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**EXAMPLES OF DIFFERENTIABLE MAPPINGS
INTO NON-LOCALLY CONVEX SPACES**

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ABSTRACT. Examples of differentiable mappings into real or complex topological vector spaces with specific properties are given, which illustrate the differences between differential calculus in the locally convex and the non-locally convex case.

1. INTRODUCTION

Beyond the familiar theories of differentiation in real or complex locally convex spaces ([6], [7]), a comprehensive theory of $C_{\mathbb{K}}^r$ -maps between open subsets of topological vector spaces over arbitrary non-discrete topological fields \mathbb{K} has recently been developed [1]. Surprisingly large parts of classical differential calculus remain intact for these maps. For example, being $C_{\mathbb{K}}^r$ is a local property, the Chain Rule holds, and $C_{\mathbb{K}}^r$ -maps admit finite order Taylor expansions [1]. Furthermore, when \mathbb{K} is a complete valued field, implicit function theorems for $C_{\mathbb{K}}^r$ -maps from topological \mathbb{K} -vector spaces to Banach spaces are available [3]. All basic constructions of infinite-dimensional Lie theory (linear Lie groups, mapping groups, diffeomorphism groups) work just as well over general topological fields, valued fields, or at least local fields [4].

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In the real locally convex case, the $C_{\mathbb{R}}^r$ -maps are precisely the C^r -maps in the sense of Michal and Bastiani (also known as Keller's C_c^r -maps [6]). The Fundamental Theorem of Calculus holds for such maps; in particular, mappings whose differentials vanish at each point have to be locally constant. A mapping into a complex locally convex space is of class $C_{\mathbb{C}}^{\infty}$ if and only if it is complex analytic in the usual sense (as in [2]). Thus, every $C_{\mathbb{C}}^{\infty}$ -map into a locally convex space is given locally by its Taylor series, and the Identity Theorem holds for such maps. Furthermore, it is known that every $C_{\mathbb{C}}^1$ -map into a complete complex locally convex space is automatically of class $C_{\mathbb{C}}^{\infty}$ (see [1] for all of this).

The purpose of this note is to describe examples showing that these facts become false for mappings with non-locally convex ranges. Thus, for suitable non-locally convex topological vector spaces E , we encounter a smooth injection $\mathbb{R} \rightarrow E$ whose derivative vanishes identically; we present a $C_{\mathbb{C}}^{\infty}$ -map $\mathbb{C} \rightarrow E$ which is not given locally by its Taylor series, around any point; we present a $C_{\mathbb{C}}^1$ -map $f: \mathbb{C} \rightarrow E$ to a metrizable, complete, non-locally convex topological vector space which is not $C_{\mathbb{C}}^2$; and we present a non-zero, compactly supported $C_{\mathbb{C}}^{\infty}$ -map $\mathbb{C} \rightarrow E$, the existence of which demonstrates that the Identity Theorem fails for suitable $C_{\mathbb{C}}^{\infty}$ -maps into non-locally convex spaces.

Mappings between open subsets of the field \mathbb{Q}_p of p -adic numbers with similar pathological properties are known in non-archimedean analysis [9]. The author also drew inspiration from [8, Ex. II.2.7], where it is shown that the map $[0, 1] \rightarrow L^{1/2}[0, 1]$, $t \mapsto \mathbf{1}_{[0,t]}$ is differentiable at each point (in the ordinary sense), with vanishing derivative.

2. THE SETTING OF DIFFERENTIAL CALCULUS

We shall not give the general definition of $C_{\mathbb{K}}^r$ -maps between open subsets of topological \mathbb{K} -vector spaces here. Rather, we shall work with a simpler definition for the special case of curves (cf. also [9]), which is equivalent to the general definition (cf. [1, Prop. 6.9]). All topological vector spaces are assumed Hausdorff.

Definition 2.1. Let \mathbb{K} be a non-discrete topological field and $f : U \rightarrow E$ be a mapping from an open subset U of \mathbb{K} to a topological \mathbb{K} -vector space E . The map f is said to be of class $C_{\mathbb{K}}^0$ if it is continuous; in this case, define $f^{<0>} := f$. We call f of class $C_{\mathbb{K}}^1$ if it is continuous and if there exists a continuous map $f^{<1>} : U \times U \rightarrow E$ such that

$$f^{<1>}(x_1, x_2) = \frac{1}{x_1 - x_2}(f(x_1) - f(x_2))$$

for all $x_1, x_2 \in U$ such that $x_1 \neq x_2$. Recursively, having defined mappings of class $C_{\mathbb{K}}^j$ and associated maps $f^{<j>} : U^{j+1} \rightarrow E$ for $j = 0, \dots, k-1$ for some $k \in \mathbb{N}$, we call f of class $C_{\mathbb{K}}^k$ if it is of class $C_{\mathbb{K}}^{k-1}$ and if there exists a continuous map $f^{<k>} : U^{k+1} \rightarrow E$ such that

$$\begin{aligned} f^{<k>}(x_1, x_2, \dots, x_{k+1}) \\ = \frac{1}{x_1 - x_2} \left(f^{<k-1>}(x_1, x_3, \dots, x_{k+1}) - f^{<k-1>}(x_2, x_3, \dots, x_{k+1}) \right) \end{aligned}$$

for all $x_1, \dots, x_{k+1} \in U^{k+1}$ such that $x_1 \neq x_2$. The map f is of class $C_{\mathbb{K}}^\infty$ (or *smooth*) if it is of class $C_{\mathbb{K}}^k$ for all $k \in \mathbb{N}_0$.

Here $f^{<k>}$ is uniquely determined, and $f^{<k>}$ is symmetric in its $k+1$ variables. Furthermore, $k! f^{<k>}(x, \dots, x) = \frac{d^k f}{dx^k}(x) =: f^{(k)}(x)$, for all $x \in U$ (cf. [1]).

3. THE FIRST EXAMPLE

Let $d\nu(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx$ be the Gauss measure on \mathbb{R} , and $\mu := \nu \otimes \nu$ be the Gauss measure on $\mathbb{C} = \mathbb{R}^2$. Let $E := L^0(\mathbb{C}, \mu)$ be the complex topological vector space of equivalence classes of measurable complex-valued functions on \mathbb{C} (modulo functions vanishing almost everywhere), equipped with the topology of convergence in measure. Thus, a basis of open zero-neighbourhoods of E is given by the sets W_k for $k \in \mathbb{N}$, where W_k consists of those equivalence classes of measurable maps $\gamma : \mathbb{C} \rightarrow \mathbb{C}$ such that $\mu(\{z \in \mathbb{C} : |\gamma(z)| \geq \frac{1}{k}\}) < \frac{1}{k}$. It is well-known that E is a metrizable, complete, non-locally convex topological vector space, which does not admit any non-zero continuous linear functionals (cf. [5]).

Given $\gamma \in L^0(\mathbb{C}, \mu)$ and a closed subset $A \subseteq \mathbb{C}$, we say that γ is *supported in A* if γ vanishes μ -almost everywhere on the complement of A . The *support of γ* is the smallest closed set in which γ is supported. In the following, $\mathbf{1}_A: \mathbb{C} \rightarrow \{0, 1\}$ denotes the characteristic function of a measurable subset $A \subseteq \mathbb{C}$.

Consider the mapping

$$f: \mathbb{C} \rightarrow E, \quad f(z) := \mathbf{1}_{A(z)},$$

where $A(z) := \{w \in \mathbb{C} : \operatorname{Re}(w) \leq \operatorname{Re}(z) \text{ and } \operatorname{Im}(w) \leq \operatorname{Im}(z)\}$. Then f has the following properties:

Proposition 3.1. *The map $f: \mathbb{C} \rightarrow E$ is of class $C_{\mathbb{C}}^{\infty}$, injective, and $f^{(j)}(z) = 0$ for all $j \in \mathbb{N}$ and $z \in \mathbb{C}$. In particular, f is not given locally by its Taylor series around any point.*

Proof. Apparently f is injective. If we can show that f is $C_{\mathbb{C}}^{\infty}$, with $f^{(j)} = 0$ for all $j \in \mathbb{N}_0$, then, given any $z_0 \in \mathbb{C}$, the Taylor series of f at z_0 will only consist of the 0th order term, and hence describes the function which is constantly $f(z_0)$. It therefore does not coincide with the injective map f on any neighbourhood of z_0 .

Thus, to complete the proof of the proposition, it suffices to establish the following assertions, by induction on $k \in \mathbb{N}_0$:

- (a) f is of class $C_{\mathbb{C}}^k$;
- (b) For all $j \in \mathbb{N}$ such that $j \leq k$, we have $f^{(j)} = 0$, and, for all $z_1, \dots, z_{j+1} \in \mathbb{C}$, the element $f^{<j>}(z_1, \dots, z_{j+1}) \in E = L^0(\mathbb{C}, \mu)$ is supported in $([x_*, x^*] + i\mathbb{R}) \cup (\mathbb{R} + i[y_*, y^*])$, where

$$(3.1) \quad \begin{aligned} x_* &:= \min\{\operatorname{Re}(z_1), \dots, \operatorname{Re}(z_{j+1})\} \\ x^* &:= \max\{\operatorname{Re}(z_1), \dots, \operatorname{Re}(z_{j+1})\} \\ y_* &:= \min\{\operatorname{Im}(z_1), \dots, \operatorname{Im}(z_{j+1})\} \\ y^* &:= \max\{\operatorname{Im}(z_1), \dots, \operatorname{Im}(z_{j+1})\}. \end{aligned}$$

The case $k = 0$. Given $z_1 \in \mathbb{C}$, let us show that f is continuous at z_1 . To this end, let $z_2 \in \mathbb{C}$; define x_*, x^*, y_*, y^* as in (3.1) (taking $j := 1$). Then the symmetric difference $A(z_1) \oplus A(z_2) := (A(z_1) \setminus A(z_2)) \cup (A(z_2) \setminus A(z_1))$ of the sets $A(z_1)$ and $A(z_2)$ is a subset of $([x_*, x^*] + i\mathbb{R}) \cup (\mathbb{R} + i[y_*, y^*])$, whence

$$\mu(A(z_1) \oplus A(z_2)) \leq |x^* - x_*| + |y^* - y_*| \leq 2|z_2 - z_1|$$

(using Fubini's Theorem). Note that $f(z_2) - f(z_1)$ is supported in $A(z_1) \oplus A(z_2)$. As the measure of this set tends to 0 as $z_2 \rightarrow z_1$, we see that $f(z_2) \rightarrow f(z_1)$ in E . Thus f is continuous.

Induction step. Let $k \in \mathbb{N}$, and suppose that (a) and (b) hold when k is replaced with $k - 1$. Then f is of class $C_{\mathbb{C}}^{k-1}$. In order that f be $C_{\mathbb{C}}^k$, with $f^{(k)} = 0$, in view of Lemma 10.5, Lemma 10.7 and Proposition 6.2 in [1], we only need to show that

$$g_n := \frac{1}{z_{n,1} - z_{n,2}} \left(f^{<k-1>}(z_{n,1}, z_{n,3}, \dots, z_{n,k+1}) - f^{<k-1>}(z_{n,2}, z_{n,3}, \dots, z_{n,k+1}) \right) \rightarrow 0$$

in E as $n \rightarrow \infty$, for every sequence $(z_n)_{n \in \mathbb{N}}$ of elements $z_n = (z_{n,1}, \dots, z_{n,k+1}) \in \mathbb{C}^{k+1}$ which converges to a diagonal element $(z, z, \dots, z) \in \mathbb{C}^{k+1}$ for some $z \in \mathbb{C}$, where $z_{n,a} \neq z_{n,b}$ whenever $a \neq b$. Given $n \in \mathbb{N}$, define $x_{n,*}, x_n^*, y_{n,*}$ and y_n^* along the lines of (3.1), using the elements $z_{n,1}, \dots, z_{n,k+1}$ (thus $j = k$). Note that, as a consequence of (b) for k replaced with $k - 1$ (valid by the induction hypothesis), each of the elements $f^{<k-1>}(z_{n,2}, z_{n,3}, \dots, z_{n,k+1})$ and $f^{<k-1>}(z_{n,1}, z_{n,3}, \dots, z_{n,k+1}) \in E$ is supported in

$$B_n := ([x_{n,*}, x_n^*] + i\mathbb{R}) \cup (\mathbb{R} + i[y_{n,*}, y_n^*]).$$

Hence also g_n is supported in B_n . Since

$$\mu(B_n) \leq 4 \max\{|z_{n,1} - z|, \dots, |z_{n,k+1} - z|\} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

we deduce that $\lim_{n \rightarrow \infty} g_n = 0$ in E , as required. Thus f is $C_{\mathbb{C}}^k$, and $k! f^{<k>}(z, \dots, z) = f^{(k)}(z) = 0$ for all $z \in \mathbb{C}$.

It only remains to prove the assertion concerning the supports. To this end, let $z_1, \dots, z_{k+1} \in \mathbb{C}$. If all of z_1, \dots, z_{k+1} coincide, then $f^{<k>}(z_1, \dots, z_{k+1}) = 0$ by the preceding, and this is an element with empty support, which therefore is contained in the desired set. Now suppose that $z_a \neq z_b$ for some a, b . By symmetry of $f^{<k>}$ in its $k + 1$ variables, we may assume that $z_1 \neq z_2$. Then

$$\begin{aligned} & f^{<k>}(z_1, \dots, z_{k+1}) \\ &= \frac{1}{z_1 - z_2} \left(f^{<k-1>}(z_1, z_3, \dots, z_{k+1}) - f^{<k-1>}(z_2, z_3, \dots, z_{k+1}) \right); \end{aligned}$$

as in the preceding part of the proof, we deduce from the induction hypothesis that this element is supported in the desired set. \square

Corollary 3.2. *Consider E as a real topological vector space. Then $g := f|_{\mathbb{R}} : \mathbb{R} \rightarrow E$ is an injective $C_{\mathbb{R}}^{\infty}$ -curve whose derivative g' vanishes identically.*

4. THE SECOND EXAMPLE

We retain μ and E as in Example 1, but consider now

$$f: \mathbb{C} \rightarrow E, \quad f(z) := \mathbf{1}_{A(z)},$$

where $A(z) := \{w \in \mathbb{C} : |z| \leq |w| \leq 1\}$. Then f has the following properties:

Proposition 4.1. *The map $f: \mathbb{C} \rightarrow E$ is of class $C_{\mathbb{C}}^{\infty}$, non-zero, and f has compact support.*

Proof. Clearly $f(z) = 0$ for all $z \in \mathbb{C}$ such that $|z| \geq 1$, entailing that f is compactly supported. Furthermore, $f \neq 0$. Given real numbers $0 \leq r \leq R$, let

$$K(r, R) := \{w \in \mathbb{C} : r \leq |w| \leq R\}$$

be the closed annulus with inner radius r and outer radius R in \mathbb{C} . Then

$$(4.1) \quad \mu(K(r, R)) \leq R - r.$$

Indeed, we have

$$\begin{aligned} \mu(K(r, R)) &= \int_r^R \int_0^{2\pi} \frac{s}{\pi} e^{-s^2} d\phi ds = e^{-r^2} - e^{-R^2} \\ &= (r - R) \cdot (-2\xi e^{-\xi^2}) = (R - r) 2\xi e^{-\xi^2} \leq R - r \end{aligned}$$

for some $\xi \in [r, R]$, using the Mean Value Theorem to pass to the second line, and using that $2te^{-t^2} \leq \sqrt{\frac{2}{e}} < 1$ for all $t \in [0, \infty[$, by an elementary calculation.

The assertion of the proposition will follow if we can prove the following claims, by induction on $k \in \mathbb{N}_0$:

- (a) f is of class $C_{\mathbb{C}}^k$;

- (b) For all $j \in \mathbb{N}$ such that $j \leq k$, the map $f^{(j)}$ vanishes, and $f^{<j>}(z_1, \dots, z_{j+1}) \in E$ is supported in the annulus $K(r_*, r^*)$, where $r_* := \min\{|z_1|, \dots, |z_{j+1}|\}$, $r^* := \max\{|z_1|, \dots, |z_{j+1}|\}$ for $z_1, \dots, z_{j+1} \in \mathbb{C}$.

In view of (4.1), an apparent adaptation of the proof of Prop. 3.1 establishes these claims. \square

Remark 4.2. Proposition 4.1 entails that the identity theorem for analytic mappings becomes invalid when analytic maps are replaced with $C_{\mathbb{C}}^{\infty}$ -maps into non-locally convex spaces.

5. THE THIRD EXAMPLE

We retain ν and μ as in Example 1 but consider the complex topological vector space $E := L^p(\mathbb{C}, \mu)$ of equivalence classes of complex-valued L^p -functions on \mathbb{C} now, where $p \in]\frac{1}{2}, 1[$. Consider

$$f: \mathbb{C} \rightarrow E, \quad f(z) := \mathbf{1}_{A(z)},$$

where $A(z) := \{w \in \mathbb{C} : \operatorname{Re}(w) \leq \operatorname{Re}(z)\}$. The map f has the following properties:

Proposition 5.1. *$f: \mathbb{C} \rightarrow E$ is of class $C_{\mathbb{C}}^1$ and f' vanishes, but f is not of class $C_{\mathbb{C}}^2$.*

Proof. Given real numbers $a \leq b$, we define $S(a, b) := \{w \in \mathbb{C} : a < \operatorname{Re}(w) \leq b\}$. Then

$$(5.1) \quad \mu(S(a, b)) = \nu(]a, b]) \leq b - a.$$

Given $z_1, z_2 \in \mathbb{C}$, where $\operatorname{Re}(z_1) \leq \operatorname{Re}(z_2)$ without loss of generality, we have

$$f(z_2) - f(z_1) = \mathbf{1}_{S(\operatorname{Re}(z_1), \operatorname{Re}(z_2))},$$

where $\mu(S(\operatorname{Re}(z_1), \operatorname{Re}(z_2))) \leq \operatorname{Re}(z_2) - \operatorname{Re}(z_1) \leq |z_2 - z_1|$. We easily deduce that f is continuous. Assuming that $z_1 \neq z_2$ here, we have

$$\begin{aligned} & \int_{\mathbb{C}} \left| \frac{f(z_2)(w) - f(z_1)(w)}{z_2 - z_1} \right|^p d\mu(w) \\ &= |z_2 - z_1|^{-p} \cdot \mu(S(\operatorname{Re}(z_1), \operatorname{Re}(z_2))) \leq |z_2 - z_1|^{1-p} \rightarrow 0 \end{aligned}$$

as $|z_2 - z_1| \rightarrow 0$, showing that $\frac{1}{z_2 - z_1}(f(z_2) - f(z_1)) \rightarrow 0$ in E whenever $|z_2 - z_1| \rightarrow 0$. Thus $f^{<1>}(z_1, z_2) := 0$ if $z_1 = z_2$, $f^{<1>}(z_1, z_2) := \frac{1}{z_2 - z_1}(f(z_2) - f(z_1))$ if $z_1 \neq z_2$ defines a continuous function $f^{<1>}: \mathbb{C} \rightarrow E$, showing that f is $C_{\mathbb{C}}^1$ with $f'(z) = f^{<1>}(z, z) = 0$ for all $z \in \mathbb{C}$.

Let $c := \frac{1}{e\sqrt{\pi}}$; then $\frac{1}{\sqrt{\pi}}e^{-t^2} \geq c$ for $t \in [0, 1]$. For $t \in [0, \frac{1}{2}]$, we have $\frac{1}{t} (f^{<1>}(t, 2t) - f^{<1>}(0, 2t)) = \frac{1}{t^2} (\mathbf{1}_{S(t, 2t)} - \frac{1}{2}\mathbf{1}_{S(0, 2t)})$, whence

$$\begin{aligned} & \int_{\mathbb{C}} \left| \frac{1}{t} (f^{<1>}(t, 2t) - f^{<1>}(0, 2t)) (w) \right|^p d\mu(w) \\ &= \left(\frac{1}{2t^2} \right)^p \cdot \mu(S(0, 2t)) = \left(\frac{1}{2t^2} \right)^p \cdot \nu([0, 2t]) \\ &\geq \left(\frac{1}{2t^2} \right)^p 2t c = 2^{1-p} t^{1-2p} c \rightarrow \infty \end{aligned}$$

as $t \rightarrow 0$. The map $E \rightarrow [0, \infty[$, $\gamma \mapsto \int_{\mathbb{C}} |\gamma(w)|^p d\mu(w)$ being continuous, we deduce that $\lim_{t \rightarrow 0} \frac{1}{t} (f^{<1>}(t, 2t) - f^{<1>}(0, 2t))$ cannot exist in E . Thus f is not $C_{\mathbb{C}}^2$. \square

Corollary 5.2. *Consider E as a real topological vector space. Then $g := f|_{\mathbb{R}} : \mathbb{R} \rightarrow E$ is a $C_{\mathbb{R}}^1$ -curve whose derivative g' vanishes identically. However, g is not $C_{\mathbb{R}}^2$.*

REFERENCES

1. W. Bertram, H. Glöckner and K.-H. Neeb, *Differential calculus over general base fields and rings*, to appear in *Expo. Math.* **22** (2004), 213-282.
2. J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*, *Studia Math.* **39** (1971), 77-112.
3. H. Glöckner, *Implicit functions from topological vector spaces to Banach spaces*, TU Darmstadt Preprint **2271**, March 2003; also math.GM/0303320
4. —, *Lie groups over non-discrete topological fields*, TU Darmstadt Preprint **2356**, July 2004; also math.GR/0408008.
5. N. J. Kalton, N. T. Peck and J. W. Roberts, "An F-space Sampler," *London Math. Soc. Lect. Note Ser.* **89**, Cambridge University Press, 1984.
6. H. H. Keller, "Differential Calculus in Locally Convex Spaces," Springer-Verlag, 1974.
7. A. Kriegl and P. W. Michor, "The Convenient Setting of Global Analysis," AMS, Providence, 1997.
8. K.-H. Neeb, "Nancy Lectures on Infinite-Dimensional Lie Groups," TU Darmstadt Preprint **2203**, March 2002.
9. W. H. Schikhof, "Ultrametric Calculus," Cambridge University Press, 1984.

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