alpha}.$$

So $x \in B$ which is open in X. If $B \cap K_{\varepsilon_1\alpha_1} \neq \varnothing$, then let $w \in B \cap K_{\varepsilon_1\alpha_1}$. Then $w \in \phi^{-1}(A)$ and so $\phi(w) \in A$. Also, $\phi(w) \in \phi(K_{\varepsilon_1\alpha_1}) = L_{\alpha_1}$. So $A \cap L_{\alpha_1} \neq \varnothing$, and thus $\alpha_1 = \alpha$. Since $w \in B \subset V_{\varepsilon_0\alpha}$ and $w \in K_{\varepsilon_1\alpha_1} = K_{\varepsilon_1\alpha} \subset V_{\varepsilon_1\alpha}$, we have $V_{\varepsilon_0\alpha} \cap V_{\varepsilon_1\alpha} \neq \varnothing$ and so $\varepsilon_1 = \varepsilon_0$ since $\{ V_{\varepsilon\alpha} : \varepsilon < \lambda \}$ is disjoint. Therefore, B meets nothing in $\mathcal{K} - \{K_{\varepsilon_0\alpha}\}$. So we see that $\mathcal{K}$ is discrete. $\square$

Next comes the key lemma.

Lemma 2.10. *Let $\phi : X \to Y$ be a covering map with Y subparacompact and such that $|\phi^{-1}(\{y\})| = \lambda$ for all $y \in Y$. Then Y has a countable closed cover consisting of evenly covered sets.*

Proof: Let $\mathcal{U}$ be an open cover of Y consisting of evenly covered sets. Let

$$\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n$$

be a σ-discrete closed refinement of $\mathcal{U}$. Let

$$F_n = \bigcup_{L \in \mathcal{L}_n} L$$

for all $n \in \mathbb{N}$. Since $\mathcal{L}_n$ is discrete and closed, F_n is closed. Thus, $\mathcal{F} = \{ F_n : n \in \mathbb{N} \}$ is a countable closed cover of Y. So we only need to show that F_n is evenly covered for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Let $\mathcal{L}_n = \{ L_\alpha : \alpha < \beta \}$ be a faithful indexing. For each $\alpha < \beta$ choose $U_\alpha \in \mathcal{U}$ such that $L_\alpha \subset U_\alpha$. Let $\mathcal{U}_n = \{ U_\alpha : \alpha < \beta \}$. By Lemma 2.9, for each $\alpha < \beta$ we can choose $\{ K_{\varepsilon\alpha} : \varepsilon < \lambda \}$, a partition of $\phi^{-1}(L_\alpha)$ into slices, such that

$$\mathcal{K} = \{ K_{\varepsilon\alpha} : \varepsilon < \lambda \text{ and } \alpha < \beta \}$$

is a faithfully indexed discrete collection of closed subsets of X. Let

$$E_\varepsilon = \bigcup_{\alpha < \beta} K_{\varepsilon\alpha} \ .$$

We will show that F_n is evenly covered by ϕ using $\{E_\varepsilon : \varepsilon < \lambda\}$. Since $\{ K_{\varepsilon\alpha} : \alpha < \beta \}$ is a discrete collection of closed subsets of X for all $\varepsilon < \lambda$ (since these are all subsets of $\mathcal{K}$), we have E_ε closed in X for all $\varepsilon < \lambda$. Suppose $E_{\varepsilon_1} \cap E_{\varepsilon_2} \neq \varnothing$. Let $x \in E_{\varepsilon_1} \cap E_{\varepsilon_2}$. Choose $\alpha_1, \alpha_2 < \beta$ such that $x \in K_{\varepsilon_1\alpha_1}$ and $x \in K_{\varepsilon_2\alpha_2}$. So $K_{\varepsilon_1\alpha_1} = K_{\varepsilon_2\alpha_2}$ since $\mathcal{K}$ is discrete, and so $\varepsilon_1 = \varepsilon_2$ and $\alpha_1 = \alpha_2$. Therefore, the E_ε's are disjoint.

Claim. $\phi^{-1}(F_n) = \bigcup_{\varepsilon<\lambda} E_\varepsilon$.

Let $x \in \phi^{-1}(F_n)$. So $\phi(x) \in F_n$. Let $\alpha < \beta$ such that $\phi(x) \in L_\alpha$. So $x \in \phi^{-1}(L_\alpha)$. Let $\varepsilon < \lambda$ such that $x \in K_{\varepsilon\alpha}$. So $x \in E_\varepsilon$.

Now suppose that $x \in E_\varepsilon$ with $\varepsilon < \lambda$. Let $\alpha < \beta$ such that $x \in K_{\varepsilon\alpha}$. So $\phi(x) \in L_\alpha \subset F_n$. Thus, $x \in \phi^{-1}(F_n)$. Therefore, the claim holds.

Fix $\varepsilon < \lambda$. Let $\eta_\varepsilon : E_\varepsilon \to F_n$ be the restriction of ϕ to E_ε. Let $\psi_{\varepsilon\alpha} : K_{\varepsilon\alpha} \to L_\alpha$ be the restriction of ϕ to $K_{\varepsilon\alpha}$. Since $\{ K_{\varepsilon\alpha} : \alpha < \beta \}$ and $\mathcal{L}_n$ satisfy the conditions of Lemma 2.6 and $\psi_{\varepsilon\alpha}$ is the restriction of η_ε to $K_{\varepsilon\alpha}$ and is a homeomorphism, we have that η_ε

is a homeomorphism. Therefore, F_n is evenly covered and so $\mathcal{F}$ is our desired cover. $\square$

Now we will show how the covering map for Lemma 2.10 can be obtained.

Lemma 2.11. *Let X be a Hausdorff space. Let G be a finite group of autohomeomorphisms acting freely on X. For $x \in X$, define $[x] = \{ g(x) : g \in G \}$. Let $Y = \{ [x] : x \in X \}$. Let $\phi : X \to Y$ by $\phi(x) = [x]$. Give Y the quotient topology. Then the following hold.*

(a) *If $A \subset X$ and $B = \phi(A)$, then $\phi^{-1}(B) = \bigcup_{g\in G} g(A)$ and $B = \phi(g(A))$ for all $g \in G$.*

(b) *ϕ is a covering map with $|\phi^{-1}(\{y\})| = |G|$ for all $y \in Y$, and ϕ is open and closed.*

Proof: The proof of (a) is straightforward and will be omitted. The proof of (b) is as follows:

Claim. ϕ is open (and closed).

Let A be open (closed) in X. Let $B = \phi(A)$. By (a),

$$\phi^{-1}(B) = \bigcup_{g\in G} g(A)$$

which is open (closed) in X. So B is open (closed) in Y since Y was given the quotient topology. Therefore, ϕ is open (and closed).

Claim. ϕ is a covering map.

By definition ϕ is continuous and surjective. Let $y \in Y$. We want to find an evenly covered open set V containing y. Let $x \in X$ such that $\phi(x) = y$. Since X is Hausdorff, by Lemma 2.2 let U be an open migrant neighborhood of x. Let $V = \phi(U)$. So $y \in V$. Also V is open since ϕ is open. We will show that V is evenly covered by ϕ using the collection $\{g(U) : g \in G\}$. First, notice that

$$\phi^{-1}(V) = \bigcup_{g\in G} g(U)$$

by (a). Since U is open, $g(U)$ is also open. Also

$$g \neq f \implies g(U) \cap f(U) = \varnothing$$

since U is migrant and so the $g(U)$'s are disjoint. Now, fix $g \in G$. Let $\psi_g : g(U) \to V$ be the restriction of ϕ to $g(U)$. Clearly,

$\phi(g(U)) \subset V$. To see that ψ_g is one–to–one, we let $v, w \in g(U)$ and notice that

$$\begin{aligned} \psi_g(v) = \psi_g(w) \quad &\Longrightarrow \quad [v] = [w] \\ &\Longrightarrow \quad v = h(w) \quad \text{for some} \quad h \in G. \end{aligned}$$

But since $g(U)$ is migrant

$$w \in g(U) \quad \text{and} \quad h(w) = v \in g(U) \implies h = e \implies v = w.$$

Therefore, ψ_g is one–to–one. To see that ψ_g is onto, let $z \in V$ and then let $w \in U$ such that $z = \phi(w)$. So $g(w) \in g(U)$ and

$$\psi_g(g(w)) = \phi(g(w)) = [g(w)] = [w] = \phi(w) = z.$$

Therefore, ψ_g is onto. Clearly,

$$g(U) \text{ open, } \phi \text{ open continuous} \implies \psi_g \text{ open continuous.}$$

Thus, ψ_g is a homeomorphism. So $\{\, g(U) : g \in G \,\}$ is a partition of $\phi^{-1}(V)$ into slices. So V is evenly covered by ϕ. Therefore, ϕ is a covering map and so the claim holds. Note that for $y \in Y$ we can choose $x \in X$ such that $\phi(x) = y$ and so

$$|\phi^{-1}(\{y\})| = |\{\, g(x) : g \in G \,\}| = |G|. \qquad \square$$

In the final result, we will use the following theorem from Nagata [4].

Theorem 2.12. *Let X be a space. Then the following are equivalent.*

(a) *X is subparacompact.*
(b) *Every open cover $\mathcal{U}$ of X has a σ-locally finite closed refinement.*

Now for the final result.

Theorem 2.13. *Let X be a Hausdorff subparacompact space. Let G be a finite group of autohomeomorphisms acting freely on X. Then X has a countable closed migrant cover.*

Additionally, if X is normal, then X has a countable open migrant cover.

Proof: Let $\lambda = |G|$. For $x \in X$ define $[x] = \{\, g(x) : g \in G \,\}$. Let $Y = \{\, [x] : x \in X \,\}$ and define $\phi : X \to Y$ by $\phi(x) = [x]$. Give Y the quotient topology. By Lemma 2.11, we have the following.

(a) If $A \subset X$ and $B = \phi(A)$, then $\phi^{-1}(B) = \bigcup_{g\in G} g(A)$ and $B = \phi(g(A))$ for all $g \in G$.
(b) ϕ is a covering map with $|\phi^{-1}(\{y\})| = |G|$ for all $y \in Y$, and ϕ is open and closed.

Claim. Y is subparacompact.

Let $\mathcal{U}_0$ be an open cover of Y. Let $\mathcal{U}$ refine $\mathcal{U}_0$ with $\mathcal{U}$ an open cover of Y consisting of evenly covered sets. (We can do this since each point in Y has an open evenly covered neighborhood and so we can intersect this neighborhood with elements of $\mathcal{U}_0$. By Lemma 2.7, these intersections are also evenly covered.) Let $\mathcal{U} = \{\, U_\alpha : \alpha < \beta \,\}$ be a faithful indexing. By Lemma 2.8, let $\{\, V_{\varepsilon\alpha} : \varepsilon < \lambda \,\}$ be a partition of $\phi^{-1}(U_\alpha)$ into slices. Let $\mathcal{V} = \{\, V_{\varepsilon\alpha} : \varepsilon < \lambda \text{ and } \alpha < \beta \,\}$. Let $\psi_{\varepsilon\alpha} : V_{\varepsilon\alpha} \to U_\alpha$ be the restriction of ϕ to $V_{\varepsilon\alpha}$. Clearly, $\mathcal{V}$ is a cover of X since

$$\begin{aligned}\bigcup \mathcal{V} &= \bigcup_{\alpha<\beta}\bigcup_{\varepsilon<\lambda} V_{\varepsilon\alpha} = \bigcup_{\alpha<\beta} \phi^{-1}(U_\alpha)\\ &= \phi^{-1}\left(\bigcup_{\alpha<\beta} U_\alpha\right) = \phi^{-1}(Y) = X.\end{aligned}$$

Let

$$\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$$

be a σ-discrete closed refinement of $\mathcal{V}$. Let $K \in \mathcal{K}$. Let $\varepsilon < \lambda$ and $\alpha < \beta$ such that $K \subset V_{\varepsilon\alpha}$. Since ϕ is closed, $\phi(K)$ is closed in Y. Also, $\phi(K) \subset \phi(V_{\varepsilon\alpha}) = U_\alpha$. Let

$$\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n \quad \text{with} \quad \mathcal{L}_n = \{\, \phi(K) : K \in \mathcal{K}_n \,\}.$$

So

$$\bigcup \mathcal{L} = \bigcup_{K\in\mathcal{K}} \phi(K) = \phi\left(\bigcup_{K\in\mathcal{K}} K\right) = \phi(X) = Y.$$

So $\mathcal{L}$ is a closed cover of Y and $\mathcal{L}$ refines $\mathcal{U}$; therefore, $\mathcal{L}$ refines $\mathcal{U}_0$.

Now to show that $\mathcal{L}_n$ is locally finite (then we will use Theorem 2.12). Let $y \in Y$. We need to find an open neighborhood B of y hitting at most a finite number of elements in $\mathcal{L}_n$. Let $x \in X$

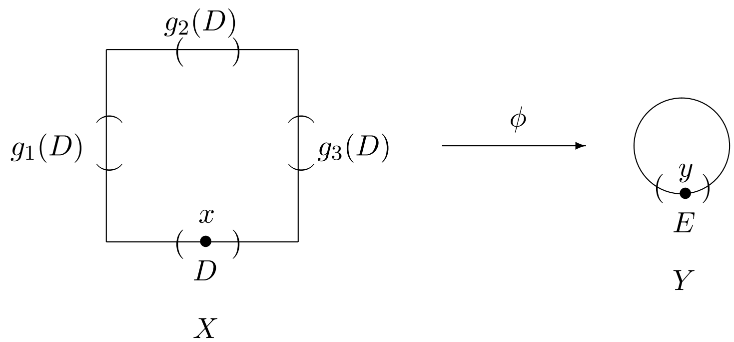

Figure 3. Covering Map with Migrant Set D

such that $y = \phi(x)$. By Lemma 2.2, let D be a migrant open neighborhood of x. Let $E = \phi(D)$. So by (a) we have

$$\phi^{-1}(E) = \bigcup_{g \in G} g(D).$$

See Figure 3 for an example of this. Since $\mathcal{K}_n$ is discrete, for each $g \in G$ let A_g be open and $K_g \in \mathcal{K}_n$ such that $g(x) \in A_g \subset g(D)$ with A_g missing every element of $\mathcal{K}_n - \{K_g\}$. Let

$$B = \bigcap_{g \in G} \phi(A_g).$$

So B is open (since ϕ is open and G is finite) and $y \in B$ since

$$y = [x] = [g(x)] = \phi(g(x)) \in \phi(A_g)$$

for all $g \in G$. Let $\widehat{\mathcal{K}}_n = \mathcal{K}_n - \{K_g : g \in G\}$. Let $\widehat{\mathcal{L}}_n = \{\phi(K) : K \in \widehat{\mathcal{K}}_n\}$. Clearly, $|\mathcal{L}_n - \widehat{\mathcal{L}}_n| \le |G|$ which is finite and so we only need to show that B hits nothing in $\widehat{\mathcal{L}}_n$. Suppose that there exists an $L \in \widehat{\mathcal{L}}_n$ with $B \cap L \neq \varnothing$. Let $K \in \widehat{\mathcal{K}}_n$ with $L = \phi(K)$. Let $b \in B \cap L$. Since $b \in L = \phi(K)$, choose $a \in K$ with $b = \phi(a)$. Now

$$\begin{aligned} b \in B \quad &\Longrightarrow \quad b \in \phi(A_g) \quad \text{for all} \quad g \in G \\ &\Longrightarrow \quad \phi^{-1}(\{b\}) \cap A_g \neq \varnothing \quad \text{for all} \quad g \in G. \end{aligned}$$

Also

$$\begin{aligned} b \in B &= \bigcap_{g\in G} \phi(A_g) \\ &\subset \bigcap_{g\in G} \phi(g(D)) \\ &= \bigcap_{g\in G} E = E \qquad \text{by (a).} \end{aligned}$$

So

$$a \in \phi^{-1}(\{b\}) \subset \phi^{-1}(E) = \bigcup_{g\in G} g(D).$$

Let $g \in G$ such that $a \in g(D)$. So

$$\begin{aligned} \phi^{-1}(\{b\}) \cap A_g &\subset \phi^{-1}(\{b\}) \cap g(D) \\ &= \{\, f(a) : f \in G \,\} \cap g(D) \\ &= \{a\} \qquad \text{since } g(D) \text{ is migrant.} \end{aligned}$$

Since we know that $\phi^{-1}(\{b\}) \cap A_g \neq \varnothing$, we must have $a \in \phi^{-1}(\{b\}) \cap A_g$ and so $a \in A_g$. Thus, $A_g \cap K \neq \varnothing$. So $K = K_g$. But this is a contradiction since $K \in \widehat{\mathcal{K}}_n$. Therefore, B hits nothing in $\widehat{\mathcal{L}}_n$. Therefore, $\mathcal{L}_n$ is locally finite. So $\mathcal{L}$ is a σ-locally finite closed refinement of $\mathcal{U}_0$ which covers Y. By Theorem 2.12, we see that Y is subparacompact.

Now to find the countable closed migrant cover of X. By Lemma 2.10, Y has a countable closed cover consisting of evenly covered sets. Let $\mathcal{F} = \{\, F_n : n \in \mathbb{N} \,\}$ be such a cover. By Lemma 2.8, let $\{\, E_{\varepsilon n} : \varepsilon < \lambda \,\}$ be a partition of $\phi^{-1}(F_n)$ into slices. Let $\eta_{\varepsilon n} : E_{\varepsilon n} \to F_n$ be the restriction of ϕ to $E_{\varepsilon n}$. Let

$$\mathcal{E} = \{\, E_{\varepsilon n} : n \in \mathbb{N} \text{ and } \varepsilon < \lambda \,\}.$$

So

$$\begin{aligned} \bigcup \mathcal{E} &= \bigcup_{n=1}^{\infty} \bigcup_{\varepsilon<\lambda} E_{\varepsilon n} = \bigcup_{n=1}^{\infty} \phi^{-1}(F_n) \\ &= \phi^{-1}\left(\bigcup_{n=1}^{\infty} F_n \right) = \phi^{-1}(Y) = X. \end{aligned}$$

So $\mathcal{E}$ is a closed cover of X. Let $n \in \mathbb{N}$ and $\varepsilon < \lambda$.

Claim. $E_{\varepsilon n}$ is migrant.

Let $x, g(x) \in E_{\varepsilon n}$. Then

$$\eta_{\varepsilon n}(x) = \phi(x) = \phi(g(x)) = \eta_{\varepsilon n}(g(x))$$

and so $x = g(x)$ which implies that $g = e$ since X is fixed point free. Therefore, $E_{\varepsilon n}$ is migrant. Since $\lambda = |G|$ is finite, $\mathcal{E}$ is countable, and so we are done.

If X is normal, then by Lemma 2.1, we can obtain a countable open migrant cover. $\square$

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