

Topology Proceedings



Web: <http://topology.auburn.edu/tp/>
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

NONSEPARABILITY AND UNIFORMITIES IN TOPOLOGICAL GROUPS

A. BOUZIAD AND J.-P. TROALLIC

ABSTRACT. Let G be a Hausdorff topological group. If the left and right uniform structures \mathcal{L}_G and \mathcal{R}_G on G coincide, then G is said to be balanced, or a SIN-group. Let $\mathcal{U}_L(G)$ (respectively $\mathcal{U}_R(G)$) denote the real Banach space of all left (respectively right) uniformly continuous bounded real-valued functions on G , and let $\mathcal{U}(G) = \mathcal{U}_L(G) \cap \mathcal{U}_R(G)$. If $\mathcal{U}_L(G) = \mathcal{U}_R(G)$, then G is said to be functionally balanced, or to be an FSIN-group. We prove that if G is not an FSIN-group, then the quotient Banach space $\mathcal{U}_R(G)/\mathcal{U}(G)$ is nonseparable. Moreover, we prove that for a large class of topological groups G , if G is not FSIN then $\mathcal{U}_R(G)/\mathcal{U}(G)$ contains a linear isometric copy of l^∞ . We also establish the equivalence between SIN and FSIN properties in various cases. In particular, we show that for any topological group G strongly functionally generated by its right precompact subsets, SIN and FSIN properties are equivalent.

1. INTRODUCTION

A topological group G is said to be a SIN-group if the left and right uniform structures \mathcal{L}_G and \mathcal{R}_G on G coincide; it is equivalent to suppose that $\mathcal{R}_G = \mathcal{L}_G \wedge \mathcal{R}_G$ (cf. Definition 3.8.2 for the notation). Let $\mathcal{U}(G)$ (respectively $\mathcal{U}_L(G)$, respectively $\mathcal{U}_R(G)$) be the real Banach space of all left and right (respectively left, respectively right) uniformly continuous bounded real-valued functions on G .

2000 *Mathematics Subject Classification.* Primary 22A05, 54E15; Secondary 22A10.

Key words and phrases. Topological group, Left (right) uniform structure, Left (right) uniformly continuous bounded real-valued function, SIN-group, FSIN-group, Left (right) thin subset, Left (right) neutral subset, Left (right) uniformly discrete subset.

If $\mathcal{U}_L(G) = \mathcal{U}_R(G)$, or, equivalently, if $\mathcal{U}_R(G) = \mathcal{U}(G)$, then G is said to be an FSIN-group. Obviously, if G is a SIN-group, then G is an FSIN-group.

In [8] (1974), Itzkowitz showed that if G is a nonunimodular locally compact topological group, then $\mathcal{U}_R(G)$ and $\mathcal{U}(G)$ are distinct. Later, in [2] (1982), Dzinotyiweyi asked if under the same condition on G the quotient Banach space $\mathcal{U}_R(G)/\mathcal{U}(G)$ is nonseparable. A positive answer to the question of Dzinotyiweyi was given in [4] (1995), not only for a nonunimodular locally compact topological group, but more generally for any locally compact topological group which is not SIN. This result was subsequently extended in [21] (1999) to any almost metrizable topological group in the sense of Pasynkov. In this paper, the following unexpected result is obtained: the quotient Banach space $\mathcal{U}_R(G)/\mathcal{U}(G)$ is nonseparable for any topological group G which is not FSIN (Section 5). It is also proved that for a large class of topological groups G , if G is not FSIN then $\mathcal{U}_R(G)/\mathcal{U}(G)$ contains a linear isometric copy of l^∞ (Section 6).

After Itzkowitz's result in 1974, it was established that SIN and FSIN properties are equivalent for large classes of topological groups (but whether or not this equivalence is always true remains an open question). All locally compact FSIN-groups are SIN; this was proved independently by Itzkowitz [10], Milnes [15] and Protasov [17] in 1991; moreover, Protasov's proof works for all almost metrizable topological groups. (All locally compact as well as all metrizable topological groups are almost metrizable.) More recently, Troallic [20] (1996), extended Protasov's result to the class of topological groups that are quasi- k -spaces, and in another direction, Megrelishvili, Nickolas and Pestov [14] (1997), proved that all locally connected FSIN-groups are SIN. In Section 3 of the present work, further information about both the locally compact and the locally connected cases is provided. In particular, a short proof is given of the fact that SIN and FSIN properties are equivalent for any locally precompact topological group; besides, this proof works for any G belonging to the larger class of ASIN-groups (cf. Definition 3.4). In Section 4, Troallic's result is extended to the vast class of topological groups that are strongly functionally generated by their right precompact subsets.

All topological groups considered in this paper are assumed to be Hausdorff. Our terminology is the same as in [3] and [19].

2. PRELIMINARIES

If G is a topological group, we always denote by e its identity element and by $\mathcal{V}_G(e)$ the set of all neighborhoods of e in G .

Definition 2.1. Let G be a topological group.

1) A subset A of G is said to be *left thin in G* if for every neighborhood V of e in G , $\cap_{x \in A} x^{-1}Vx$ is a neighborhood of e in G . If G is left thin in itself, or equivalently [6, 4.14, (g)], if the left and right uniform structures \mathcal{L}_G and \mathcal{R}_G on G are the same, then G is said to be a *SIN-group*. For every $V \in \mathcal{V}_G(e)$, let $V_{\mathcal{L}}$ (respectively $V_{\mathcal{R}}$) be the set of all $(x, y) \in G \times G$ such that $x^{-1}y \in V$ (respectively $xy^{-1} \in V$). The set of all $V_{\mathcal{L}}$ (respectively $V_{\mathcal{R}}$), as V runs through $\mathcal{V}_G(e)$, is a fundamental system of entourages of the uniform space (G, \mathcal{L}_G) (respectively (G, \mathcal{R}_G)) [6, 4.11].

2) A subset A of G is said to be *right neutral in G* if for every neighborhood V of e in G , there exists a neighborhood U of e in G such that $AU \subset VA$.

3) G is said to be a *functionally SIN-group* (or an *FSIN-group*, for short) if every left uniformly continuous bounded real-valued function on G is right uniformly continuous (or, equivalently, if every right uniformly continuous bounded real-valued function on G is left uniformly continuous).

The following result by Protasov and Saryev [18] (cf. also [10]) is a well-known criterion for a topological group to be FSIN.

Theorem 2.2. *Let G be a topological group. Then G is an FSIN-group if and only if every subset of G is right neutral in G .*

Definition 2.3. Let G be a topological group and let $V \in \mathcal{V}_G(e)$. A subset A of G is said to be *right uniformly discrete in G* (with respect to V) if Va and Vb are disjoint whenever $a, b \in A$ and $a \neq b$.

It has been remarked by Previts and Wu [16] that a topological group G is FSIN if and only if all its right uniformly discrete subsets are right neutral. Previts and Wu's proof mainly makes use of the following property: If A is a subset of G and if W is a symmetric neighborhood of e in G , then there is $B \subset A$ such that B is right uniformly discrete with respect to W and $A \subset W^2B$. Previously, the particular case of this property when $A = G$, was considered by Megrelishvili, Nickolas and Pestov [14] and used by Hernández [5]

to prove that G is SIN if and only if all its right uniformly discrete subsets are left thin. The following Lemma 2.4 is of the same sort as Hernández's result. Let us remark that it has as an immediate corollary that a subset of G is left thin in G if and only if all its right uniformly discrete subsets are left thin in G . This lemma will be used in the proof of Theorem 3.10 below.

Lemma 2.4. *Let G be a topological group. Let $A \subset G$ and $V \in \mathcal{V}_G(e)$. Let us suppose that $\bigcap_{a \in A} a^{-1}Va \notin \mathcal{V}_G(e)$. Let W be a symmetric neighborhood of e in G such that $W^5 \subset V$. Then there is a right uniformly discrete subset B of A with respect to W such that $\bigcap_{b \in B} b^{-1}Wb \notin \mathcal{V}_G(e)$.*

Proof. By Zorn's lemma, there is a maximal right uniformly discrete subset B of G with respect to W such that $B \subset A$. The maximality of B implies the inclusion $A \subset W^2B$. To obtain the lemma, it suffices to show that $\bigcap_{b \in B} b^{-1}Wb \subset \bigcap_{a \in A} a^{-1}Va$. Let $u \in \bigcap_{b \in B} b^{-1}Wb$ and $a \in A$. There is $w_1 \in W^2$ and $b \in B$ such that $a = w_1b$ and $w_2 \in W$ such that $u = b^{-1}w_2b$. Consequently, $u = a^{-1}w_1w_2w_1^{-1}a \in a^{-1}W^5a \subset a^{-1}Va$. \square

Remark 2.5. Let A be a right precompact subset of a topological group G . Then it is well known that A is left thin in G . An elementary direct proof of this fact is given for example in [6, 4.9]. Since every right uniformly discrete subset of A is necessarily finite, it is also an obvious corollary of Lemma 2.4.

The following Lemma 2.6, which is of the same sort as Previts and Wu's result, is to the concept of neutrality what Lemma 2.4 is to the concept of thinness. It implies that all the subsets of $A \subset G$ are right neutral in G if and only if all the right uniformly discrete subsets of A are right neutral in G . It should be noted that the beginning of the proof of Lemma 5.3 below is similar to the proof of Lemma 2.6.

Lemma 2.6. *Let G be a topological group. Let $A \subset G$ and $V \in \mathcal{V}_G(e)$. Let us suppose that $AU \not\subset VA$ for all $U \in \mathcal{V}_G(e)$. Let W be a symmetric neighborhood of e in G such that $W^3 \subset V$. Then there is a right uniformly discrete subset B of A with respect to W such that $BU \not\subset WB$ for all $U \in \mathcal{V}_G(e)$.*

Proof. By Zorn's lemma, there is a maximal right uniformly discrete subset B of G with respect to W such that $B \subset A$. The maximality of B implies the inclusion $A \subset W^2B$. Let $U \in \mathcal{V}_G(e)$. If the inclusion $BU \subset WB$ holds, then $W^2BU \subset W^3B$, and consequently $AU \subset VA$, which contradicts the hypothesis. \square

3. EQUIVALENCE BETWEEN SIN AND FSIN PROPERTIES: TWO BASIC CASES

If a topological group G has the SIN property, then it obviously has the FSIN property; whether or not the converse is true is still an open question raised by Itzkowitz [9]. In two fundamental cases of completely different character, the answer to Itzkowitz's question is known to be in the affirmative: all locally compact and all locally connected FSIN-groups are SIN. Recall that this was proved independently by Itzkowitz [9], Milnes [15] and Protasov [17] in the locally compact case, and by Megrelishvili, Nickolas and Pestov [14] in the locally connected case.

Our main goal in this section is to provide further information about these two cases. First, a short proof is given of the fact that SIN and FSIN properties are equivalent for any G in the class of ASIN-groups, a class larger than that of all locally precompact topological groups. Next, in the second part of the section, the concept of well-chainedness is highlighted and a substantial improvement of the result by Megrelishvili, Nickolas and Pestov is obtained.

We begin with an elementary lemma.

Lemma 3.1. *Let A and B be two left thin subsets of a topological group G . Then AB is left thin in G .*

Proof. Let $V \in \mathcal{V}_G(e)$. We have $\cap_{a \in A} a^{-1}Va \in \mathcal{V}_G(e)$ and

$$\cap_{b \in B} b^{-1}(\cap_{a \in A} a^{-1}Va)b \in \mathcal{V}_G(e).$$

Consequently, $\cap_{c \in AB} c^{-1}Vc \in \mathcal{V}_G(e)$. \square

Definition 3.2. Let G be a topological group and let $V \in \mathcal{V}_G(e)$. We shall say that a subset A of G is *Roelcke uniformly discrete in G (with respect to V)* if aV and Vb are disjoint whenever $a, b \in A$ and $a \neq b$.

Lemma 3.3. *Let A be a Roelcke uniformly discrete subset of a topological group G . Then the following statements are equivalent:*

- (1) A is left thin in G .
- (2) A is right neutral in G .

Proof. Obviously, (1) \Rightarrow (2). Conversely, let us suppose condition (2) is satisfied. Since A is a Roelcke uniformly discrete subset of G , there is $W \in \mathcal{V}_G(e)$ such that $(aW) \cap (Wb) = \emptyset$ for all $a, b \in A$ with $a \neq b$. Let $V \in \mathcal{V}_G(e)$. Since A is right neutral, there is $U \in \mathcal{V}_G(e)$ such that $U \subset W$ and $AU \subset (V \cap W)A$. Let $a \in A$. Since $aU \subset (V \cap W)A$ and $(aU) \cap (Wb) = \emptyset$ for all $b \in A \setminus \{a\}$, we have $aU \subset (V \cap W)a$. Consequently, $\bigcap_{x \in A} x^{-1}Vx \in \mathcal{V}_G(e)$, which proves condition (1). \square

Definition 3.4. Let G be a topological group. Then G is said to be an *ASIN-group* if e has at least a left thin neighborhood in G [19, 10.16]. Obviously, if G is a SIN-group, then G is an ASIN-group. “ASIN” stands for “almost SIN”. As recalled above, every right precompact subset of G is left thin in G ; consequently, if G is locally precompact, then G is ASIN. In another direction, if G is extremally disconnected, then G contains an open abelian subgroup [13], which easily implies that G is ASIN [19].

Proposition 3.5. *Let G be an ASIN-group. Then the following statements are equivalent:*

- (1) G is a SIN-group.
- (2) Every Roelcke uniformly discrete subset of G is left thin in G .

Proof. Obviously, (1) \Rightarrow (2). Suppose that condition (2) holds, and let us prove (1). Let V be a symmetric neighborhood of e in G . By Zorn’s lemma, there is a maximal Roelcke uniformly discrete subset A of G with respect to V . The maximality of A implies the equality $G = VAV$. Since A is left thin in G (by hypothesis), and since V can be chosen left thin in G , it follows from Lemma 3.1 that G is SIN. \square

Taking into account criterion 2.2 of Protasov and Saryev, the following theorem is now an immediate corollary of 3.3 and 3.5.

Theorem 3.6. *Let G be an ASIN-group. Then G is a SIN-group if and only if G is an FSIN-group.*

Corollary 3.7. *A locally precompact topological group is a SIN-group if and only if it is an FSIN-group.*

Definition 3.8. 1) A uniform space (X, \mathcal{U}) is said to be *well-chained* if for each $x, y \in X$ and each entourage $V \in \mathcal{U}$, there exists an integer $n > 0$ such that $(x, y) \in V^n$, or, equivalently, if every uniformly continuous function of X into a discrete uniform space is constant. If X is connected, then X is well-chained.

2) Let G be a topological group. The infimum of the left and right uniform structures \mathcal{L}_G and \mathcal{R}_G on G is denoted by $\mathcal{L}_G \wedge \mathcal{R}_G$. For every neighborhood V of e in G , let us put $\tilde{V} = \{(x, y) \in G \times G \mid y \in VxV\}$; then $\{\tilde{V} \mid V \in \mathcal{V}_G(e)\}$ is a fundamental system of entourages of the uniform space $(G, \mathcal{L}_G \wedge \mathcal{R}_G)$ [19, 2.5].

3) We shall say that a topological group G is *locally well-chained* if the set of neighborhoods of e in G which are well-chained uniform subspaces of (G, \mathcal{L}_G) (or (G, \mathcal{R}_G)) is a fundamental system of neighborhoods of e in G . Every locally connected topological group is locally well-chained. More generally, every dense subgroup of a locally connected topological group is locally well-chained. We shall say that G is *Roelcke locally well-chained* if the set of neighborhoods of e in G which are well-chained uniform subspaces of $(G, \mathcal{L}_G \wedge \mathcal{R}_G)$ is a fundamental system of neighborhoods of e in G .

Lemma 3.9. *Let G be a topological group. Let U be a well-chained uniform subspace of $(G, \mathcal{L}_G \wedge \mathcal{R}_G)$ such that $e \in U$. Let $V \in \mathcal{V}_G(e)$, A a right V^3 -uniformly discrete subset of G , and $a \in A$ such that $aU \subset VA$. Then $aU \subset Va$.*

Proof. Let $u \in U$. As U is well-chained relative to $\mathcal{L}_G \wedge \mathcal{R}_G$, and as $a^{-1}Va \in \mathcal{V}_G(e)$, there are $u_1, \dots, u_n \in U$ such that $u_1 = e, u_n = u$ and $u_{i+1} \in (a^{-1}Va)u_i(a^{-1}Va)$ for $1 \leq i < n$. Let us suppose that $au_i \in Va$ and let us show that $au_{i+1} \in Va$ ($1 \leq i < n$); as $au_1 \in Va$ it will follow that $au \in Va$. We have $au_{i+1} = v_1au_ia^{-1}v_2a$ with $v_1, v_2 \in V$, therefore $au_{i+1} \in v_1Vv_2a \subset V^3a$. As $au_{i+1} \in VA$ and as A is right V^3 -uniformly discrete, we obtain $au_{i+1} \in Va$. \square

Theorem 3.10. *Let G be a topological group. Then the following statements are equivalent:*

- (1) G is a locally well-chained SIN-group.
- (2) G is a Roelcke locally well-chained FSIN-group.

Proof. Obviously, (1) \Rightarrow (2). Conversely, suppose condition (2) is satisfied and let us prove (1). It suffices to prove that G is SIN since in this case, Roelcke locally well-chainedness of G and locally well-chainedness of G are equivalent notions. Let A be a right uniformly discrete subset of G ; by Lemma 2.4, it suffices to show that A is left thin in G . Let $V \in \mathcal{V}_G(e)$. Let $W \in \mathcal{V}_G(e)$ be such that $W \subset V$ and such that A is right W^3 -uniformly discrete in G . Since G is FSIN and since $(G, \mathcal{L}_G \wedge \mathcal{R}_G)$ is locally well-chained at e , there is a well-chained neighborhood U of e in G such that $AU \subset WA$. By Lemma 3.9 above, we have $aU \subset Wa$ for all $a \in A$, and consequently, since $W \subset V$, $\cap_{a \in A} a^{-1}Va$ belongs to $\mathcal{V}_G(e)$. \square

Corollary 3.11. *A locally connected topological group is a SIN-group if and only if it is an FSIN-group.*

4. AN IMPROVEMENT OF COROLLARY 3.7

In Section 3, a short proof has been given of the fact that SIN and FSIN properties are equivalent for any G in the class of locally precompact topological groups. In this section, an analogous result is obtained for the considerably larger class of topological groups that are strongly functionally generated by their right precompact subsets (Theorem 4.2). This class contains all topological groups that are quasi- k -spaces; consequently, Theorem 4.2 improves the result of [20]. Recall here that all locally compact as well as all metrizable topological groups are quasi- k -spaces; more generally, any almost metrizable topological group is a quasi- k -space. The method of proof used in Theorem 4.2 is an improvement of the method used in [20].

Definition 4.1. Let us recall that a topological space X is said to be *strongly functionally generated by a class \mathcal{C} of subspaces of X* if the following condition is satisfied: for each discontinuous function f of X into the real line \mathbb{R} (or equivalently any Tychonoff space Y), there is $A \in \mathcal{C}$ such that the restriction $f|_A$ is discontinuous. If X is strongly functionally generated by the class of all countably compact subspaces of X , then X is said to be a *quasi- $k_{\mathbb{R}}$ -space*.

Theorem 4.2. *Let G be a topological group that is strongly functionally generated by the class of all its right precompact subsets. Then the following statements are equivalent:*

- (1) G is a SIN-group.
- (2) G is an FSIN-group.

Proof. Obviously, (1) \Rightarrow (2). Conversely, suppose that condition (2) is satisfied and let us prove (1). Let us assume that G is not a SIN-group or, equivalently, let us assume that the set \mathcal{H} of all inner automorphisms of G is not equicontinuous relative to the left uniform structure on G . Then, since G is strongly functionally generated by the class \mathcal{C} of all its right precompact subsets, there is a right precompact subset A of G such that the set $\mathcal{H} \upharpoonright A$ of restrictions to A of mappings in \mathcal{H} is not equicontinuous. Let z be a point of A such that $\mathcal{H} \upharpoonright A$ is not equicontinuous at z and let $K = Az^{-1}$; then K is a right precompact subset of G , e belongs to K and $\mathcal{H} \upharpoonright K$ is not equicontinuous at e . Let V be a symmetric neighborhood of e in G such that for every neighborhood U of e in G , there is a point u in $U \cap K$ and a point h in G such that huh^{-1} does not belong to V ; then for this V , the following property (*) is satisfied:

$$(*) \forall U \in \mathcal{V}_G(e), \exists x, y \in G, x^{-1}y \in U \cap K \text{ and } xy^{-1} \notin V.$$

Let W be a symmetric neighborhood of e in G such that $W^5 \subset V$. By induction, we build a sequence (x_n, y_n) of points of $G \times G$ in the following way. By applying (*) we choose (x_1, y_1) such that $x_1^{-1}y_1$ belongs to K and such that $x_1y_1^{-1}$ does not belong to W^5 . Let us assume that $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ are given; by applying condition (*), we then choose (x_{n+1}, y_{n+1}) such that

$$x_{n+1}^{-1}y_{n+1} \in K, \tag{i}$$

$$x_{n+1}^{-1}y_{n+1} \in (\cap_{i=1}^n x_i^{-1}Wx_i) \cap (\cap_{i=1}^n y_i^{-1}Wy_i), \tag{ii}$$

$$x_{n+1}y_{n+1}^{-1} \notin W^5. \tag{iii}$$

Let $A = \{x_n \mid n \in \mathbb{N}\}$ and $B = \{x_i \mid i \in \mathbb{I}\}$, \mathbb{I} being an infinite subset of the set \mathbb{N} . Let $U \in \mathcal{V}_G(e)$ and let us verify that $BU \not\subset WA$; it will imply in particular that $AU \not\subset WA$, and consequently that G is not FSIN by Protasov and Saryev's criterion (cf. Theorem 2.2). Since K is right precompact, there exist $p, q \in \mathbb{I}$ such that $p < q$ and $(x_p^{-1}y_p)(x_q^{-1}y_q)^{-1} \in U$. Let us suppose that $BU \subset WA$; then since $y_p y_q^{-1} x_q \in BU$, $y_p y_q^{-1} x_q \in WA$. By (ii), $y_p y_q^{-1} x_q \in W y_p$; consequently, $WA \cap W y_p \neq \emptyset$ and there is $r \in \mathbb{N}$ such that $x_r y_p^{-1} \in W^2$. Necessarily, $p \neq r$ by (iii). Let us suppose that $r > p$.

By (ii), $y_r \in x_r(y_p^{-1}W y_p) = (x_r y_p^{-1})(W y_p) \subset W^3 y_p$; consequently, $x_r y_r^{-1} \in (W^2 y_p)(y_p^{-1}W^3) = W^5$ which contradicts (iii). The same contradiction holds if $p > r$ by symmetry of (ii). \square

Definition 4.3. Let X be a topological space. A subset A of X is said to be *relatively pseudocompact*, or *bounded*, in X provided that every continuous real-valued function defined on X is bounded on A . Every countably compact subspace of X is relatively pseudocompact in X . Every relatively pseudocompact subset of X is precompact with respect to any uniform structure on X defining the topology of X .

Corollary 4.4. *Let G be a topological group that is strongly functionally generated by the class of all its relatively pseudocompact subsets. Then G is a SIN-group if and only if G is an FSIN-group.*

Corollary 4.5. *Let G be a topological group that is a quasi- $k_{\mathbb{R}}$ -space. Then G is a SIN-group if and only if G is an FSIN-group.*

5. NONSEPARABILITY OF $\mathcal{U}_R(G)/\mathcal{U}(G)$

Let G be any topological group. Recall that the real Banach space of all left (respectively right) uniformly continuous bounded functions of G into \mathbb{R} is denoted by $\mathcal{U}_L(G)$ (respectively $\mathcal{U}_R(G)$). The real Banach space $\mathcal{U}_L(G) \cap \mathcal{U}_R(G)$ of all uniformly continuous bounded functions of G into \mathbb{R} is denoted by $\mathcal{U}(G)$. The aim of this section is to prove that if G is not FSIN, then the quotient Banach space $\mathcal{U}_R(G)/\mathcal{U}(G)$ is nonseparable. Let us note that this result had already been proved in [4] in the particular case when G is locally compact (and consequently in the particular case when G is locally precompact); it was subsequently extended in [21] to any almost metrizable G by use of similar methods. Here, the result is obtained for an arbitrary topological group which is not FSIN in Theorem 5.5. It should be pointed out that our proof of Theorem 5.5 distinguishes, for one of its steps, the locally precompact case from the others.

We begin with two preliminary statements. The first (Lemma 5.1) is proved in 4.2 above; it is used to obtain 5.5 in the locally precompact case. (Lemma 5.1 is more general than needed in 5.5; it will be used with its full generality in Section 6.) The second (Lemma 5.3) is used to obtain 5.5 in all the other cases.

Lemma 5.1. *Let G be a topological group which is strongly functionally generated by the set of all its right precompact subsets. Let us suppose that G is not an FSIN-group (or equivalently, by 4.2, that G is not a SIN-group). Then there exist $V \in \mathcal{V}_G(e)$ and an infinite subset A of G such that $BU \not\subset VA$ for all infinite subsets B of A and all $U \in \mathcal{V}_G(e)$.*

Definition 5.2. Let G be a topological group and let $V \in \mathcal{V}_G(e)$. A sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of G is said to be *right uniformly discrete in G (with respect to V)* if VA_i and VA_j are disjoint whenever $i, j \in \mathbb{N}$ and $i \neq j$. A sequence $(x_n)_{n \in \mathbb{N}}$ of points of G is said to be *right uniformly discrete in G (with respect to V)* if the sequence $(\{x_n\})_{n \in \mathbb{N}}$ is right uniformly discrete in G with respect to V .

Lemma 5.3. *Let G be a topological group, $E \subset G$ and $V \in \mathcal{V}_G(e)$. Let us suppose that G is not locally precompact. Then there exist $S \in \mathcal{V}_G(e)$ and a right uniformly discrete sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of G with respect to S such that:*

- (1) $E \subset VA_n$ and $A_n \subset VE$ for all $n \in \mathbb{N}$,
- (2) A_n is right uniformly discrete with respect to S for all $n \in \mathbb{N}$.

Proof. Let W be a symmetric neighborhood of e in G such that $W^3 \subset V$. By Zorn's lemma, there is a maximal right uniformly discrete subset A of G with respect to W such that $A \subset E$. The maximality of A implies the inclusion $E \subset W^2A$. As W is symmetric, the inclusion $W \subset W^2w$ holds for all $w \in W$; it implies $W^2 \subset W^3w$ ($w \in W$). The following condition (*) follows from these inclusions.

$$(*) \quad \forall w \in W, E \subset VwA.$$

Condition (*) is used at the end of the proof.

Let $R \in \mathcal{V}_G(e)$ be such that $R^2 \subset W$. As R is not right precompact, there exist $S \in \mathcal{V}_G(e)$ and a right uniformly discrete sequence $(d_n)_{n \in \mathbb{N}}$ in R with respect to S . Of course, S can be chosen such that $S \subset R$. Let $A_n = d_nA$ for all $n \in \mathbb{N}$. Let us show that this S and this sequence $(A_n)_{n \in \mathbb{N}}$ satisfy the conditions announced.

First, if $p, q \in \mathbb{N}$ and if $SA_p \cap SA_q \neq \emptyset$, there exist $s, t \in S$ and $a, b \in A$ such that $sd_p a = td_q b$; as $sd_p, td_q \in SR \subset R^2 \subset W$, and as A is right W -uniformly discrete, we have $a = b$, and consequently $sd_p = td_q$; the sequence $(d_n)_{n \in \mathbb{N}}$ of points in G being right uniformly discrete with respect to S , we have $p = q$.

In the same way, A_n is right uniformly discrete with respect to S ($n \in \mathbb{N}$). Indeed, if $a', b' \in A_n$ and if $Sa' \cap Sb' \neq \emptyset$, there exist $s', t' \in S$ and $a'', b'' \in A$ such that $a' = d_n a''$, $b' = d_n b''$, and $s' d_n a'' = t' d_n b''$; as $s' d_n, t' d_n \in SR \subset R^2 \subset W$, and as A is right uniformly discrete with respect to W , we have $a'' = b''$, and consequently $a' = b'$.

Next, since $d_n \in W$, the inclusion $E \subset V d_n A = V A_n$ holds by condition (*) ($n \in \mathbb{N}$). Finally, $A_n = d_n A \subset RE \subset WE \subset VE$. \square

Remark 5.4. Let G be a topological group, and E a nonempty subset of G . Suppose that for some symmetric neighborhood V of e in G , there exists a right uniformly discrete sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of G such that $E \subset V A_n$ for all $n \in \mathbb{N}$. Then V is not right precompact. Indeed, let $a \in E$, and for all $n \in \mathbb{N}$, let $v_n \in V$ and $x_n \in A_n$ such that $a = v_n x_n$. Since $x_n = v_n^{-1} a$, $V a$ contains a right uniformly discrete sequence and consequently, V is not right precompact.

Theorem 5.5. *Let G be a topological group which is not an FSIN-group. Then the Banach space $\mathcal{U}_R(G)$ contains a linear isometric copy K of l^∞ such that*

$$\inf\{\|2k + h\| \mid h \in \mathcal{U}_L(G)\} \geq \|k\| \quad \text{for all } k \in K.$$

In particular, the quotient Banach space $\mathcal{U}_R(G)/\mathcal{U}(G)$ is nonseparable.

Proof. (1) Let us prove that there exist a sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of G and $V \in \mathcal{V}_G(e)$ such that:

- (i) $\forall U \in \mathcal{V}_G(e), \forall n \in \mathbb{N}, A_n U \not\subset V(\cup_{k \in \mathbb{N}} A_k)$,
- (ii) for all $n \in \mathbb{N}$, A_n is right uniformly discrete with respect to V ,
- (iii) the sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of G is right uniformly discrete with respect to V .

(a) First, let us suppose that G is locally precompact. So, by Lemma 5.1, there exist $V' \in \mathcal{V}_G(e)$ and an infinite subset A of G such that $BU \not\subset V' A$ for all infinite subset B of A and all $U \in \mathcal{V}_G(e)$. Obviously, such a set A is not right precompact; consequently, by taking a subset of A if necessary, we may assume that there is $W \in \mathcal{V}_G(e)$ such that A is right uniformly discrete with respect to W .

Put $V = V' \cap W$ and let us choose any partition $(A_n)_{n \in \mathbb{N}}$ of A into infinite subsets of A ; then $(A_n)_{n \in \mathbb{N}}$ and V satisfy conditions (i), (ii) and (iii) above.

(b) Let us suppose now that the topological group G is not locally precompact. Let E be a subset of G which is not right neutral, and let $W' \in \mathcal{V}_G(e)$ be such that $EU \not\subset W'E$ for all $U \in \mathcal{V}_G(e)$. Let V' be a neighborhood of e in G such that $V'^3 \subset W'$. By Lemma 5.3, there is $S \in \mathcal{V}_G(e)$ and a right S -uniformly discrete sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of G such that $E \subset V'A_n$, $A_n \subset V'E$ and such that A_n is right S -uniformly discrete ($n \in \mathbb{N}$). Let $n \in \mathbb{N}$ and $U \in \mathcal{V}_G(e)$; then $EU \subset (V'A_n)U = V'(A_nU)$ and $V'(V'(\cup_{k \in \mathbb{N}} A_k)) \subset V'^3E \subset W'E$; as $EU \not\subset W'E$, we necessarily have $A_nU \not\subset V'(\cup_{k \in \mathbb{N}} A_k)$. Let us put $V = V' \cap S$; then $(A_n)_{n \in \mathbb{N}}$ and V satisfy conditions (i), (ii) and (iii) above.

(2) Let us consider a sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of G and $V \in \mathcal{V}_G(e)$ such that conditions (i), (ii), (iii) above are satisfied.

Let $g \in \mathcal{U}_R(G)$ be such that $g(G) \subset [0, 1]$, $g(e) = 1$ and $g(x) = 0$ for all $x \in G \setminus V$ (cf. [7, p. 7]). For each $n \in \mathbb{N}$, define the mapping $f_n : G \rightarrow \mathbb{R}$ by:

$$f_n(x) = \sum_{a \in A_n} g(xa^{-1}) \quad \text{for all } x \in G,$$

and for each $c = (c_n)_{n \in \mathbb{N}} \in l^\infty$, define the mapping $f_c : G \rightarrow \mathbb{R}$ by:

$$f_c = \sum_{n \in \mathbb{N}} c_n f_n.$$

Then it is easy to verify that f_c is well defined and that f_c belongs to $\mathcal{U}_R(G)$. Moreover, if $A = \cup_{n \in \mathbb{N}} A_n$, then $f_c(x) = 0$ for all $x \in G \setminus VA$.

It is immediate that the function $c \rightarrow f_c$ of l^∞ into $\mathcal{U}_R(G)$ is a linear isometry, and that the function $c \rightarrow 2f_c + \mathcal{U}(G)$ of l^∞ into $\mathcal{U}_R(G)/\mathcal{U}(G)$ is linear. Let us consider $c = (c_n)_{n \in \mathbb{N}} \in l^\infty$ and $h \in \mathcal{U}_L(G)$; in order to obtain the theorem, it suffices now to verify that $\|2f_c + h\| \geq \|c\|$.

Let $\epsilon > 0$. Let U be a symmetric neighborhood of e in G such that $x^{-1}y \in U$ implies $|h(x) - h(y)| < \epsilon$ ($x, y \in G$). Let $n \in \mathbb{N}$ be such that $\|c\| - \epsilon/2 < |c_n|$. By condition (i) above, $A_nU \not\subset VA$. Let $a \in A_n$ and $u \in U$ such that $au \notin VA$; then $f_c(au) = 0$ and $f_c(a) = c_n$. Moreover, $(au)^{-1}a = u^{-1} \in U$, so $|h(au) - h(a)| < \epsilon$.

Consequently,

$$\begin{aligned}
 2\|2f_c + h\| &\geq |(2f_c + h)(au)| + |(2f_c + h)(a)| \\
 &\geq |(2f_c + h)(au) - (2f_c + h)(a)| \\
 &\geq |2(f_c(au) - f_c(a)) + h(au) - h(a)| \\
 &\geq 2|f_c(au) - f_c(a)| - |h(au) - h(a)| \\
 &\geq 2|c_n| - \epsilon \\
 &\geq 2\|c\| - 2\epsilon,
 \end{aligned}$$

which implies $\|2f_c + h\| \geq \|c\|$. \square

6. COPY OF l^∞ IN $\mathcal{U}_R(G)/\mathcal{U}(G)$

Our goal in this section is to show that for a vast class of topological groups G , the inequality of the above Theorem 5.5 is in fact an equality. Let us recall that such a result had already been obtained in [4] in the case when G is locally compact (and consequently in the case when G is locally precompact), and in [21] when G is almost metrizable and complete. Here, a new approach of the problem allows us to improve significantly these results in 6.2. The proof of 6.2 is based on two lemmas. The first is Lemma 5.1 above; the second (Lemma 6.1) had already been used in [4] and [21]; it is an easy consequence of Katětov's extension theorem of bounded uniformly continuous functions [11, 12].

Lemma 6.1. *Let G be a topological group, let $(A_n)_{n \in \mathbb{N}}$ be a sequence of subsets of G and let $c = (c_n)_{n \in \mathbb{N}} \in l^\infty$. Suppose there is $V \in \mathcal{V}_G(e)$ such that for all $p, q \in \mathbb{N}$, with $p \neq q$, the relation $(VA_pV) \cap (VA_qV) = \emptyset$ holds. Then there exists a mapping $h_c \in \mathcal{U}(G)$ such that $h_c(x) = c_n$ for all $x \in A_n, n \in \mathbb{N}$.*

We are now in a position to prove the following result 6.2. Since every almost metrizable and complete topological group satisfies conditions (1) and (2) of 6.2 [19], the main result of [21] is a special case of it.

Theorem 6.2. *Let us consider statement 5.5 above, and let us suppose that the following additional conditions are satisfied:*

- (1) G is strongly functionally generated by the set of all its right precompact subsets,
- (2) every $(\mathcal{L}_G \wedge \mathcal{R}_G)$ -precompact subset of G is right precompact.

Then, in the conclusion, “ \geq ” can be replaced by “ $=$ ” (and “FSIN-group” can be replaced by “SIN-group” because of 4.2). Moreover, the quotient Banach space $\mathcal{U}_R(G)/\mathcal{U}(G)$ contains a linear isometric copy of l^∞ .

Proof. By (1) and Lemma 5.1, there exist $V \in \mathcal{V}_G(e)$ and an infinite subset A of G such that $BU \not\subset VA$ for all infinite subsets B of A and all $U \in \mathcal{V}_G(e)$. Obviously, such a set A is not right precompact, and by (2), it is not $(\mathcal{L}_G \wedge \mathcal{R}_G)$ -precompact. Therefore, by taking a subset of A if necessary, we may assume that there is $W \in \mathcal{V}_G(e)$ such that A is Roelcke uniformly discrete with respect to W , that is to say such that $aW \cap Wb = \emptyset$ whenever $a, b \in A$ and $a \neq b$. We may also suppose that the above neighborhood V of e is symmetric and is such that $V^4 \subset W$; we then have

$$(V^2aV) \cap (V^2bV) = \emptyset \quad \text{for all } a, b \in A \text{ such that } a \neq b. \quad (\mathcal{C})$$

Let us choose a partition $(A_n)_{n \in \mathbb{N}}$ of A into infinite subsets of A ; then $(A_n)_{n \in \mathbb{N}}$ and V satisfy conditions (i), (ii) and (iii) in the proof of 5.5, and, for each $c = (c_n)_{n \in \mathbb{N}} \in l^\infty$, one can define the function $f_c \in \mathcal{U}_R(G)$ associated with $(A_n)_{n \in \mathbb{N}}$ and V as in the proof of 5.5. Let us verify that

$$\inf\{\|2f_c + h\| \mid h \in \mathcal{U}(G)\} = \|c\|,$$

which will prove 6.2. By 5.5, the inequality $\inf\{\|2f_c + h\| \mid h \in \mathcal{U}(G)\} \geq \|c\|$ holds; to obtain 6.2, it suffices to show that there is $h_c \in \mathcal{U}(G)$ such that $\|2f_c + h_c\| \leq \|c\|$. By (\mathcal{C}) the sequence $(V^2A_nV)_{n \in \mathbb{N}}$ is pairwise disjoint; hence, by 6.1, there is a function $h_c \in \mathcal{U}(G)$ such that $h_c(x) = -c_n$ for all $x \in VA_n$ and all $n \in \mathbb{N}$; moreover, one can choose h_c such that $\|h_c\| = \|c\|$.

Let $x \in G$; if $x \notin \cup_{n \in \mathbb{N}}VA_n$, then

$$|(2f_c + h_c)(x)| = |h_c(x)| \leq \|h_c\| = \|c\|;$$

if $x \in \cup_{n \in \mathbb{N}}VA_n$, then $x \in VA_m$ for one and only one $m \in \mathbb{N}$ and, for some $b \in A_m$, we have

$$|(2f_c + h_c)(x)| = |c_m| |2g(xb^{-1}) - 1| \leq |c_m| \leq \|c\|;$$

hence the inequality $\|2f_c + h_c\| \leq \|c\|$ holds. □

Remarks 6.3. (1) Let G be a topological group. If G is locally precompact, then it is easy to verify that conditions (1) and (2) of 6.2 hold. On the other hand, if G is an ASIN-group, or if the uniform

space $(G, \mathcal{L}_G \wedge \mathcal{R}_G)$ is complete, it is shown in [19] that condition (2) of 6.2 is satisfied. Recall also that it has been noted in Section 5 that condition (1) of 6.2 is satisfied when G is strongly functionally generated by the class of all its relatively pseudocompact subsets (in particular when G is a quasi- $k_{\mathbb{R}}$ -space).

(2) Let us point out that in [1], Chou proved results of the same sort as 6.2 in the context of weak almost periodicity. For example, the following statement is a corollary of Theorem 5.2 in [1]: let G be a noncompact locally compact topological group and let $W(G)$ be the real Banach space of all continuous bounded real-valued functions on G which are weakly almost periodic; if G is a SIN-group, then the quotient Banach space $\mathcal{U}(G)/W(G)$ contains a linear isometric copy of l^∞ .

REFERENCES

- [1] C. Chou, *Weakly almost periodic functions and almost convergent functions on a group*, Trans. Amer. Math. Soc. **206** (1975), 175-200.
- [2] H.A.M. Dzinotyiweyi, *Nonseparability of quotient spaces of function algebras on topological semigroups*, Trans. Amer. Math. Soc. **272** (1982), 223-235.
- [3] R. Engelking, *General Topology* (Heldermann Verlag, Berlin, Revised and completed ed., 1989).
- [4] G. Hansel and J.P. Trollic, *Nonseparability and uniform structures in locally compact groups*, Proc. Amer. Math. Soc. **123** (1995), 1613-1621.
- [5] S. Hernández, *Topological characterization of equivalent uniformities in topological groups*, Topology Proceedings **25** (Spring 2000), 181-188.
- [6] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I* (Springer-Verlag, New York, 2nd ed., 1979).
- [7] J.R. Isbell, *Uniform Spaces* (Amer. Math. Soc., Providence, 1964).
- [8] G.L. Itzkowitz, *Continuous measures, Baire category, and uniform continuity in topological groups*, Pacific J. Math. **54** (1974), 115-125.
- [9] G. L. Itzkowitz, *Uniformities and uniform continuity on topological groups*, in: *General Topology and Applications* (Staten Island, NY, 1989); *Lecture Notes in Pure and Appl. Math.* **134** (Dekker, New York, 1991) 155-178.
- [10] G. L. Itzkowitz, *Projective limits and balanced topological groups*, *Topology Appl.* **110** (2001), 163-183.
- [11] M. Katětov, *On real-valued functions in topological spaces*, *Fund. Math.* **38** (1951), 85-91.
- [12] M. Katětov, *Correction to "On real-valued functions in topological spaces"*, *Fund. Math.* **40** (1953), 203-205.
- [13] V.I. Malychin, *Extremally disconnected and close to them groups*, *Dokl. Akad. Nauk SSSR* **220** (1975), 27-30.

- [14] M. Megrelishvili, P. Nickolas and V. Pestov, *Uniformities and uniformly continuous functions on locally connected groups*, Bull. Austral. Math. Soc. **56** (1997), 279-283.
- [15] P. Milnes, *Uniformity and uniformly continuous functions for locally compact groups*, Proc. Amer. Math. Soc. **109** (1990), 567-570.
- [16] W.H. Previts and T.S. Wu, *Notes on balanced groups*, Bull. Instit. Math. Acad. Sinica **29** (2001), 105-123.
- [17] I. Protasov, *Functionally balanced groups*, Math. Notes **49** (1991), 614-616.
- [18] I. Protasov and A. Saryev, *Subgroup of closed subsets of a topological group*, Izv. Akad. Nauk TadzhSSR. Ser. Fiz.-Tekh. Nauk **3** (1988), 21-25.
- [19] W. Roelcke and S. Dierolf, *Uniform Structures on Topological Groups and their Quotients* (McGraw-Hill, New York, 1981).
- [20] J.P. Trollic, *Sequential criteria for equicontinuity and uniformities on topological groups*, Topology Appl. **68** (1996), 83-95.
- [21] J.P. Trollic, *Equicontinuity, uniform continuity and sequences in topological groups*, Topology Appl. **93** (1999), 179-190.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE ROUEN, UMR CNRS
6085, F-76820 MONT SAINT AIGNAN, FRANCE
E-mail address: `Ahmed.Bouziad@univ-rouen.fr`

UMR CNRS 6085, FACULTÉ DES SCIENCES ET TECHNIQUES, UNIVERSITÉ
DU HAVRE, 25 RUE PHILIPPE LEBON, F-76600 LE HAVRE, FRANCE
E-mail address: `jean-pierre.trollic@univ-lehavre.fr`