Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 28, No.2, 2004
Pages 401-424

TAMPERING WITH PSEUDOCOMPACT GROUPS

W.W. COMFORT*

ABSTRACT. This paper is an extended version of the Invited Address presented by the author at the 2003 Summer Conference on General Topology and Its Applications (Howard University, Washington, DC).

Let $\mathbb{C}$, Ω, $\mathbb{CC}$, $\mathbb{P}$ and $\mathbb{TB}$ denote, respectively, the class of Hausdorff topological groups which are compact, ω-bounded, countably compact, pseudocompact, and totally bounded, and let $\mathbb{X}, \mathbb{Y} \in \{\mathbb{C}, \Omega, \mathbb{CC}, \mathbb{P}, \mathbb{TB}\}$.

In *Part I* the author attempts a survey of the literature concerning the following $5 + (2 \times 5 \times 5) = 55$ questions and their Abelian analogues: 1. Is there an algebraic characterization of those groups G which admit a group topology $\mathcal{T}$ such that $(G, \mathcal{T}) \in \mathbb{X}$? 2. If $(G, \mathcal{T}) \in \mathbb{X}$, does G admit a proper dense subgroup $H \in \mathbb{Y}$? 3. If $(G, \mathcal{T}) \in \mathbb{X}$ does G admit a group topology $\mathcal{U}$, properly refining $\mathcal{T}$, such that $(G, \mathcal{U}) \in \mathbb{Y}$? Emphasis is both on what is known and on some of the most attractive or compelling unsolved questions.

Part II studies more intensively the extensive literature concerning Questions 2 and 3 in the case that $\mathbb{X}$ and $\mathbb{Y}$ are the class of nonmetrizable pseudocompact (Abelian) groups. Not even a ZFC-consistent answer is known in either case, but the questions are shown to be unexpectedly related. Most of the new results cited derive from (as yet unpublished) joint work with Jorge Galindo.

Here is a sample: If a nonmetrizable pseudocompact Abelian group is either torsion-free or countably compact, then it admits either a proper, dense, pseudocompact subgroup or a strictly finer pseudocompact group topology.

2000 *Mathematics Subject Classification.* Primary 54H11, 22A05; Secondary 22-02, 54-02.

Key words and phrases. Compact group, countably compact group, ω-bounded group, pseudocompact group, totally bounded group. free topological group, free Abelian topological group. refinement topology, dense subgroup. extremal pseudocompact topological group.

*I am grateful to the following mathematicians for reviewing portions of a pre-publication version of this paper with considerable care and attention. Each caught some errors, and in other ways improved the exposition: Dikran Dikranjan, Jorge Galindo, Jan van Mill, Kenneth A. Ross, Michael Tkachenko, Javier Trigos-Arrieta, and the referee.

0. Prelude

The sequence of Summer Conferences on General Topology and Its Applications is playing an increasingly important role in the development of our discipline, and it is a signal honor to be invited to give the opening address this year. It is a pleasure to see many personal friends and co-authors, also a number of my former graduate students and their own mathematical descendants, in the audience.

Part 1. Some General Questions

1. Introduction

1.1. Discussion. The question was raised by Markov, as long ago as 1944, whether every infinite group admits a nondiscrete Hausdorff group topology. It is not difficult to see (as in 3.2 below) that the answer is Yes for Abelian groups. That the answer is No in general is a deeper result, established by Hesse [41] and by Ol′shanskiĭ [49] in the late 1970's. I will concentrate today mainly on Abelian groups, with a view to pursuing this natural extension of Markov's question: Given a group G and a class $\mathbb{X}$ of spaces, does G admit a Hausdorff group topology $\mathcal{T}$ such that $(G, \mathcal{T}) \in \mathbb{X}$? A related pair of questions, now given $(G, \mathcal{T}) \in \mathbb{X}$ and a class $\mathbb{Y}$ of spaces (perhaps with $\mathbb{Y} = \mathbb{X}$) is this: Does G admit

(a) a proper dense subgroup $D \subseteq G$ such that $(D, \mathcal{T}) \in \mathbb{Y}$; and/or

(b) a group topology $\mathcal{U}$ strictly refining $\mathcal{T}$ such that $(G, \mathcal{U}) \in \mathbb{Y}$?

Of course, these questions can be asked about any classes $\mathbb{X}$ and $\mathbb{Y}$ of topological spaces whatever, but many choices $(\mathbb{X}, \mathbb{Y})$ give rise to questions which are transparent or trivial or otherwise uninteresting. In *Part I*, I will survey some aspects of what is known about these questions for the following classes of spaces.

$\mathbb{C}$: the class of compact topological groups.

Ω: the class of ω-bounded topological groups.

$\mathbb{CC}$: the class of countably compact topological groups.

$\mathbb{P}$: the class of pseudocompact topological groups.

$\mathbb{TB}$: the class of totally bounded topological groups.

The symbol $\mathbb{A}$ denotes the class of Abelian groups, and we write $\mathbb{CA} := \mathbb{C} \cap \mathbb{A}$, $\Omega\mathbb{A} := \Omega \cap \mathbb{A}$, and so forth.

By a *space* is meant a Tychonoff space, i.e., a completely regular Hausdorff space, and every topological group hypothesized here is assumed to be a space in that sense.

For a space $(X, \mathcal{T})$ and $A \subseteq X$, we denote by A or by $(A, \mathcal{T})$ the set A with the topology inherited from $(X, \mathcal{T})$. For X a set and κ a cardinal, we write $[X]^\kappa := \{A \subseteq X : |X| = \kappa\}$; the notations $[X]^{<\kappa}$ and $[X]^{\leq\kappa}$ are defined analogously.

Little new mathematical material will appear in *Part I*, though I hope the organization will coherently illuminate the present state of the art and will expose some applications as natural and interesting. Then in *Part II*, I summarize and advance slightly what is known about two well known, heavily investigated questions, namely questions (a) and (b) above in the case $\mathbb{X} = \mathbb{Y} = \mathbb{PA}$.

The definitions of a compact space, and of a countably compact space, are familiar to the reader. A space X is *ω-bounded* if each $A \in [X]^{\leq\omega}$ has $\overline{A}^X$ compact; and, as defined by Hewitt [42], a space X is *pseudocompact* if each continuous real-valued function on X is bounded. A topological group $(G, \mathcal{T})$ is *totally bounded* if for $\emptyset \neq U \in \mathcal{T}$ there is $F \in [G]^{<\omega}$ such that $G = \cup_{x \in F}\, xU$. According to Weil [66], these are exactly the groups G which embed (densely) into a compact group; this latter group, the so-called *Weil completion* of G, is unique in an obvious sense and is here denoted $\overline{G}$.

The class-theoretic relations $\mathbb{C} \subseteq \Omega \subseteq \mathbb{CC} \subseteq \mathbb{P} \subseteq \mathbb{TB}$ are readily established. The last of these inclusions, noted in [20], suggests the following question.

2. Which Totally Bounded Groups are Pseudocompact?

The following useful result, responding to that question, will be used often hereafter.

Theorem 2.1 ([20]). *For $G \in \mathbb{TB}$, the following conditions are equivalent.*

(a) $G \in \mathbb{P}$;

(b) G is G_δ-dense in $\overline{G}$;

(c) $\overline{G} = \beta(G)$.

In the development of [20], a key tool in the proof of Theorem 2.1 is the fact that a topological group G is pseudocompact if and only if each continuous function $f : G \to \mathbb{R}$ is constant on the cosets of a (suitably chosen) closed, normal, G_δ subgroup of G. A more direct approach, based on the behavior of suitably restricted sequences in G, was given subsequently by de Vries [64]. In any event, here are three immediate consequences of Theorem 2.1. To the best of my knowledge, the first of these was first recorded explicitly by H. Wilcox [68]; the latter two are in [20].

Corollary 2.2. *Every divisible, pseudocompact group G is connected.*

Proof. The compact group $\overline{G}$ is divisible, hence is connected by Mycielski's theorem (cf. [48] or [43](24.25)), so G is connected since $\overline{G} = \beta(G)$. □

Corollary 2.3. *If $G \in \mathbb{P}$ and H is a dense subgroup of G, then $H \in \mathbb{P}$ if and only if H is G_δ-dense in G.*

Proof. H is G_δ-dense in G if and only if H is G_δ-dense in $\overline{G} = \overline{H}$. □

Corollary 2.4. *If $\{G_i : i \in I\} \subseteq \mathbb{P}$ then $G := \Pi_{i\in I}\, G_i \in \mathbb{P}$.*

Proof. Surely $G \in \mathbb{TB}$, and the uniqueness aspect of Weil's theorem yields $\overline{G} = \Pi_{i\in I}\, \overline{G_i}$; evidently G is G_δ-dense in $\Pi_{i\in I}\, \overline{G_i}$. □

2.5. Discussion. Typically when a theorem answers a question, a more delicate question (demanding a more subtle theorem or a counterexample) immediately suggests itself. But when Ken Ross and I proved Theorem 2.1 we did not pursue additional paths. We perceived the result as taking the form "if and only if", thus leaving no natural issue for further investigation. Nothing could have been more wrong: Later workers found useful, in some cases sweeping and surprising, generalizations, of Theorem 2.1. Here are four examples. It continues to surprise me, whenever I revisit these results, that in each case the groups G_i are arbitrary (Hausdorff) topological groups with no topological assumptions; in each case, $F_i \subseteq G_i$.

Theorem 2.6 (Tkachenko [54], [55])**.** *If F_i is functionally bounded in G_i (in the sense that each continuous $f : G_i \to \mathbb{R}$ is bounded on F_i), then $\Pi_{i\in I}\, F_i$ is functionally bounded in $\Pi_{i\in I}\, G_i$.*

Theorem 2.7 (Uspenskiĭ [61])**.** *If F_i is a pseudocompact retract of G_i, then $\Pi_{i\in I} F_i$ is pseudocompact.*

Theorem 2.8 (Uspenskiĭ [61])**.** *If F_i is a pseudocompact G_δ subset of G_i, then $\Pi_{i\in I} F_i$ is pseudocompact.*

Theorem 2.9 (Trigos-Arrieta [60])**.** *If F_i is pseudocompact and regular-closed in G_i, then $\Pi_{i\in I} F_i$ is pseudocompact.*

In later work [23], Trigos-Arrieta and I produced "localized" versions of Theorem 2.1 and of several of the other results given in [20].

The arguments of [20] leaned heavily on both the algebraic structure and the topological disposition of a totally bounded group G within its Weil completion $\overline{G}$. The following theorem shows that for many purposes the algebraic structure of G is immaterial; the arguments of [40] parallel more closely those of de Vries [64] than of [20].

Theorem 2.10 (Hernández and Sanchis [40])**.** *Let X be a G_δ-dense subset of a compact group K. Then X is pseudocompact, and $K = \beta(X)$.*

In view of previous experience, one hesitates to say that this next result is "the last word" in this line of inquiry; but at the least it may be safely asserted that it subsumes and encompasses several of the results previously cited in this section.

Theorem 2.11 (Arhangel′skiĭ [2])**.** *A dense subset of a pseudocompact group is C-embedded in its G_δ-closure.*

3. The Classes $\mathbb{X}'$

For $\mathbb{X} \in \{\mathbb{C}, \Omega, \mathbb{CC}, \mathbb{P}, \mathbb{TB}\}$ or $\mathbb{X} \in \{\mathbb{CA}, \Omega\mathbb{A}, \mathbb{CCA}, \mathbb{PA}, \mathbb{TBA}\}$, let $\mathbb{X}'$ denote the class of groups G which admit a (Hausdorff) group topology $\mathcal{T}$ such that $(G, \mathcal{T}) \in \mathbb{X}$. We have of course

$\mathbb{C}' \subseteq \Omega' \subseteq \mathbb{CC}' \subseteq \mathbb{P}' \subseteq \mathbb{TB}'$ and $\mathbb{CA}' \subseteq \Omega\mathbb{A}' \subseteq \mathbb{CCA}' \subseteq \mathbb{PA}' \subseteq \mathbb{TBA}' \subseteq \mathbb{A}$.

The obvious questions in this context are these.

Question 3.1. (a) Given $\mathbb{X} \in \{\mathbb{C}, \Omega, \mathbb{CC}, \mathbb{P}, \mathbb{TB}\}$, which groups G satisfy $G \in \mathbb{X}'$?

(b) Given $\mathbb{X} \in \{\mathbb{CA}, \Omega\mathbb{A}, \mathbb{CCA}, \mathbb{PA}, \mathbb{TBA}\}$, which groups G satisfy $G \in \mathbb{X}'$?

To the best of my knowledge, several of those 10 questions have never been addressed explicitly in the literature. With no pretense at completeness, let me now give (in no special order) the flavor of the present state of our knowledge in a few instances.

Here and in what follows, for $\mathbb{X}$ as above and α a cardinal we write $(G, \mathcal{T}) \in \mathbb{X}(\alpha)$ if $(G, \mathcal{T}) \in \mathbb{X}$ and $w(G, \mathcal{T}) = \alpha$; and for G a group we write $G \in \mathbb{X}'(\alpha)$ if G admits a (Hausdorff) group topology $\mathcal{T}$ such that $(G, \mathcal{T}) \in \mathbb{X}(\alpha)$.

3.2. The class $\mathbb{TBA}'$. The answer here is short, sweet and definitive: For every group $G \in \mathbb{A}$ the group $\mathrm{Hom}(G, \mathbb{T})$ separates points of G, so $\mathrm{Hom}(G, \mathbb{T})$ induces on G a Hausdorff totally bounded group topology. Restated: $\mathbb{TBA}' = \mathbb{A}$.

3.3. The class $\mathbb{CA}'$. It was Halmos [36] who first requested an algebraic characterization of those Abelian groups which admit a compact group topology. Here again there is a complete and happy solution, resting in part on the basics of Pontrjagin duality theory. A self-contained treatment, including all details of arguments of Łos, Hulanicki, D. K. Harrison, Kakutani and others, is available in the monograph of Hewitt and Ross [43](25.25).

3.4. The class $\mathbb{P}'$. If a group G with $|G| = \gamma \geq \omega$ satisfies $G \in \mathbb{P}'(\alpha)$ then, since G with the witnessing topology is G_δ-dense (hence dense) in $\overline{G}$ and $w(G) = w(\overline{G})$, we have $\log(\alpha) = d(\overline{G}) \leq |G| \leq 2^\alpha$; further, since every pseudocompact space is a Baire space, the condition $|G| = \kappa$ is impossible if $\mathrm{cf}(\kappa) = \omega$. That establishes the following theorem, first noted and recorded by van Douwen [34].

Theorem 3.5 (van Douwen [34])**.** *Let $\alpha \geq \omega$ and $\gamma \geq \omega$. In order that some group G with $|G| = \gamma$ satisfy $G \in \mathbb{P}'(\alpha)$, it is necesary that $\log(\alpha) \leq \gamma \leq 2^\alpha$; if $\mathrm{cf}(\log(\alpha)) = \omega$, it is necessary that $\gamma > \log(\alpha)$.*

Two clusters of papers ([12], [13]; [26], [27], [25], [28]) appearing in the 1990's addressed with some success the issue of characterizing the groups in the classes $\mathbb{PA}'(\alpha)$. It was shown, for example, given $\alpha \geq \omega$ and $\gamma \geq \omega$, that if some group G with $|G| = \gamma$ satisfies $G \in \mathbb{P}'(\alpha)$, then both $\oplus_\gamma F$ (F an arbitrary finite Abelian group) ([13](3.3); [27](8.1), [28]) and $\oplus_\gamma \mathbb{Q}$ ([12](3.5); [27](8.1), [28]) are in $\mathbb{PA}'(\alpha)$; arguments exploiting the properties of a *variety* of groups furnished [28] the parallel conclusion for the group $\oplus_\gamma \mathbb{Z}$. The papers [13], [28] contain complete algebraic characterizations of the torsion groups in the classes $\mathbb{PA}'(\alpha)$, though the conditions are too technical to warrant reproduction here. A simple example will illustrate some of the subtleties. Let γ be one of the cardinals forbidden by Theorem 3.5, say γ is a strong limit cardinal with $\mathrm{cf}(\gamma) = \omega$, and let

$$G_0 := \oplus_\gamma \mathbb{Z}(p) \oplus \oplus_{2^\gamma} \mathbb{Z}(p^2) \text{ and } G_1 := \oplus_{2^\gamma} \mathbb{Z}(p) \oplus \oplus_\gamma \mathbb{Z}(p^2),$$

with p prime. The group $(\mathbb{Z}(p^2))^{2^\gamma}$ contains algebraically a G_δ-dense copy of the group $\oplus_{2^\gamma} \mathbb{Z}(p^2)$, indeed one may arrange $\oplus_{2^\gamma} \mathbb{Z}(p^2) \subseteq G_0 \subseteq (\mathbb{Z}(p^2))^{2^\gamma}$; hence $G_0 \in \mathbb{PA}'(2^\gamma)$ by Theorem 2.1, indeed according to [13](3.14) we have $G_0 \in \mathbb{PA}'(\alpha)$ for all α such that $2^\gamma \leq \alpha \leq 2^{2^\gamma}$. But $G_1 \notin \mathbb{P}'$, since in any pseudocompact group topology on G_1 the subgroup $p \cdot G_1 = \oplus_\gamma \mathbb{Z}(p)$ will also be pseudocompact, contrary to Theorem 3.5.

While the characterization of the torsion groups, and the torsion-free groups, and the divisible groups, and the free Abelian groups, which are in $\mathbb{PA}'$ is in hand [28], the pieces have not yet quite meshed to characterize fully the groups in $\mathbb{PA}'$.

3.6. The Class $\mathbb{CC}'$. The question of characterizing algebraically those groups G which admit a countably compact group topology appears to be considerably more subtle than the corresponding question for pseudocompactness. Among other things, the latter issue (as indicated in 3.5 above and the works cited there) lends itself to theorems in ZFC, while so far as is known today the former issue may be axiom-sensitive. In what follows I denote by $F(\gamma)$ [resp., $FA(\gamma)$] the free group [resp., the free Abelian group] on γ-many symbols. It was shown by Tkachenko [53], [56] using CH, that $FA(\mathfrak{c}) \in \mathbb{CC}'$. The result was extended by Tkacenko and Yaschenko [57]: Assuming MA, every torsion-free $G \in \mathbb{A}$ with $|G| = \mathfrak{c}$ satisfies $G \in \mathbb{CC}'$; in the same vein, it is consistent with the axioms of ZFC that $FA(2^\mathfrak{c}) \in \mathbb{CC}'$ [46]. Further, extending earlier work of Dikranjan and Tkachenko [32], Dikranjan and Shakhmatov [30] have given a simple algebraic condition which, consistently, characterizes those groups $G \in \mathbb{A}$ with $|G| \leq 2^\mathfrak{c}$ such that $G \in \mathbb{CC}'$.

The result in this context which I personally find most pleasing, because it is definitive in the sense that it uses only the axioms of ZFC, is this of Dikranjan and Shakhmatov [26]: no group of the form $F(\gamma)$ is in $\mathbb{CC}'$. The same paper gives a complete algebraic characterization of the groups in $\mathbb{CA}(\omega)$, and contains several other results about those Abelian groups which admit a separable group topology in the classes $\mathbb{X}$ of Question 3.1(b).

Here are some of the most interesting unsolved questions in this area of inquiry.

Question 3.7. Is the statement $FA(\mathfrak{c}) \in \mathbb{CC}'$ a theorem of ZFC?

Question 3.8. What are the groups in $\mathbb{CC}'$? In $\mathbb{CCA}'$?

Question 3.9 ([29])**.** Are the classes $\mathbb{CC}'$, $\mathbb{CCA}'$ closed under arbitrary products? Finite products? If $G \in \mathbb{CCA}'$, is $G \times G \in \mathbb{CCA}'$?

Consistent answers to portions of Question 3.9 are available (in the positive direction). Specifically, it is proved in [32] under [MA] that the class $\mathbb{CCA}'$ is closed under products of $\mathfrak{c}$-many factors each of size $\mathfrak{c}$, and in the forcing model built in [30] the same class is closed under products of (at most) $2^{\mathfrak{c}}$-many factors each of size not exceeding $2^{\mathfrak{c}}$. Question 3.9 is particularly tantalizing in view of the examples of van Douwen [33], K. P. Hart and van Mill [38], Tomita [58], [59] and others to the effect that in appropriately augmented axiom systems there are groups in $\mathbb{CCA}$ whose product is not in $\mathbb{CCA}$. When viewed in the light of the class-theoretic containments $\mathbb{C} \subseteq \mathbb{CC} \subseteq \mathbb{P}$, Tychonoff's theorem and Corollary 2.4 above, these examples are perceived by some workers as anomalous or unexpected.

3.10. <u>The Inclusions $\mathbb{C}' \subseteq \Omega' \subseteq \mathbb{CC}' \subseteq \mathbb{P}' \subseteq \mathbb{TB}'$</u>. From what has preceded it is clear, I hope, that among the classes $\{\mathbb{C}, \Omega, \mathbb{CC}, \mathbb{P}, \mathbb{TB}\}$ it is only for $\mathbb{X} = \mathbb{C}$ and $\mathbb{X} = \mathbb{TB}$ that the question "Which Abelian groups are in $\mathbb{X}'$?" has been fully answered. In the interest of completeness, let us note now that, at least consistently, the inclusions

$$\mathbb{CA}' \subseteq \Omega\mathbb{A}' \subseteq \mathbb{CCA}' \subseteq \mathbb{PA}' \subseteq \mathbb{TBA}'$$

are all proper. Here the search for witnessing examples must be conducted with due respect for the constraints established in [32]: Among divisible groups G with $|G| = \mathfrak{c}$, the conditions $G \in \mathbb{PA}'$ and $G \in \mathbb{CA}'$ are equivalent (assuming MA).

(a) $\mathbb{CA}' \neq \Omega\mathbb{A}'$ [ZFC]. Let α be a strong limit cardinal such that $\mathrm{cf}(\alpha) > \omega$, and let G be the Σ-product in a group of the form K^α with K compact metric, $|K| > 1$. Then $G \in \Omega$ (with $G \in \Omega\mathbb{A}$ if K is chosen Abelian) and $|G| = \alpha^\omega = \alpha$, but $G \notin \mathbb{C}'$ since the cardinality of every infinite every compact group G has the form 2^κ (indeed with $\kappa = w(G)$) (cf. [44](28.58)).

(b) $\Omega\mathbb{A}' \neq \mathbb{CCA}'$ [ZFC + MA]. (The essentials of the following argument appear in Dikranjan [24].) According to the result cited from [53] and [56] (or from [57]), from CH (or from MA) we have $FA(\mathfrak{c}) \in \mathbb{CC}'$. But every subgroup of $FA(\mathfrak{c})$ has algebraically the form $FA(\gamma)$ for some $\gamma \leq \mathfrak{c}$, and no such group admits a compact group topology [56]. Thus in no group topology does $FA(\mathfrak{c})$ admit a nondegenerate compact subgroup; in particular, $FA(\mathfrak{c}) \notin \Omega'$.

I suppose that $\Omega\mathbb{A}' \neq \mathbb{CCA}'$ is a theorem of ZFC. It may be mentioned in passing that in any model where $\Omega\mathbb{A}' = \mathbb{CCA}'$ holds, Questions 3.9 have an affirmative answer. The argument is elementary: The product of any set of ω-bounded spaces is ω-bounded, so for each set $\mathcal{S} \subseteq \mathbb{CCA}' = \Omega\mathbb{A}'$ there is on $\Pi\mathcal{S}$ an ω-bounded group topology—that is, $\Pi\mathcal{S} \in \Omega\mathbb{A}' = \mathbb{CCA}'$.

(c) $\mathbb{CCA}' \neq \mathbb{PA}'$ [ZFC]. According to [12] or [27], [28], the group $H := \oplus_\mathfrak{c}\, \mathbb{Z}$ or $H := \oplus_\mathfrak{c}\, \mathbb{Q}$ embeds as a dense pseudocompact subgroup of the compact group $\mathbb{T}^\mathfrak{c}$. Now for a fixed prime p let A be any subgroup of $\mathbb{T}^\mathfrak{c}$ algebraically of the form $A = \oplus_\omega\, \mathbb{Z}(p)$ and set $G := \langle H \cup A \rangle = H \oplus A$. The map $G \to G$ given by $x \to p \cdot x$ is continuous in any group topology on G, and A is its (closed) kernel; since $A \notin \mathbb{CC}'$, we have $G \notin \mathbb{CC}'$.

For many results on the algebraic structure of the groups in $\mathbb{CCA}'$, including some results in ZFC, see [30].

(d) $\mathbb{PA}' \neq \mathbb{TBA}'$ [ZFC]. As indicated in 3.4 above, the cardinality of a pseudocompact group cannot be a strong limit cardinal with countable cofinality. But according to the argument in 3.2, every Abelian group admits a totally bounded group topology.

4. Extremal Behavior

Definition 4.1. Let $\mathbb{X}$ and $\mathbb{Y}$ be classes of topological groups. A group $G = (G, \mathcal{T}) \in \mathbb{X}$ is

(a) r-$(\mathbb{X}, \mathbb{Y})$-*extremal* if no group topology $\mathcal{U}$ on G strictly refining $\mathcal{T}$ satisfies $(G, \mathcal{U}) \in \mathbb{Y}$;

(b) s-$(\mathbb{X}, \mathbb{Y})$-*extremal* if no proper dense subgroup D of G satisfies $(D, \mathcal{T}) \in \mathbb{Y}$.

(c) *doubly* $(\mathbb{X}, \mathbb{Y})$-*extremal* if G is both r-$(\mathbb{X}, \mathbb{Y})$-extremal and s-$(\mathbb{X}, \mathbb{Y})$-extremal.

The symbols r and s are intended to evoke the words *refinement* and *subgroup*.

The foregoing definition is given in broad generality in order to bring into focus a plethora of problems ripe for investigation, namely:
For $\mathbb{X}, \mathbb{Y} \in \{\mathbb{C}, \Omega, \mathbb{CC}, \mathbb{P}, \mathbb{TB}\}$ or $\mathbb{X}, \mathbb{Y} \in \{\mathbb{CA}, \Omega\mathbb{A}, \mathbb{CCA}, \mathbb{PA}, \mathbb{TBA}\}$, are there r-$(\mathbb{X}, \mathbb{Y})$-extremal and/or s-$(\mathbb{X}, \mathbb{Y})$-extremal groups?
Again without pretense at completeness, I will note from the literature a bit of what is known for certain choices $\mathbb{X}, \mathbb{Y}$. The case $\mathbb{X} = \mathbb{Y} = \mathbb{P}$ is postponed to *Part II*.

4.2. On r-$(\mathbb{C}, \mathbb{CC})$-extremality. Although a compact topology admits no proper compact refinement, it might be expected that compact groups would generally admit a proper countably compact group refinement. The following theorem puts a severe limitation on such constructions—indeed it shows that consistently no such refinement can exist.

Theorem (Arhangel′skiĭ [1]). *If $(G, \mathcal{T}) \in \mathbb{C}$ and G admits a proper refinement $\mathcal{U} \supseteq \mathcal{T}$ such that $(G, \mathcal{U}) \in \mathbb{CC}$, then*

(a) there exists a compact group K and a discontinuous, sequentially continuous homomorphism $h : (G, \mathcal{T}) \to K$; and

(b) $|G|$ is Ulam-measurable.

After reading a pre-publication copy of [1], Dieter Remus and I were able to prove a partial converse, as follows.

Theorem ([14]). *If $G \in \mathbb{C}$ with $|G|$ Ulam-measurable, and if either G is Abelian or connected, then*

(a) there exists a compact group K and a discontinuous, sequentially continuous homomorphism $h : (G, \mathcal{T}) \to K$; and

(b) G admits a proper countably compact group refinement.

This result was vastly strengthened in unpublished work reported by Uspenskiĭ [62].

Theorem ([62]). *Let $(G, \mathcal{T}) \in \mathbb{C}$ with $|G|$ Ulam-measurable. Then there is a group topology $\mathcal{U}$ on G such that $\mathcal{U} \supseteq \mathcal{T}$, $\mathcal{U} \neq \mathcal{T}$, and $\mathcal{U}|A = \mathcal{T}|A$ for every $A \subseteq G$ of non-Ulam-measurable cardinality.*

Remark. The papers [1] and [14] cited above both depend in part on work of Varopoulos [63]. That paper was the progenitor of several subsequent papers in general topology on the existence of discontinuous, sequentially continuous functions; Varopoulos was one of the first workers to recognize the relation between the existence of such functions and the existence of (Ulam-) measurable cardinals.

4.3. On r-$(\mathbb{CA}, \mathbb{P})$- and s-$(\mathbb{CA}, \Omega)$-extremality.

Theorem (a) ([16]). *No nonmetrizable compact Abelian group is* r-$(\mathbb{CA}, \mathbb{P})$*-extremal*;

and

(b) ([5](p. 141)). *No nonmetrizable compact Abelian group is* s-$(\mathbb{CA}, \Omega)$*-extremal.*

Proof. The theorem asserts simply that every compact Abelian group G of uncountable weight admits a proper pseudocompact group refinement and a proper ω-bounded dense subgroup. Both statements derive from Pontrjagin duality. Indeed such G satisfies $w(G) = |\widehat{G}| > \omega$, so $\widehat{G}$ contains either the free Abelian group $\oplus_{\omega^+} \mathbb{Z}$ or a group of the form $\oplus_{\omega^+} \mathbb{Z}(p)$; thus there is a continuous surjective homomorphism $h : G \twoheadrightarrow K$ with either $K = \mathbb{T}^{(\omega^+)}$ or $K = (\mathbb{Z}(p))^{(\omega^+)}$. Such a group K admits a proper pseudocompact refinement, and this is easily "pulled back" by h to such a refinement for G; further, the Σ-product $D \subseteq K$ is a proper ω-bounded subgroup, and $h^{-1}(D)$ is then proper and ω-bounded in G. □

4.4. On s-$(\mathbb{C}, \mathbb{CC})$-extremality. Pontrjagin duality being unavailable when G is (compact but) nonabelian, the methods of Theorem 4.3 do not apply to show the nonexistence of (possibly nonabelian) nonmetrizable r-$(\mathbb{C}, \mathbb{P})$- or s-$(\mathbb{C}, \mathbb{P})$-extremal groups. Using the fact that every compact group of weight α admits group quotients of arbitrary infinite weight $\kappa \leq \alpha$, however, Itzkowitz and Shakhmatov [45] were able to prove the following theorem, one of the few established "anti-extremal" results without an Abelian hypothesis.

Theorem ([45]). *No (nonmetrizable) compact group G with $|G| > \mathfrak{c}$ is* s-$(\mathbb{C}, \mathbb{CC})$*-extremal.*

4.5. On r-$(\mathbb{C}, \mathbb{TB})$-extremality. It is a theorem of van der Waerden [65] that the (compact, metrizable) matrix groups $V(n) := SO(2n+1, \mathbb{R})$ $(0 < n < \omega)$ have the property that every homomorphism from $V(n)$ to a compact group K is continuous. In recognition of this nontrivial result, compact groups with this continuity property are called *van der Waerden* groups. In our terminology, these are exactly the r-$(\mathbb{C}, \mathbb{TB})$-extremal groups. Since an infinite compact Abelian group G admits many discontinuous homomorphisms to $\mathbb{T}$ (as follows from the relations

$|\widehat{G}| = w(G) < 2^{wG} = |G| < 2^{|G|} = |\mathrm{Hom}(G, \mathbb{T})|$),

there is no infinite Abelian van der Waerden group. It is known that if a compact group G admits for some integer n infinitely many irreducible unitary representations of dimension n, then G is of Haar measure zero in its Bohr compactification $b(G)$ [47]; since a van der Waerden group G satisfies $G = b(G)$ it has for each n (at most) finitely many such representations, hence is metrizable. See also [37] for additional information, including nontrivial examples of totally disconnected van der Waerden groups.

4.6. On s-$(\Omega, \mathbb{CC})$-extremality. Let $\alpha \geq \omega$, let K be a compact, metrizable group, and let G be the Σ-product in K^α. Then, as noted in [18](3.3), for every G_δ-dense subset $D \subseteq G$ and $p \in G$, there is a sequence $x(k) \in D$ such that $x(k) \to p$. (Given such D and p, and writing $s(x) := \{\xi < \alpha : x_\xi \neq e_K\}$ for $x \in G$, one chooses $x(0) \in D$ such that $x(0)_\xi = p_\xi$ for all $\xi \in s(p)$ and then, if $x(k) \in D$ has been chosen for all $k \leq n < \omega$, one chooses $x(n+1) \in D$ such that $x(n+1)_\xi = p_\xi$ for all $\xi \in s(p) \cup \cup_{k \leq n} s(x(k))$.) Thus for every $\alpha \geq \omega$ some group $G \in \Omega(\alpha)$ has no proper, G_δ-dense, countably compact subspace, in particular G is s-$(\Omega, \mathbb{CC})$-extremal. As is pointed out in [9](1.1), an appropriate choice of K yields G which is a torsion group or torsion-free, connected or zero-dimensional, Abelian or nonabelian.

4.7. On $(\mathbb{PA}, \mathbb{TBA})$-extremality. To see that no infinite $(G, \mathcal{T}) \in \mathbb{PA}$ is r-$(\mathbb{PA}, \mathbb{TBA})$-extremal, it is enough to recall that the maximal totally bounded group topology $\mathcal{T}^\#$ on G (which necessarily contains $\mathcal{T}$) is not pseudocompact [21](2.2). That also such $(G, \mathcal{T})$ is not s-$(\mathbb{PA}, \mathbb{TBA})$-extremal is shown in [10](3.4); the authors there note that this result was proved independently by E. A. Reznichenko (unpublished).

4.8. On r- and s-$(\mathbb{TBA}, \mathbb{TBA})$-extremality. As is clear from Theorem 5.8 below, every Abelian group G admits not only a maximal totally bounded group topology but a largest such topology; it is the topology $\mathcal{T}^\#$ induced on G by $\mathrm{Hom}(G, \mathbb{T})$. Thus an Abelian group $(G, \mathcal{T}) \in \mathbb{TBA}$ is r-$(\mathbb{TBA}, \mathbb{TBA})$-extremal if and only if $\mathcal{T} = \mathcal{T}^\#$. As for s-extremality in the class $\mathbb{TBA}$, it is shown in [10](2.6) that every torsion-free group $G \in \mathbb{TBA}(> \omega)$ has a proper dense subgroup (necessarily in $\mathbb{TBA}$); in the other direction every subgroup of G is closed in the topology $\mathcal{T}^{\#}$, so every group of the form $(G, \mathcal{T}^{\#}) \in \mathbb{TBA}$ is $(\mathbb{TBA}, \mathbb{TBA})$-extremal.

Part II. Extremal Pseudocompact Groups

In §5 I will attempt to expose in coherent manner a number of results from the literature, including several in papers I have written with T. Soundararajan, Lewis C. Robertson, Jan van Mill, and Helma Gladdines. Then in §6 I turn to new results, all with Jorge Galindo, which have been submitted with detailed proofs for publication [7] and which appear in summary form in [35]; my debt and gratitude to Galindo for instruction and the development of these new techniques are very great.

Henceforth the brief expressions *r-extremal*, *s-extremal* and *doubly extremal* will abbreviate r-$(\mathbb{P}, \mathbb{P})$-extremal, s-$(\mathbb{P}, \mathbb{P})$-extremal, and doubly $(\mathbb{P}, \mathbb{P})$-extremal, respectively.

5. Results From the Literature: A Brief Survey

Since (as is easily shown) a normal pseudocompact space is countably compact, a metrizable pseudocompact space is necessarily compact. Since a compact space admits no proper dense compact subspace, and a continuous bijection between compact spaces is a homeomorphism, every pseudocompact metrizable group is doubly extremal. It should be noted frankly at the start of our treatment concerning nonmetrizable pseudocompact groups, however, that none of the following three questions have been settled in full generality. One may anticipate "absolute" answers, i.e., answers in ZFC, but in fact at present the literature in each case lacks even a consistent response, even when the questions are restricted to the context of Abelian groups.

Question 5.1. Is there a nonmetrizable r-extremal group?

Question 5.2. Is there a nonmetrizable s-extremal group?

Question 5.3. Are r- and s-extremality equivalent conditions?

Of course if the answer to 5.1 and 5.2 is "No", then the answer to 5.3 is "Yes". As we show *via* citations to the literature, much evidence points in that direction. So far as is now known, however, there may exist in ZFC r- but not s-extremal groups, s- but not r-extremal groups, and/or doubly extremal groups.

Two tools have proved to be useful in the investigation of totally bounded groups in general and of pseudocompact groups in particular. I have in mind here, respectively, point-separating subgroups of $\mathrm{Hom}(G, \mathbb{T})$ and, for topological groups G, the set of closed normal G_δ subgroups of G. In what follows we will refer frequently to 5.4–5.9.

Notation 5.4. For G a topological group, $\Lambda(G)$ denotes the set of closed, normal G_δ-subgroups of G.

Theorem 5.5 ([22])**.** *A topological group G is pseudocompact if and only if G/N is compact metric for every $N \in \Lambda(G)$.*

Corollary 5.6 ([18])**.** *If $G \in \mathbb{P}$ and $N \in \Lambda(G)$, then $N \in \mathbb{P}$.*

A Galois-type correspondence between totally bounded group topologies on an Abelian group G and point-separating subgroups of $\mathrm{Hom}(G, \mathbb{T})$ is given in the following theorem. The theme has been developed further in [4], [50], [51], [52], [11], [15], and [3].

Notation 5.7. Let G be an Abelian group and let A be a point-separating subgroup of $\mathrm{Hom}(G, \mathbb{T})$. Then $\mathcal{T}_A$ denotes the (Hausdorff) group topology induced on G by A.

Theorem 5.8 ([19])**.** *Let G be an Abelian group. Then*

(a) each topology $\mathcal{T}_A$ as in 5.7 makes $(G, \mathcal{T}_A)$ a totally bounded topological group;

(b) each totally bounded group topology $\mathcal{T}$ on G arises as in 5.7: if $A := \widehat{(G, \mathcal{T})}$, then $\mathcal{T} = \mathcal{T}_A$; and

(c) different point-separating subgroups $A, B \subseteq \mathrm{Hom}(G, \mathbb{T})$ induce different totally bounded group topologies $\mathcal{T}_A, \mathcal{T}_B$ on G.

That $\mathcal{T}_A \neq \mathcal{T}_B$ in (c) follows from (a) and (b), since $\widehat{(G, \mathcal{T}_A)} = A \neq B = \widehat{(G, \mathcal{T}_B)}$.

5.9. Discussion. The class-theoretic notation introduced earlier makes it possible to cite efficiently several questions arising naturally from Theorem 5.8.

Question. Let $G \in \mathbb{A}$ and $\mathbb{X} \in \{\mathbb{C}, \Omega, \mathbb{CC}, \mathbb{P}, \mathbb{TB}\}$. Which point-separating subgroups $A \subseteq \mathrm{Hom}(G, \mathbb{T})$ make $(G, \mathcal{T}_A) \in \mathbb{X}$?

A bit is known about that question when $\mathbb{X} = \mathbb{P}$ (cf. [18](6.5), [39](3.4)), but to the best of my knowledge no successful, comprehensive study has been made of those five questions. Here are four simple preliminary comments.

(a) According to Theorem 5.8(a), when $\mathbb{X} = \mathbb{TB}$ the answer is "All such A".

(b) For some $\mathbb{X}$ and G, there are no such A: We have noted in Theorem 3.5 that some $G \in \mathbb{A}$ are not in $\mathbb{P}'$ (hence, not in Ω' or $\mathbb{C}'$; see also 3.3).

(c) When $(G, \mathcal{T}_A) \in \mathbb{CA}$, no subgroup $B \subseteq \mathrm{Hom}(G, \mathbb{T})$ properly containing A, or properly contained in A, makes $(G, \mathcal{T}_B) \in \mathbb{C}$.

(d) When $(G, \mathcal{T}_A) \in \mathbb{X} \in \{\mathbb{C}, \Omega, \mathbb{CC}, \mathbb{P}, \mathbb{TB}\}$, every (point-separating) subgroup $B \subseteq A$ makes $(G, \mathcal{T}_B) \in \mathbb{X}$ (with necessarily $B = A$ when $\mathbb{X} = \mathbb{C}$).

For the rest of this talk I restrict attention to pseudocompact Abelian groups and to questions about r- and/or s-extremal topologies on such groups. As indicated earlier, a pseudocompact metrizable space is compact, so pseudocompact metrizable groups are doubly extremal and need concern us no further. The condition $G \in \mathbb{PA}(> \omega)$, which occurs frequently in what follows, is our shorthand for the condition that G is a nonmetrizable pseudocompact Abelian group; according to Theorem 5.8, such G have the form $G = (G, \mathcal{T}_A)$ with $A := \widehat{G}$ a point-separating subgroup of $\mathrm{Hom}(G, \mathbb{T})$, $|A| = w(G) > \omega$.

Concerning r-extremality, Theorem 5.8 allows a useful change of perspective: A topological group $(G, \mathcal{T}_A) \in \mathbb{PA}$ is r-extremal if (and only if) A is maximal among subgroups of $\mathrm{Hom}(G, \mathbb{T})$ which induce a pseudocompact group topology.

The continuous image of a pseudocompact space is pseudocompact, hence if metrizable is compact. It is clear, then, that if a homomorphism $h \in \mathrm{Hom}(G, \mathbb{T})$ is an element of some subgroup $A \subseteq \mathrm{Hom}(G, \mathbb{T})$ such that $(G, \mathcal{T}_A)$ is pseudocompact, then $h[G]$ is a compact subgroup of $\mathbb{T}$. A characterization of those (discontinuous) homomorphisms h which can be adjoined to A when $(G, \mathcal{T}_A)$ is pseudocompact without destroying pseudocompactness is given in the following Theorem. Here and later, for simplicity I wrote $A(h) := \langle A \cup \{h\} \rangle$, where (of course) $A(h) = A$ if already $h \in A$.

Theorem 5.10 ([7]). *Let* $(G, \mathcal{T}_A) \in \mathbb{PA}$ *and let* $h \in \mathrm{Hom}(G, \mathbb{T})\backslash A$. *Then*

(a) if $A \cap \langle h \rangle = \{0\}$, *then* $(G, \mathcal{T}_{A(h)}) \in \mathbb{P}$ *if and only if* $h[G]$ *is closed in* $\mathbb{T}$ *and* $\ker(h)$ *is* G_δ*-dense in* $(G, \mathcal{T}_A)$*;*

(b) if $A \cap \langle h \rangle = \langle nh \rangle$, *then* $(G, \mathcal{T}_{A(h)}) \in \mathbb{P}$ *if and only if* $h[G]$ *is closed in* $\mathbb{T}$ *and* $\ker(h)$ *is* G_δ*-dense in* $\ker(nh)$.

Theorem 5.10 is the workhorse lemma on which much of [7] is based. In 5.11(b) and 5.12 we note two immediate consequences; 5.11(a), a result of some years standing, is included here to make the point that results on r- and s-extremality have a tendency to occur in parallel pairs.

In what follows, for every Abelian group G and $0 < n < \omega$ we define $\phi_n : G \to G$ by $\phi_n(x) := n \cdot x \in G$, and $t_n(G) := \ker(\phi_n) = \{x \in G : n \cdot x = 0\}$.

Corollary 5.11. *Let* $(G, \mathcal{T}_A) \in \mathbb{PA}(> \omega)$ *be connected and nondivisible. Then*

(a) [9] G *is not s-extremal; and*

(b) [7] G *is not r-extremal.*

Proof. For some prime p the group $\phi_p[G]$ is a proper subgroup of G.

(a) $\overline{G}$ is compact and connected and hence divisible, so $\phi_p[\overline{G}] = \overline{G}$. Now G is G_δ-dense in $\overline{G}$, so $\phi_p[G]$ is G_δ-dense in $\phi_p[\overline{G}] = \overline{G}$; hence $\phi_p[G]$ is G_δ-dense in G.

(b) Since $p \cdot G/\phi_p[G] = \{0\}$, the group $G/\phi_p[G]$ has the form $\oplus_{i\in I} \mathbb{Z}(p)$ with $I \neq \emptyset$; thus there is a surjective homomorphism $h : G \twoheadrightarrow \mathbb{Z}(p) \subseteq \mathbb{T}$. Further, $h[G] = \mathbb{Z}(p)$ is compact, and $A \cap \langle h \rangle = \{0\}$ since A is torsion-free while $ph = 0$. Thus Theorem 5.10(a) applies. □

Corollary 5.12. *If* $(G, \mathcal{T}_A) \in \mathbb{PA}$ *is not r-extremal, then some* $N \in \Lambda(G)$ *is not s-extremal.*

Proof. [We have noted already in Corollary 5.6 that indeed $N \in \mathbb{PA}$.] Now, let $h \in \mathrm{Hom}(G, \mathbb{T})\backslash A$ with $(G, \mathcal{T}_{A(h)})$ pseudocompact. Either $A \cap \langle h \rangle = \{0\}$, in which case $\ker(h)$ is (proper and) G_δ-dense in G itself, or $A \cap \langle h \rangle = \langle nh \rangle$ with $0 < n < \omega$, in which case $\ker(h)$ is proper and G_δ-dense in $\ker(nh) \in \Lambda(G)$. □

It would be pleasing to be able to report a version of Corollary 5.12 with "r-" and "s-" exchanged, but no such result is known. In any event, Corollary 5.12 is unnecessarily weak. It turns out that for every $G \in \mathbb{PA}$, whether or not assumed r-extremal, some $N \in \Lambda(G)$ is not s-extremal. The proof depends on the following two lemmas, of which the first is easily verified using Theorem 5.10. Again, parallel results appear concerning our two kinds of extremal behavior.

Lemma 5.13. *Let $(G, \mathcal{T}_A) \in \mathbb{PA}$ and let H be a closed, pseudocompact subgroup of G.*

(a) If G/H is not r-extremal, then G is not r-extremal.

(b) If G/H is not s-extremal, then G is not s-extremal.

Lemma 5.14. *Let $G \in \mathbb{PA}$ be either r- or s-extremal, and for $0 < n < \omega$ and $N \in \Lambda(G)$ set $N_n := \mathrm{cl}_G\, \phi_n[N]$. Then $N_n \in \Lambda(G)$.*

Proof. From [9](3.2) we have $\phi_n[\overline{N}] \in \Lambda(\phi_n[\overline{G}])$ and hence $N_n = \phi_n[\overline{N}] \cap G \in \Lambda(G_n)$, so from $w(G/N_n) = w(G/G_n) + w(G_n/N_n)$ it follows that $w(G/N_n) > \omega$ if and only if $w(G/G_n) > \omega$. Since a pseudocompact, Abelian torsion group of uncountable weight is neither r- nor s-extremal [18](7.5), Lemma 5.13 shows $\omega = w(G/G_n) = w(G/N_n)$ and hence $N_n \in \Lambda(G)$. □

Corollary 5.15. *If $G \in \mathbb{PA}$ is either r- or s-extremal then for every $N \in \Lambda(G)$ some $M \in \Lambda(N) \subseteq \Lambda(G)$ is connected.*

Proof. According to Lemma 5.14 (applied to N in place of G), it is enough to take $M := \cap_n \mathrm{cl}_N\, \phi_n[N]$. □

As usual, we say that a space X is *hereditarily disconnected* if every connected $S \subseteq X$ satisfies $|S| \leq 1$. It is known that every pseudocompact, Abelian torsion group is zero-dimensional and hence hereditarily disconnected (cf. [18](7.5)), but there are hereditarily disconnected groups $G \in \mathbb{PA}$ which are not zero-dimensional (cf. [9](7.7)). Thus the following consequence of Corollary 5.15 improves the statement from [18](7.5) that a zero-dimensional group $G \in \mathbb{PA}(> \omega)$ is neither r- nor s-extremal.

Theorem 5.16. *(a) A hereditarily disconnected group $G \in \mathbb{PA}(> \omega)$ is neither r- nor s-extremal.*

(b) For every $G \in \mathbb{PA}(> \omega)$, some $N \in \Lambda(G)$ is not s-extremal.

(c) If there is $G \in \mathbb{PA}(>\omega)$ for which every $N \in \Lambda(G)$ is r-extremal, then there is connected $G' \in \Lambda(G)$ with the same property and with $w(G') = w(G)$.

In parts (b) and (c) of Theorem 5.16, the r- and s- symmetry which has characterized most of the theorems cited earlier is lacking. Despite some efforts by Galindo and me [7], [35], we do not know if the statement obtained from 5.16(b) by replacing "s-extremal" by "r-extremal" is valid.

The papers [7] and [35] enumerate additional techniques and conditions useful for investigating r- and s-extremal behavior.

6. Doubly Extremal Pseudocompact Groups

We begin this section with two easily-stated criteria, one of them necessary that a group $G \in \mathbb{PA}(>\omega)$ be r-extremal, the other that such a group be s-extremal. All results here are from [7], where detailed proofs are available.

Theorem 6.1 ([7])**.** *Let $G = (G, \mathcal{T}_A) \in \mathbb{PA}(>\omega)$.*

(a) If $t_p(G)$ is G_δ-dense in $t_p(\overline{G})$ for all primes p, then G is not s-extremal.

(b) If (for some prime p) $t_p(G)$ is not dense in $t_p(\overline{G})$, then G is not r-extremal.

Proof. [Outline]. (a) If (a) fails then by Corollary 5.15 some $N \in \Lambda(G)$ is connected and not s-extremal. If N is divisible then G splits algebraically as $G = N \times G/N$, and this relation is topological also when all groups carry the associated G_δ ("P-space") topology, G/N now being discrete; if $D \subseteq N$ is chosen to witness that N is not s-extremal, then $D \times G/N$ proves that G is not s-extremal. If N is not divisible choose a prime p such that $\phi_p(N)$ is a proper, G_δ-dense subgroup of N; there is a proper subgroup L of G such that algebraically $G/\phi_p(N) = N/\phi_p(N) + L/\phi_p(N)$,
and then $\phi_p(N) + L$ is proper and G_δ-dense in G.

(b) The continuous characters of the compact group $t_p(\overline{G})/\mathrm{cl}_{\overline{G}}\, t_p(G)$ separate points and extend continuously over $\overline{G}$, so there is $\chi \in \widehat{\overline{G}}$ such that $t_p(G) \subseteq \ker(\chi)$ while $t_p(\overline{G}) \subseteq \ker(\chi)$ fails. Writing $f := \chi | G \in A$ there is $h \in \mathrm{Hom}(G, \mathbb{T})$ such that $f = ph$, and one shows without difficulty that $h \notin A$ (indeed $A \cap \langle h \rangle = \langle f \rangle$) and $(G, \mathcal{T}_{A(h)})$ is pseudocompact. □

Theorem 6.1 indicates that there are difficulties in finding nonmetrizable doubly extremal groups, as follows.

Corollary 6.2 ([7])**.** *If* $G \in \mathbb{PA}(>\omega)$ *and all the groups* $t_p(G)$ *(p prime) are pseudocompact, then* G *is not doubly extremal.*

It follows in particular that if $G \in \mathbb{PA}(>\omega)$ is torsion-free, or if every closed subgroup of G is pseudocompact (as is the case, for example, if G is normal in the usual topological sense), then G is not doubly extremal. In particular:

Corollary 6.3 ([7])**.** *No group* $G \in \mathbb{CCA}(>\omega)$ *is doubly* $(\mathbb{CCA}, \mathbb{PA})$*-extremal—that is, such* G *has either a proper dense pseudocompact subgroup or a proper pseudocompact group refinement.*

The condition on a pseudocompact group G that each of its closed subgroups is pseudocompact is strictly weaker than the condition that G is countably compact. For an example one may choose any nonmetrizable compact connected Abelian group K and set $G := \{x \in K : \langle x \rangle \text{ is metrizable}\}$. Such groups, introduced and examined in [67], have been studied in detail in [31].

6.4. Discussion. We recapitulate briefly: An s-extremal group $G \in \mathbb{PA}(>\omega)$ must satisfy $w(G) > \mathfrak{c} = |G| = r_0(G)$ [8], must have some $N \in \Lambda(G)$ connected, and if also r-extremal must for some prime p have $t_p(G)$ dense but not G_δ-dense in $t_p(\overline{G})$. (So again, for emphasis: such G, if doubly extremal, is neither torsion-free nor countably compact.) Unfortunately, groups satisfying these several conditions exist which are neither r- nor s-extremal. To see this, first choose for each prime p a dense but not G_δ-dense subgroup H_p of the group $(\mathbb{Z}(p))^{(\mathfrak{c}^+)} = t_p(\mathbb{T}^{(\mathfrak{c}^+)})$ such that $|H_p| = \mathfrak{c}$. Next, arguing as in [12](4.4 and 1.6) and [28](4.4), choose a G_δ-dense, torsion-free subgroup H of $\mathbb{T}^{(\mathfrak{c}^+)}$ such that $|H| = \mathfrak{c}$. Then for any set $\mathbb{S}$ of primes set $G_\mathbb{S} := H + \oplus_{p\in\mathbb{S}} H_p$. Then $G_\mathbb{S} \in \mathbb{PA}(\mathfrak{c}^+)$ with $|G_\mathbb{S}| = r_0(G_\mathbb{S}) = \mathfrak{c}$, $G_\mathbb{S}$ is connected (since $\beta(G_\mathbb{S}) = \overline{G_\mathbb{S}} = \mathbb{T}^{(\mathfrak{c}^+)}$), and $t_p(G_\mathbb{S}) = H_p$ is dense but not G_δ-dense in $t_p(\overline{G_\mathbb{S}})$ exactly when $p \in \mathbb{S}$; when $\mathbb{S} \neq \emptyset$ the group $G_\mathbb{S}$ is nondivisible and hence neither r- nor s-extremal. The upshot is that the known conditions necessary for doubly extremal behavior are collectively inadequate to ensure (simple, singly) extremal behavior.

7. Postlude

I appreciate your attention. I hope that some of the questions raised but left unsettled here will pique your interest. Several mathematicians I admire have told me independently over the years in a variety of contexts that "the place to begin" (or to continue) a program of investigation is with the simplest concrete unsolved question one can think of. In that spirit I close by repeating a very explicit question of which versions have been asked frequently, initially (I believe) in 1985 [9](8.6):

Question 7.1. Is some dense pseudocompact subgroup (of cardinality $\mathfrak{c}$) of $\mathbb{T}^{(\mathfrak{c}^+)}$ r-extremal? s-extremal? doubly extremal?

Thank you.

References

[1] A. V. Arhangel′skiĭ, *On countably compact topologies on compact groups and dyadic compacta*, Topology and Its Applications **57** (1994), 163–181.

[2] A. V. Arhangel′skiĭ, *On a theorem of W. W. Comfort and K.A. Ross*, Comment. Math. Univ. Carolinae **40** (1999), 133–151.

[3] Alessandro Berarducci, Dikran Dikranjan, Marco Forti, and Stephen Watson, *Cardinal invariants and independence results in the poset of precompact group topologies*, J. Pure and Applied Algebra **126** (1998), 19–49.

[4] S. Berhanu, W. W. Comfort, and J. D. Reid, *Counting subgroups and topological group topologies*, Pacific J. Math. **116** (1985), 217–241.

[5] W. W. Comfort, *Some questions and answers in topological groups*, In: Memorias del Seminario Especial de Topología, Volumen 5 (Javier Bracho and Carlos Prieto, eds.), pp. 131–149. Instituto de Matemáticas de la Universidad Nacional Autónoma de México, Mexico City, 1983.

[6] W. W. Comfort and Jorge Galindo, *Pseudocompact topological group refinements of maximal weight*, Proc. Amer. Math. Soc. **131** (2003), 1311–1320.

[7] W. W. Comfort and Jorge Galindo, *Extremal pseudocompact topological groups*, J. Pure and Applied Algebra (2005). To appear.

[8] W. W. Comfort, Helma Gladdines, and Jan van Mill, *Proper pseudocompact subgroups of pseudocompact Abelian groups*, In: Papers on General Topology and Applications, *Annals of the New York Academy of Sciences* **728** (New York) (Susan Andima, Gerald Itzkowitz, T. Yung Kong, Ralph Kopperman, Prabud Ram Misra, Lawrence Narici, and Aaron Todd, eds.), pp. 237–247, 1994. [Proc. June, 1992 Queens College Summer Conference on General Topology and Applications.]

[9] W. W. Comfort and Jan van Mill, *Concerning connected, pseudocompact Abelian groups*, Topology and Its Applications **33** (1989), 21–45.

[10] W. W. Comfort and Jan van Mill, *Some topological groups with, and some without, proper dense subgroups*, Topology Appl. **41** (1991), 3–15.

[11] W. W. Comfort and Dieter Remus, *Long chains of Hausdorff topological group topologies*, J. Pure and Applied Algebra **70** (1991), 53–72.

[12] W. W. Comfort and Dieter Remus, *Imposing pseudocompact group topologies on Abelian groups*, Fundamenta Math. **142** (1993), 221–240.

[13] W. W. Comfort and Dieter Remus, *Abelian torsion groups with a pseudocompact group topology*, Forum Mathematicum **6** (1994), 323–337.

[14] W. W. Comfort and Dieter Remus, *Compact groups of Ulam measurable cardinality: partial converses to theorems of Arhangel′skiĭ and Varopoulos*, Mathematica Japonica **39** (1994), 203–210.

[15] W. W. Comfort and Dieter Remus, *Long chains of topological group topologies—a continuation*, Topology and Its Applications **75** (1997), 51–79. Correction: **96** (1999), 277–278.

[16] W. W. Comfort and Lewis C. Robertson, *Proper pseudocompact extensions of compact Abelian group topologies*, Proc. Amer. Math. Soc. **86** (1982), 173–178.

[17] W. W. Comfort and Lewis C. Robertson, *Cardinality constraints for pseudocompact and for totally dense subgroups of compact Abelian groups*, Pacific J. Math. **119** (1985), 265–285.

[18] W. W. Comfort and Lewis C. Robertson, *Extremal phenomena in certain classes of totally bounded groups*, Dissertationes Math. **272** (1988), 48 pages. [Rozprawy Mat. Polish Scientific Publishers, Warszawa.]

[19] W. W. Comfort and Kenneth A. Ross, *Topologies induced by groups of characters*, Fundamenta Math. **55** (1964), 283–291.

[20] W. W. Comfort and Kenneth A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math. **16** (1966), 483–496.

[21] W. W. Comfort and Victor Saks, *Countably compact groups and finest totally bounded topologies*, Pacific J. Math. **49** (1973), 33–44.

[22] W. W. Comfort and T. Soundararajan, *Pseudocompact group topologies and totally dense subgroups*, Pacific J. Math. **100** (1982), 61–84.

[23] W. W. Comfort and F. J. Trigos-Arrieta, *Locally pseudocompact topological groups*, Topology Appl. (1995), 263–280.

[24] Dikran Dikranjan, *Countably compact groups satisfying the open mapping theorem*, Topology Appl. **98** (1999), 81–129.

[25] D. Dikranjan and D. Shakhmatov, *Pseudocompact topologies on groups*, Topology Proc. **17** (1992), 335–342.

[26] D. N. Dikranjan and D. B. Shakhmatov, *Pseudocompact topologizations of groups*, Zbornik radova Filozofskog fakulteta u Nišu Serija Matematika **4** (1990), 83–93.

[27] D. N. Dikranjan and D. B. Shakhmatov, *Algebraic structure of the pseudocompact groups*, 1991, Report 91–19, pp. 1–37. York University, Ontario, Canada.

[28] Dikran N. Dikranjan and Dmitri B. Shakhmatov, *Algebraic structure of pseudocompact groups*, Memoirs Amer. Math. Soc. **133** (1998), 83 pages.

[29] Dikran Dikranjan and Dmitri Shakhmatov, *Compact-like group topologies on Abelian groups*, In: Zoltan T. Balogh Memorial Topology Conference, pp. 120–22. Miami University (Oxford, Ohio), 2002.

[30] Dikran N. Dikranjan and Dmitri B. Shakhmatov, *Forcing hereditarily separable compact-like group topologies on Abelian groups*, Topology and Its Applications. To appear.

[31] Dikran Dikranjan and Michael Tkačenko, *Sequentially complete groups: dimension and minimality*, J. Pure and Applied Algebra **157** (2001), 215–239.

[32] Dikran N. Dikranjan and Mikhail Tkačenko, *Algebraic structure of small countably compact Abelian groups*, Forum Math. **15** (2003), 811–837.

[33] E. K. van Douwen, *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (1980), 417–427.

[34] E. K. van Douwen, *The weight of a pseudocompact (homogeneous) space whose cardinality has countable cofinality*, Proc. Amer. Math. Soc. **80** (1980), 678–682.

[35] Jorge Galindo, *Dense pseudocompact subgroups and finer pseudocompact group topologies*, Scientiae Math. Japonicae **55** (2001), 627–640.

[36] Paul Halmos, *Comments on the real line*, Bull. Amer. Math. Soc. **50** (1944), 877–878.

[37] Joan E. Hart and Kenneth Kunen, *Bohr compactifications of non-abelian groups*, Topology Proc. (2002), 593–626.

[38] K. P. Hart and J. van Mill, *A countably compact group H such that $H \times H$ is not countably compact*, Trans. Amer. Math. Soc. **323** (1991), 811–821.

[39] Salvador Hernández and Sergio Macario, *Dual properties in totally bounded Abelian groups*, Archiv der Math. **80** (2002), 271–283.

[40] Salvador Hernández and Manuel Sanchis, *G_δ-open functionally bounded subsets in topological groups*, Topology Appl. **53** (1993), 289–299.

[41] G. Hesse, *Zur Topologisierbarkeit von Gruppen*, Ph.D. thesis, Universität Hannover, Hannover (Germany), 1979.

[42] Edwin Hewitt, *Rings of real-valued continuous functions I*, Trans. Amer. Math. Soc. **64** (1948), 45–99.

[43] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis, volume I*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, volume 115, Springer Verlag, Berlin-Göttingen-Heidelberg, 1963.

[44] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis, volume II*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 152, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[45] Gerald Itzkowitz and Dimitri Shakhmatov, *Dense countably compact subgroups of compact groups*, Mathematica Japonica **45** (1997), 497–501.

[46] P. B. Kozmider, A. H. Tomita, and S. Watson, *Forcing countably compact group topologies on a larger free Abelian group*, Topology Proceedings **25** (2000), 563–574.

[47] W. Moran, *On almost periodic compactifications of locally compact groups*, J. London Math. Soc. (2) **3** (1971), 507–512.

[48] Jan Mycielski, *Some properties of connected compact groups*, Colloq. Math. **5** (1958), 162–166.

[49] A. Yu. Ol'shanskiĭ, *A remark on a countable non-topologizable group*, Vestnik Mosk. Gos. Univ., Ser. I, Mat. Mekh. **3** (1980), 103. [In Russian.]

[50] Dieter Remus, *Zur Struktur des Verbandes der Gruppentopologien*, Ph.D. thesis, Universität Hannover, Hannover (Germany), 1983. [English summary: *Resultate Math.* **6** (1983), 151–152.]

[51] Dieter Remus, *The number of T_2-precompact group topologies on free groups*, Proc. Amer. Math. Soc. **95** (1985), 315–319.

[52] Dieter Remus, *Die Anzahl von T_2-präkompackten Gruppentopologien auf unendlichen abelschen Gruppen*, Rev. Roumaine Math. Pures Appl. **31** (1986), 803–806.

[53] M. G. Tkachenko, *On pseudocompact topological groups*, Interim Report of the Prague Topological Symposium **2**, page 18, Math. Institute Czech. Acad. Sci., Prague, 1987.

[54] M. G. Tkachenko, *Compactness type properties in topological groups*, Czechoslovak Math. J. **38** (1988), 324–341.

[55] M. G. Tkachenko, *Generalization of the Comfort-Ross theorem, I*, Ukrainian Math. J. **41** (1989), 334–338. [Russian original in: Украинскии Математическии Журнал **41** (1989), 377–382.]

[56] M. G. Tkachenko, *Countably compact and pseudocompact topologies on free Abelian groups*, Soviet Math. **34:5** (1990), 79–86. [Russian original in: Известия Высш. Уцкебн. Завед, Мат. 1 **34**: 5 (1990), 68–75.]

[57] Mikhail Tkachenko and Ivan Yaschenko, *Independent group topologies on Abelian groups*, Topology Appl. **122** (2002), 425–451.

[58] Artur Hideyuki Tomita, *On finite powers of countably compact groups*, Comment. Math. Univ. Carolinae. **37** (1996), 617–626.

[59] Artur Hideyuki Tomita, *A group under $MA_{\text{countable}}$ whose square is countably compact but whose cube is not*, Topology and Its Applications **91** (1999), 91–104.

[60] F. Javier Trigos-Arrieta, *Products of locally pseudocompact topological groups*, In: Papers on General Topology and Applications, *Annals of the New York Academy of Sciences* **659**, pp. 182–187, 1992.

[61] V. V. Uspenskiĭ, *Topological groups and Dugundji compacta*, Math. USSR Sbornik **67:2** (1990), 555–580. [Russian original in: Матем. Сборник **180**:8 (1989), 1092–1118.]

[62] V. V. Uspenskiĭ, *On sequentially continuous homomorphisms of topological groups*, 1993. Unpublished manuscript.

[63] N. Th. Varopoulos, *A theorem on the continuity of homomorphisms of locally compact groups*, Proc. Cambridge Phil. Soc. **60** (1964), 449–463.

[64] J. de Vries, *Pseudocompactness and the Stone-Čech compactification for topological groups*, Nieuw Archief voor Wiskunde (3) **23** (1975), 35–48.

[65] B. L. van der Waerden, *Stetigkeitssätze für halbeinfache Liesche Gruppen*, Math. Zeitschrift **36** (1933), 780–786.

[66] André Weil, *Sur les Espaces à Structure Uniforme et sur la Topologie Générale*, Publ. Math. Univ. Strasbourg, vol. 551, Hermann & Cie, Paris, 1938.

[67] Howard J. Wilcox, *Pseudocompact groups*, Pacific J. Math. **19** (1966), 365–379.

[68] Howard J. Wilcox, *Dense subgroups of compact groups*, Proc. Amer. Math. Soc. **28** (1971), 578–580.

Wesleyan University, Middletown, CT, 06459 USA
E-mail address: `wcomfort@wesleyan.edu`