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CERTAIN ANALYTIC PREIMAGES OF PSEUDOCIRCLES ARE PSEUDOCIRCLES

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ABSTRACT. In the plane of complex numbers, the inverse image of a pseudocircle X under the map $z \to z^n$ is again a pseudocircle provided zero is in the bounded complementary domain of X.

1. INTRODUCTION

The question of what properties of plane continua are preserved by lifting to preimages under analytic functions has interested me for many years. Since Vladimir N. Akis [1] has shown that an analytic function which keeps invariant a non-separating plane continuum has a fixed point in that continuum, the relationship between the properties of plane continua and properties of their images or preimages under analytic functions has assumed new importance.

Herein a pseudocircle is a planar hereditarily indecomposable circle-like continuum which is not chainable. The first such continuum was constructed by R. H. Bing in [2]. In [3], Lawrence Fearnley proved that any two pseudocircles are homeomorphic. For further background on pseudocircles, the reader may wish to consult [8], [4], [5], [6], and the papers cited in these.

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2. Notation and results

Lemma 1. *Let M, O be subsets of $\mathbf{R}^n$ with M compact, O open in $\mathbf{R}^n$, and $M \subseteq O$. Then there exists $\epsilon > 0$ such that whenever $L \subseteq M$ and the diameter of L is less than ϵ, it follows that the convex hull of L is a subset of O.*

Proof: The set M admits a finite open cover by sets of the form $U \cap M$ where $U \subseteq O$ and U is a convex open set. Let ϵ be the Lebesque number of this covering. □

In the results and proofs below, the following notation and terminology will be used. The field of complex numbers, with its usual topology, is denoted $\mathbf{C}$; n is a fixed integer greater than 1; $f : \mathbf{C} \to \mathbf{C}$ is the map $f(z) = z^n$; $X \subseteq \mathbf{C}$ is a continuum which separates $\mathbf{C}$ and has 0 in a bounded complementary domain B; U is the unbounded complementary domain of X; D_1 and D_2 are circular disks centered at 0 with $D_2 \subseteq B$ and $X \subseteq$ Int D_1; A is the annulus $D_1 \backslash \text{Int } D_2$ (thus, $X \subseteq \text{Int } A$). A half-plane of $\mathbf{C}$, the boundary line of which contains 0, is called a *standard* half-plane. The closed convex hull of any set $J \subseteq \mathbf{C}$ will be denoted $H(J)$. Finally, if $H(J) \subseteq A$, a *piece* of $f^{-1}(J)$ is a set of the form $f^{-1}(J) \cap K$, where K is a component of $f^{-1}(H(J))$.

Lemma 2. *Suppose J is a set and $H(J) \subseteq A$. Then J lies in the intersection of A with some standard half-plane.*

Proof: Let S be a circle centered at zero with $S \subseteq \text{Int} D_2$. Let x denote the point of $H(J)$ nearest to 0. By rotating if necessary, assume x is on the positive real axis. Let T_1 and T_2 be the tangent lines to S through x, labeled so that T_1 has negative slope and T_2 has positive slope. Each of $T_1 \bigcap D_1$ and $T_2 \bigcap D_1$ is a chord of the circle $Bd(D_1)$, tangent to S. Note that looking outward from 0, every line T tangent to S has a left and a right half-line from the point of tangency with S; furthermore, every such line is uniquely determined by its point of tangency with S. Let $T(z)$ be the line tangent to S at z. Define two sets $L, R \subseteq S$ by:

$$L = \{z \in S | T(z) \text{ intersects } H(J) \text{ on its right side}\}$$

$$R = \{z \in S | T(z) \text{ intersects } H(J) \text{ on its right side}\}$$

Let t_1, t_2 be the points of tangency of T_1, T_2, respectively, with S. Then $t_1 \in R$ and $t_2 \in L$. Since $H(J)$ is compact it follows that R and L are both closed in S, and $R \cap L = \phi$, since if for some $z \in S$, the tangent line $T(z)$ to S at z meets $H(J)$ at a point on each side, then the line segment joining them contains z, and so $z \in H(J)$, which is impossible since $H(J) \subseteq A$ and $z \notin A$.

Now, let M be the minor arc of S from t_1 to t_2. $M \not\subseteq R \cup L$ since M is connected. Let $m \in M \backslash (R \cup L)$. Then $T(m)$ does not intersect $H(J)$. Since x is on the opposite side of $T(m)$ from t_1 and t_2 and hence on the opposite side from 0, and $H(J)$ is connected, it follows that $H(J)$ is a subset of the standard half-plane of C containing x and with boundary line parallel to $T(m)$. Since $H(J) \subseteq A$ by hypothesis, the proof is done. □

Lemma 3. *The set $f^{-1}(X)$ is a continuum with 0 in its bounded complementary domain, just as X is.*

Proof: Let $\hat{X}$ denote the union of X with all its complementary domains, if any, except for B and U. That is, $\hat{X} = \mathbf{C} \backslash (B \cup U)$. □

Then $f^{-1}(B) \cup f^{-1}(\hat{X}) \cup f^{-1}(U) = \mathbf{C}$. Also, $f^{-1}(B)$ and $f^{-1}(U)$ are open and disjoint from each other and from $f^{-1}(\hat{X})$. Thus, $f^{-1}(\hat{X})$ separates $\mathbf{C}$; and 0 is in the bounded complementary domain, $f^{-1}(B)$, of $f^{-1}(\hat{X})$. However, $f^{-1}(B)$ and $f^{-1}(U)$ are complementary domains of $f^{-1}(X)$ also. To see this suppose J is an arc from a point in $f^{-1}(B)$ to a point not in $f^{-1}(B)$. Then $J \cap f^{-1}(B)$ is open in J and $J \cap M$ is also open in J, where $M = \bigcup\{f^{-1}(W) | W$ is a complementary domain of X in $\mathbf{C}$ and $W \neq B\}$.

Since $J \not\subseteq f^{-1}(B)$, if $J \cap f^{-1}(X) = \phi$, it follows that $(J \cap f^{-1}(B)) \cup (J \cap M)$ is a separation of J, which is impossible.

Thus, $f^{-1}(X)$ separates $\mathbf{C}$ between 0 and ∞, and so some component K of $f^{-1}(X)$ also does. Suppose $K \neq f^{-1}(X)$. Then $f^{-1}(X)$ has another component $K_1 \neq K$, and $f(K_1) = f(K) = X$, since $f|f^{-1}(D_1) : f^{-1}(D_1) \to D_1$ is open and hence confluent.

Let L denote the nonnegative real axis in $\mathbf{C}$, and let $\{L_j\}_{j=1}^{n}$ denote the collection of rays from 0 such that $f^{-1}(L) = \bigcup_{j=1}^{n} L_j$. Let $a, b \in L \cap X$ be the points such that a is the point of $L \cap X$ nearest to 0, while b is the point of $L \cap X$ furthest from 0. Let

$\tilde{L} \subseteq L$ be the line segment from 0 to a, while $\hat{L} \subseteq L$ is the ray with endpoint b. For each j let $\tilde{L}_j = f^{-1}(\tilde{L}) \cap L_j$ and $\hat{L}_j = f^{-1}(\hat{L}) \cap L_j$. Then, $f^{-1}(a) \subseteq \bigcup_{j=1}^{n} \tilde{L}_j$ and $f^{-1}(b) \subseteq \bigcup_{j=1}^{n} \hat{L}_j$. Therefore, for some $j, k, K_1 \cap \tilde{L}_j \neq \phi$ and $K_1 \cap \hat{L}_k \neq \phi$. It follows that $\tilde{L}_j \cup K_1 \cup \hat{L}_k$ is an unbounded connected set containing 0 and missing K, a contradiction. Therefore, $f^{-1}(X)$ is connected.

Lemma 4. *If X is circularly chainable, so is $f^{-1}(X)$.*

Proof: Let $\epsilon > 0$ be arbitrary. A circular chain cover of $f^{-1}(X)$ consisting of open sets of diameter less than ϵ can be constructed as follows: Note that for any standard half-plane H, each branch of f^{-1} is uniformly continuous on $H \cap A$. Using this fact and Lemma 1, there exists $\delta > 0$ such that both the following hold.

i) Whenever $x, y \in f^{-1}(A)$ and the angular separation from 0 between x and y is less than $\frac{\pi}{n}$ and $d(f(x), f(y)) < \delta$, then $d(x, y) < \epsilon$.

ii) Whenever $M \subseteq X$ and $\operatorname{dia} M < 3\delta$, it follows that $H(M) \subseteq \operatorname{Int} A$.

Now, let $\{U_i\}_{i=1}^{p}$ be a circular chain of open sets, each of diameter less than δ, covering X. Addition of subscripts of these U_i*'s* will always be taken modulo p. Note that by ii) and Lemma 2, the union of any three consecutive links of $\{U_i\}_{i=1}^{p}$ is contained in some standard half plane. Define an open cover $\mathcal{V}$ of $f^{-1}(X)$ by $\mathcal{V} = \{V|$ for some j, V is a piece of $f^{-1}(U_j)\}$.

By i), it follows that each $V \in \mathcal{V}$ has diameter less than ϵ. It remains to be shown that $\mathcal{V}$ is a circular chain.

Since for every $j, H(U_{j-1} \cup U_j \cup U_{j+1})$ is evenly covered by $f|A$, each $V \in \mathcal{V}$ meets exactly two other members of $\mathcal{V}$. Hence, the nerve of $\mathcal{V}$ is a 2-regular graph. Such a finite graph is either a simple cycle or a union of disjoint cycles. Since X is connected, the nerve is a single cycle, so that $\mathcal{V}$ is a circular chain. □

Corollary to Proof. *If X is circularly chainable and $\epsilon > 0$, then there exists $\delta > 0$ such that whenever $\{U_j\}_{j=1}^{p}$ is a circular chain of p links consisting of open subsets of X, each of diameter less than δ, then the collection $\{V|$ for some j, V is a piece of $f^{-1}(U_j)\}$ is a circular chain of np links each of which is an open subset of $f^{-1}(x)$ of diameter less than ϵ. Furthermore, this circular chain* *can be indexed as* $\{V_k\}_{k=1}^{np}$ *in such a way that for every nonnegative integer* $m < n$, V_{mp+j} *is a piece of* $f^{-1}(U_j)$ *(and, of course, such that for each* $k, V_k \cap V_{k+1} \neq \phi$*, where* $k+1$ *is taken modulo np.)*

Proof: This follows from examination of the structure of the circular chain $\mathcal{V}$ constructed in the proof of Lemma 4. □

Lemma 5. *If* X *is circularly chainable and hereditarily indecomposable, then for each proper subcontinuum* $L \subseteq f^{-1}(X), f|L : L \to f(L)$ *is a homeomorphism.*

Proof: Suppose X is circularly chainable and hereditarily indecomposable, and suppose $L \subseteq f^{-1}(X)$ is a proper subcontinuum. Then L is chainable.

By Theorem 14 of [9, p. 234], L cannot be continuously mapped onto X. Thus, $f(L)$ is a proper subcontinuum of X. Since X is hereditarily indecomposable, $f(L)$ is nowhere dense in X. Since $f|f^{-1}(X) : f^{-1}(X) \to X$ is an open mapping, $f^{-1}(f(L))$ is nowhere dense in $f^{-1}(X)$.

Let $\epsilon > 0$ be small enough that any circular chain of open sets, each of diameter less than ϵ, covering $f^{-1}(X)$ has at least one link missing $f^{-1}f(L))$. Let $\delta > 0$ be the number guaranteed by the Corollary to the proof of Lemma 4 corresponding to ϵ, and let $\{U_j\}_{j=1}^{p}$ be a circular chain of mesh less than δ covering X. Let $\{V_k\}_{k=1}^{np}$ be the circular chain cover of $f^{-1}(X)$ obtained from $\{U_j\}_{j=1}^{p}$ as in this same Corollary. Then, for some k, $V_k \cap L = \phi$. By renumbering $\{V_k\}_{k=1}^{np}$ and $\{U_j\}_{j=1}^{p}$, it can be guaranteed that $V_p \cap f^{-1}(f(L)) = \phi$. Thus, $U_p \cap f(L) = \phi$. It follows that for $1 \leq m \leq n, V_{mp} \cap f^{-1}(f(L)) = \phi$, so that the continuum L is contained in the union of some subchain $\{V_{mp+j}\}_{j=1}^{p-1}$ of $\{V_j\}_{j=1}^{np}$. Again with no loss of generality, assume $L \subseteq \bigcup_{j=1}^{p-1} V_j$.

Let $r : f^{-1}(A) \to f^{-1}(A)$ denote the rigid rotation through an angle of $\frac{2\pi}{n}$. Then r is a generator of the group of covering transformations for $f|f^{-1}(A)$, so $f \circ r^m = f$ for each integer m. Since r permutes the set $\{V_{mp}\}_{m=0}^{n-1}$, it also permutes the subchains into which the removal of the V_{mp}'s partition $\{V_j\}_{j=1}^{np}$. Hence, each such subchain $\{V_{mp+j}\}_{j=1}^{p-1}$ contains a component of $f^{-1}f(L)$; call this component L_m. Then for $x \in L$ and $0 \leq m < n$, $r^m(x) \in L_m$.

Since there are n such points $r^m(x)$ and each L_m contains one, there can be no point except x in $f^{-1}(f(x)) \cap L$. Thus, $f|L$ is one to one. $\square$

3. Main result

Theorem. *Suppose X is a pseudocircle in $\mathbf{C}$ with the point zero in its bounded complementary domain. Define $f : \mathbf{C} \to \mathbf{C}$ by $f(z) = z^n$, where n is any nonzero integer. Then $f^{-1}(X)$ is also a pseudocircle.*

Proof: Let X be a pseudocircle in the complex plane $\mathbf{C}$ with 0 in its bounded complementary domain. Let $f : z \to z^n$, as above. By Lemma 5, every proper subcontinuum of $f^{-1}(X)$ is homeomorphic to a proper subcontinuum of X. Since every proper subcontinuum of X is a pseudoarc, so is every proper subcontinuum of $f^{-1}(X)$. Thus, $f^{-1}(X)$ is hereditarily indecomposable. Since by Lemma 4, $f^{-1}(X)$ is circularly chainable, it follows from the principal result in [3] that $f^{-1}(X)$ is homeomorphic with X. The Theorem follows for $n < 0$ since the map $g : z \to 1/z$ is a homeomorphism on $\mathbf{C}\backslash\{0\}$. $\square$

The notation in these questions is the same as used above.

Question 1. Suppose X is any hereditarily indecomposable continuum in $\mathbf{C}$ which is irreducible with respect to the property of separating 0 from ∞. Is $f^{-1}(X)$ always a hereditarily indecomposable continuum?

Question 2. Suppose X is a hereditarily indecomposable continuum in $\mathbf{C}$ and that the pseudocircle X' is a proper subcontinuum of X. Is it possible that $f^{-1}(X)$ can also be hereditarily indecomposable? (It is possible for $f^{-1}(X)$ to be decomposable.)

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