Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 27-48

BEST APPROXIMATION AND WRAPPINGS

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Abstract. In this paper, we introduce the concept of pre-wrapping that allows us to give a general and unified description of best approximation tools. With topological arguments we give useful characterizations of the proximinality of a set and we study properties that cover and improve classical results concerning the semicontinuity of the multivalued function of best approximation.

1. Introduction

In this paper, X will always be a nonempty set and we will denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X. First we introduce the notion of pre-wrapping in order to estimate the gap between points in X, in a similar way that metric balls can be used to estimate the distance between points in a metric space. From now on, the inclusion symbol $\subset$ must be understood in the general sense, that is, proper or not.

Definition 1.1. A *pre-wrapping* for X is a function $\xi : X \times [0,\infty) \to \mathcal{P}(X)$ such that for all $x \in X$ and $r, s \geq 0$

$$\xi(x,0) = \{x\}\,,\ \xi(x,r) \subset \xi(x,r+s) \text{ and } \bigcup_{t>r} \xi(x,t) = X.$$

2000 *Mathematics Subject Classification.* Primary 41A65; Secondary 54D30, 54H99.

Key words and phrases. best approximation, compactness, proximinality, semicontinuity, wrappings.

[1]Partially supported by Junta de Andalucía, Research Group FQM 0178.

[2]Partially supported by Junta de Andalucía, Research Group FQM 0268.

Moreover, if $A \in \mathcal{P}(X)$ and ξ is a pre-wrapping for X, we denote

$$F_\xi(x,A) = \{r \geq 0 : A \cap \xi(x,r) \neq \emptyset\}\,,\ d_\xi(x,A) = \inf F_\xi(x,A),$$

and

$$P_\xi(x,A) = A \bigcap \Big(\bigcap_{r \in F_\xi(x,A)} \xi(x,r) \Big).$$

The subset A is called the approximant class and the non-negative number $d_\xi(x,A)$ is called the ξ distance of x to A. If $P_\xi(x,A) \neq \emptyset$, any point $y \in P_\xi(x,A)$ is called a *best approximant* for x *by elements of* A. If $P_\xi(x,A) \neq \emptyset$ for all $x \in X$, then A is said to be *proximinal* . ($P_\xi(x,A)$ is also called a *projection* of x.) The properties associated with ξ will be called geometric properties.

Recall that for a topological space (X,τ), a set $A \in \mathcal{P}(X)$ is a *neighborhood* of a point $x \in X$, if there exists an open set U such that $x \in U \subset A$. If (X,τ) is a topological space, a *wrapping* for X is a pre-wrapping ξ such that, for all $x \in X$ and $r > 0$, $\xi(x,r)$ is a closed neighborhood of X. This notion was introduced by Samuel G. Moreno, et al. in [4] to study problems related with best approximation. Actually, their definition is slightly different from the one presented here.

In this paper, we associate to any pre-wrapping ξ for X a topology τ_ξ on X. Our main purpose is to show that this topology provides a natural frame to study problems related with best approximation. This will be accomplished as follows. In the second section, we characterize, by means of geometrical properties, those pre-wrappings which are wrappings for (X,τ_ξ). As a by-product, we obtain that, in such case, (X,τ_ξ) satisfies the first axiom of countability, that is, (X,τ_ξ) is a first countable space. In the third section we present a characterization of the proximinal sets $A \in \mathcal{P}(X)$ by means of the ξ covering property, and show how some known and new results concerning proximinality can be derived from this characterization. The ξ covering property seems to be very technical but, as can be seen in the fourth section, it may be useful in studying other problems (for instance, to prove that P_ξ is an upper semicontinuous function). The last section is devoted to showing the connections between the properties of the pre-wrappings introduced in the paper.

Usually, the best approximation theory is developed in the context of normed or topological vector spaces (see [2] and [6]). As was shown in [4], some of the problems of these theories can be formulated by means of wrappings. There are several reasons to consider wrappings. For instance, if ξ is a wrapping for (X, τ), then $\tau_\xi \subset \tau$, where τ_ξ is a topology that ξ induces on X (see Definition 2.1). This allows us to work with a weaker topology. We remark that, in some cases, $\tau = \tau_\xi$, but in all the situations considered here, τ_ξ is a first countable topology while, in general, τ does not have this property. On the other hand, it follows from the present approach that some important results for best approximation theory depend deeply on the geometrical properties of the involved wrapping.

2. Pre-wrappings and Topologies

In this section we associate a topology to a pre-wrapping and study how some geometrical properties of the pre-wrapping are related to topological ones.

Definition 2.1. Let ξ be a pre-wrapping for X and let τ_ξ be the collection formed by the empty set and all sets $U \in \mathcal{P}(X)$ for which, for every $x \in U$, there exists $r = r(x) > 0$ such that $\xi(x, r) \subset U$. The collection τ_ξ will be called *the topology induced by ξ on X.*

It is easy to verify that τ_ξ is a topology on X. In what follows, if there is no confusion, all topological notions are taken in the τ_ξ-topology. One of the reasons to consider the topology τ_ξ is explained by the following result.

Proposition 2.2. *Let ξ be a pre-wrapping for X. If τ is a topology on X such that ξ is a wrapping with respect to τ, then $\tau_\xi \subset \tau$.*

Proof: Assume that ξ is a wrapping for X with respect to τ. It is sufficient to prove that each τ_ξ-open set is a τ-neighborhood of each of its points. Let $U \in \mathcal{P}(X)$ be a τ_ξ-open set. For each $x \in U$, there is $r = r(x) > 0$ such that $\xi(x, r) \subset U$. Taking into account that $\xi(x, r)$ is a τ-neighborhood of x and that $U = \cup_{x \in U} \xi(x, r)$, it follows that U is τ-open. □

Note that, in studying wrappings, the sets $\xi(x, r)$ $(r > 0)$ should be closed neighborhoods of x. Let us look for some properties on ξ which guarantee the closedness of all these sets.

For $x \in X$ and $r > 0$ denote

$$V_\xi(x,r) = \{y \in X : \text{ there exists } s > 0 \text{ such that } \xi(y,s) \subset \xi(x,r)\}.$$

Definition 2.3. If for all $x \in X$, $r > 0$, and $z \in X \setminus \xi(x,r)$, there exists $s > 0$ such that $\xi(z,s) \cap \xi(x,r) = \emptyset$, we say that ξ has the *outer property.* If for all $x \in X$, $r > 0$, and $y \in V_\xi(x,r)$, there exists $t > 0$, for which $\xi(y,t) \subset V_\xi(x,r)$, we say that ξ has the *inner property*. By an IO pre-wrapping, we mean a pre-wrapping with the inner and the outer property.

Proposition 2.4. *Let ξ be a pre-wrapping for X. The following assertions are equivalent:*

(i) *The function ξ is a wrapping for (X, τ_ξ).*
(ii) *The function ξ is an IO pre-wrapping.*

Proof: First, suppose that ξ is a wrapping for (X, τ_ξ). That is, for all $x \in X$ and $r > 0$, $\xi(x,r)$ is a closed neighborhood of x. For $x \in X$ and $r > 0$, the set $X \setminus \xi(x,r)$ is τ_ξ-open. By construction of the topology τ_ξ, for any $z \in X \setminus \xi(x,r)$, there exists $s > 0$ such that $\xi(z,s) \cap \xi(x,r) = \emptyset$. This proves that ξ has the outer property. On the other hand, if $y \in V_\xi(x,r)$, there is $s > 0$ such that $\xi(y,s) \subset \xi(x,r)$. Since $\xi(y,s)$ is a τ_ξ-neighborhood of y, there exists a τ_ξ-open set U containing y. By definition of the τ_ξ topology there exists $t > 0$ such that $\xi(y,t) \subset U$. In order to prove that ξ has the inner property it is sufficient to show that $\xi(y,t) \subset V_\xi(x,r)$. If $z \in \xi(y,t)$, there exists $s(z) > 0$ such that

$$\xi(z, s(z)) \subset U \subset \xi(y,s) \subset \xi(x,r).$$

Hence, $z \in V_\xi(x,r)$ and so ξ has the inner property.

Now, assume that (ii) holds and fix $x \in X$ and $r > 0$. Since ξ has the outer property, then $X \setminus \xi(x,r)$ is a τ_ξ-open set. Thus, $\xi(x,r)$ is a τ_ξ closed set. On the other hand, it follows from the inner property that $V_\xi(x,r)$ is a τ_ξ-open set. Since $V_\xi(x,r) \subset \xi(x,r)$, this last set is a closed neighborhood of x. □

A family $\{U_\alpha\}_{\alpha \in I}$ of neighborhoods of a point $x \in X$ is a *fundamental system of neighborhoods for x* if, for every neighborhood U of x, there is $\alpha \in I$ such that $U_\alpha \subset U$. If every point $x \in X$ has a countable fundamental system of neighborhoods then X is said to be a first countable space ([1, p. 12]).

Proposition 2.5. *If ξ is an IO pre-wrapping for X, then (X, τ_ξ) is a first countable space.*

Proof: If U is a neighborhood of a point $x \in X$, there exists $s > 0$ such that $\xi(x, s) \subset U$. Therefore, for $n \geq 1/s$, $\xi(x, 1/n)$ is a neighborhood of x contained in U. □

For any space X there is always a pre-wrapping defined by

$$\xi(x, 0) = \{x\} \text{ and } \xi(x, r) = X \text{ for } r > 0.$$

This simple pre-wrapping will be called trivial.

Proposition 2.6. *If (X, τ) is a first countable space, then there exists a non trivial pre-wrapping ξ for X.*

Proof: For $x \in X$, let $\{U_n(x)\}_{n=1}^{\infty}$ be a fundamental system of neighborhoods for x. We can define a pre-wrapping ξ as follows: $\xi(x, 0) = \{x\}$, $\xi(x, r) = \cap_{n=1}^{m} U_n(x)$ for $\frac{1}{m+1} \leq r < \frac{1}{m}$, and $\xi(x, r) = X$ if $r \geq 1$. □

Remark 2.7. Let (X, d) be a metric space. The function $\xi : X \times [0, \infty) \to \mathcal{P}(X)$ defined by

$$\xi(x, r) = \{y \in X : d(x, y) \leq r\},$$

is a pre-wrapping for X with the inner and the outer properties. In this case the metric topology coincides with τ_ξ. Notice that, if $A \in \mathcal{P}(X)$ and $x \in X$, then

$$d(x, A) = \inf \{d(x, a) : \ a \in A\} = d_\xi(x, A).$$

To close this section we give some examples of nontrivial pre-wrappings which are not wrappings.

1) Let (X, d) be a metric space. If $S(x, r)$ stands for the open ball of center x and radius r, i. e., if $S(x, r) = \{y \in X : d(x, y) < r\}$, we define the pre-wrapping $\xi : X \times [0, \infty) \to \mathcal{P}(X)$ by

$$\xi(x, r) = \begin{cases} \{x\}, & \text{if } r = 0, \\ S(x, r), & \text{if } r > 0. \end{cases}$$

This pre-wrapping verifies the inner property (in fact, for positive r, we have that $V_\xi(x, r) = \xi(x, r)$), and does not verify the outer one (for arbitrary $x \in X$ and arbitrary $r > 0$, pick a point in $\overline{\xi(x, r)} \setminus \xi(x, r)$). The function ξ is not a wrapping because the sets $\xi(x, r)$ for $r > 0$ fail to satisfy the condition of being closed.

We also note that the topology induced by ξ on X obviously coincides with the metric topology.

2) Let (X,d) be a metric space. If $B(x,r)$ stands for the closed ball of center x and radius r, we define the function $\xi : X \times [0,\infty) \to \mathcal{P}(X)$ by

$$\xi(x,r) = \begin{cases} \{x\}, & \text{if } 0 \leq r \leq 1, \\ B(x,r-1), & \text{if } r > 1. \end{cases}$$

It is easy to check that ξ is an IO pre-wrapping which is not a wrapping because, although $\xi(x,r)$ are closed sets, for $0 < r \leq 1$, it fails to satisfy the condition that the sets $\xi(x,r)$ have a nonempty interior.

Finally, we give two examples related to first countable non metrizable spaces.

3) Let us consider the *Khalimsky line*, i. e., the set of integers $\mathbb{Z}$ equipped with the topology having as a basis the set

$$\mathcal{B} = \{\{2i-1, 2i, 2i+1\} : i \in \mathbb{Z}\} \cup \{\{2i+1\} : i \in \mathbb{Z}\}.$$

If we denote by $g(r)$ the smallest integer greater than or equal to r, and define the function $\xi : \mathbb{Z} \times [0,\infty) \to \mathcal{P}(\mathbb{Z})$ by

$$\xi(j,r) = \bigcup_{k=0}^{2g(r)} \{j - g(r) + k\},$$

then ξ is a pre-wrapping which is not a wrapping because, for example, the open set $\xi(0,1) = \{-1,0,1\}$ is not closed.

We recall that the topology described above is induced by the quasi-metric d defined as follows:

$$d(j,k) = \begin{cases} 0, & \text{if } j = k, \\ 0, & \text{if } j = 2n \text{ and } k = 2n \pm 1, \text{ where } n \in \mathbb{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

4) In the *Sorgenfrey line* (the set of real numbers endowed with the topology generated by $\{[a,b) \subset \mathbb{R} : a, b \in \mathbb{R} \text{ with } a < b\}$), we define the function $\xi : \mathbb{R} \times [0,\infty) \to \mathcal{P}(\mathbb{R})$ by

$$\xi(x,r) = \begin{cases} \{x\}, & \text{if } r = 0 \\ (x - r, x + r^2), & \text{if } r > 0. \end{cases}$$

This pre-wrapping is not a wrapping because for positive r, the open sets $\xi(x,r)$ are not closed (in fact, $\overline{(x-r,x+r^2)} = (x-r,x+r^2])$.

3. Proximinality

The main purpose of this section is to give a characterization of the proximinality of a set A, when the pre-wrapping ξ has the IO property and it also satisfies the intersection property (that will be defined as a kind of monotony condition). First, we will state that a proximinal subset is closed in the τ_ξ topology, and after two results concerning proximinality, we give an adapted definition of approximatively compactness to obtain a corresponding proximinality result. By the analogy with the results in this section, we conclude with a well known uniqueness result for complete metric spaces.

Definition 3.1. Let ξ be a pre-wrapping for X and $A \in \mathcal{P}(X)$. A nonempty set $B \subset A$ is called (ξ, A)-open if for all $x \in B$ there exists $\delta = \delta(x) > 0$ such that $\xi(x,\delta) \cap A \subset B$.

So $B \subset A$ is (ξ, A)-open if and only if it is open in the relative topology that A inherits from (X, τ_ξ).

Henceforth, the notation $\varepsilon_n \downarrow \varepsilon$ means that $\{\varepsilon_n\}$ is a decreasing sequence of positive reals with limit ε.

Definition 3.2. Let ξ be a pre-wrapping for X, $A \in \mathcal{P}(X)$, and $x \in X$. We say that A has the ξ *covering property* for x if, for every increasing sequence $\{A_n\}$ of nonempty (ξ, A)-open subsets of A, and every $\varepsilon_n \downarrow \varepsilon$ such that $\xi(x,\varepsilon_n) \cap A_n = \emptyset$ for all n, it holds that either $A \neq \cup_n A_n$ or there exists N such that $\xi(x,\varepsilon_n) \cap A = \emptyset$ for $n > N$.

Note that a nonempty set A has the ξ covering property for all $x \in A$. So, throughout what follows, we will say that the set A has the ξ *covering property* if it has the ξ *covering property* for all $x \in X \setminus A$.

Definition 3.3. A pre-wrapping ξ for X has the *intersection property* if for all $x \in X$ and $r \geq 0$,

$$\xi(x,r) = \bigcap_{t>r} \xi(x,t).$$

Proposition 3.4. *Let ξ be an IO pre-wrapping for X with the intersection property and let $A \in \mathcal{P}(X)$. If A has the ξ covering property, then it is τ_ξ-closed.*

Proof: First we recall that (X, τ_ξ) is a first countable space. Suppose A is not τ_ξ-closed. Then there exists a sequence $\{x_n\}$ in A such that $x_n \to x \in X \backslash A$. We pick a natural number m and a subsequence of the natural numbers $\{l_n\}$ such that $\{A_n = A \backslash \xi(x, \frac{1}{m+l_n})\}$ is an increasing sequence of nonempty subsets of A. It is clear that the sets A_n are (ξ, A)-open, since they are the complements of the τ_ξ-closed subsets $\xi(x, \frac{1}{m+l_n})$ in the relative topology for A. Choosing $\varepsilon_n = \frac{1}{m+l_n}$, we have $\xi(x, \varepsilon_n) \cap A_n = \emptyset$ for all n. Nevertheless, $A = \cup_n A_n = A \backslash \cap_n \xi(x, \varepsilon_n) = A \backslash \{x\}$, and also $\xi(x, \varepsilon_n) \cap A \neq \emptyset$ for all n. □

Proposition 3.5. *Let ξ be an IO pre-wrapping for X with the intersection property. If a set $A \in \mathcal{P}(X)$ is proximinal, then it is τ_ξ-closed.*

Proof: Suppose that A is not a τ_ξ-closed set. Since (X, τ_ξ) is a first countable space, there exists a sequence $\{x_n\}$ in A which converges to a point $x \in X \setminus A$. Of course, we can assume that $x_n \neq x_m$ for $n \neq m$. For each $\varepsilon > 0$, there exists N such that, for $n > N$, $x_n \in \xi(x, \varepsilon)$ (this last set is a neighborhood of x). Thus, $d_\xi(x, A) = 0$. Taking into account that A is proximinal and that ξ has the intersection property, we obtain

$$P_\xi(x, A) = A \cap \xi(x, 0) = A \cap \{x\}.$$

By the hypotheses, $P_\xi(x, A) \neq \emptyset$; hence, $x \in A$ and we have a contradiction. □

Definition 3.6. Let ξ be an IO pre-wrapping for X. An increasing sequence $\{A_n\}$ of subsets of X is said to be *I monotone* if the set of all $x \in X$ such that for every n there exists an integer $\phi(n) > n$ for which $d_\xi(x, A_{\phi(n)}) < d_\xi(x, A_n)$ is nonempty.

Theorem 3.7. *Let ξ be an IO pre-wrapping for X with the intersection property and let $A \in \mathcal{P}(X)$. The following assertions are equivalent:*

(i) *A is proximinal.*

(ii) *For every I monotone sequence* $\{A_n\}$ *of subsets of* A,

$$\cup_n A_n \neq A.$$

(iii) *For every* $x \in X$ *and every sequence* $\{A_n\}$ *of subsets of* A *such that* $d_\xi(x, A_{n+1}) < d_\xi(x, A_n)$ *and* $\lim_n d_\xi(x, A_n) = d_\xi(x, A)$, *it holds that* $\cup_n A_n \neq A$.

(iv) A *has the* ξ *covering property.*

Proof: (i) $\Rightarrow$ (ii) Suppose that A is proximinal. It follows from Proposition 3.5 that A is τ_ξ-closed. Let $\{A_n\}$ be an I monotone sequence of subsets of A and fix $x \in X \setminus \cup_n A_n$ such that, for every n, there exists an integer $\phi(n) > n$ for which $d_\xi(x, A_{\phi(n)}) < d_\xi(x, A_n)$. Denote $\alpha = d_\xi(x, \cup_n A_n) = \inf d_\xi(x, A_n)$. Clearly, $\alpha \leq d_\xi(x, A_{\phi(n)}) < d_\xi(x, A_n)$; hence, $A_n \cap \xi(x, \alpha) = \emptyset$. Therefore, $\xi(x, \alpha) \cap (\cup_n A_n) = \emptyset$. Taking into account that A is proximinal, there exists

$$a \in (A \cap \xi(x, d_\xi(x, A))) \subset (A \cap \xi(x, \alpha)) \subset (A \setminus \cup_n A_n)\,.$$

So $\cup_n A_n \neq A$.

(ii) $\Rightarrow$ (iii) Fix $x \in X$ and a sequence $\{A_n\}$ of subsets of A such that $d_\xi(x, A_{n+1}) < d_\xi(x, A_n)$ and $\lim_n d_\xi(x, A_n) = d_\xi(x, A)$. Consider $B_n = \cup_{m=1}^n A_m$. Then $\{B_n\}$ is an I monotone sequence. By (ii), $\cup_n A_n = \cup_n B_n \neq A$.

(iii) $\Rightarrow$ (iv) Fix $x \in X$, an increasing sequence $\{A_n\}$ of (ξ, A)-open subsets of A and $\varepsilon_n \downarrow \varepsilon$ such that $A_n \cap \xi(x, \varepsilon_n) = \emptyset$ for all n. We will consider three cases.

Case 1. If $\varepsilon < d_\xi(x, A)$ then, by the definition of $d_\xi(x, A)$, there exists N such that, for $n > N$, $\xi(x, \varepsilon_n) \cap A = \emptyset$.

Case 2. If $d_\xi(x, A) < \varepsilon$ then, for $\beta \in (d_\xi(x, A), \varepsilon)$,

$$\emptyset \neq A \cap \xi(x, \beta) \subset (A \cap (\cap_n \xi(x, \varepsilon_n))) \subset A \setminus \cup_n A_n.$$

Therefore, $\cup_n A_n \neq A$.

Case 3. Assume that $d_\xi(x, A) = \varepsilon$. We first suppose that there exists a positive integer N such that, for each positive integer p, $d_\xi(x, A_N) = d_\xi(x, A_{N+p})$. Fix $a \in A \cap \xi(x, \varepsilon_{N+1})$. If $p \geq N$, $a \notin A_p$. In fact, $d_\xi(x, A_p) = d_\xi(x, A_N) \geq \varepsilon_N > \varepsilon_{N+1}$. On the other hand, if $1 \leq p < N$, $A_p \cap \xi(x, \varepsilon_{N+1}) \subset A_p \cap \xi(x, \varepsilon_p) = \emptyset$. Thus $x \notin \cup_n A_n$ and we obtain that $\cup_n A_n \neq A$.

If our first supposition fails, then there exists a sequence $\{\phi(n)\}$ of positive integers such that $d_\xi(x, A_{\phi(n+1)}) < d_\xi(x, A_{\phi(n)})$ for each n.

Let $\beta = \lim_n d_\xi(x, A_{\phi(n)})$. Note that $\beta \geq \alpha$ and that $\xi(x,\beta) \cap A_{\phi(n)} = \emptyset$ for all n. If $\beta > \alpha$ then $\xi(x,\beta) \cap A \neq \emptyset$ and so $A \neq \cup_n A_{\phi(n)} = \cup_n A_n$. Finally, if $\beta = \alpha$, it follows that $\{A_{\phi(n)}\}$ is a sequence that verifies the hypotheses of (iii); hence, $\cup_n A_n = \cup_n A_{\phi(n)} \neq A$.

(iv) $\Rightarrow$ (i) Assume that A has the ξ covering property for x. For each positive integer n denote $B_n = \xi(x, d_\xi(x,A) + 1/n) \cap A$. If there exists an infinite set Λ of positive integers p such that $B_p = A$, $p \in \Lambda$, then $P_\xi(x, A) = A$ and (i) holds.

Therefore, in what follows, we assume that $B_n \neq A$ for each n. Taking into account that B_n is a τ_ξ-closed set (since A and $\xi(x, d_\xi(x,A)+1/n)$ are τ_ξ-closed sets), and considering the induced topology on A, we have that $A_n = A \setminus B_n$ is a (ξ, A)-open set. Moreover, for each n, $A_n \subset A_{n+1}$ and by definition

$$A_n \cap \xi\Big(x, d_\xi(x,A) + \tfrac{1}{n}\Big) = \emptyset.$$

Since A has the ξ covering property for x, and

$$\xi\Big(x, d_\xi(x,A) + \tfrac{1}{n}\Big) \cap A \neq \emptyset$$

for each n, we have that $A \neq \cup_n A_n$. Therefore,

$$A \setminus \cup_n A_n = \bigcap_{n=1}^{\infty} B_n \neq \emptyset.$$

Thus, by using the intersection property, we obtain

$$\begin{aligned} P_\xi(x,A) &= A \cap \xi(x, d_\xi(x,A)) \\ &= A \cap \left(\bigcap_{n=1}^{\infty} \xi\Big(x, d_\xi(x,A) + \frac{1}{n}\Big) \right) = \bigcap_{n=1}^{\infty} B_n \neq \emptyset. \quad \square \end{aligned}$$

Condition (iii) in the theorem above is frequently used in proving that a given set is proximinal.

Let us obtain from Theorem 3.7 some known results related to proximinality. Recall that a topological space (X, τ) is countably compact ([1], p. 202) if every countably infinite subset of X has an accumulation point. Hence, (X, τ) is countably compact if and only if each non-increasing sequence of closed nonempty subsets of X has nonempty intersection.

Proposition 3.8. *Let ξ be an IO pre-wrapping for X with the intersection property and $A \in \mathcal{P}(X)$. If there exists a topology η for X with the property that for each $x \in X$ there exists $\delta_n \downarrow 0$ such that*

a) *$A \cap \xi(x, d_\xi(x, A) + \delta_1)$ is η-countably compact,*

b) *for $n > 1$ the set $A \cap \xi(x, d_\xi(x, A) + \delta_n)$ is η-closed,*

then A is proximinal.

Proof: We will prove that A has the ξ covering property. Fix $x \in X$, an increasing sequence $\{A_n\}$ of nonempty (ξ, A)-open subsets of A, and $\varepsilon_n \downarrow \varepsilon$ such that $\xi(x, \varepsilon_n) \cap A_n = \emptyset$ for all n. If $\varepsilon < d_\xi(x, A)$, as in the proof of Theorem 3.7 , it follows that there exists N such that, for $n > N$, $\xi(x, \varepsilon_n) \cap A = \emptyset$. If $d_\xi(x, A) < \varepsilon$, it is clear that $A \neq \cup_n A_n$. Finally assume that $\varepsilon = d_\xi(x, A)$. Passing to sequence (if necessary, we can assume that $d_\xi(x, A) \leq \varepsilon_n \leq \delta_n$), fix $x_n \in \xi(x, \varepsilon_n) \cap A$. We can assume that the set $B = \{x_n : n \in N\}$ is infinite. Let y be a limit point of B (in the τ topology). Since each set $A \cap \xi(x, d_\xi(x, A) + \delta_{n+1})$ is η-closed, we have that $y \notin \cup_n A_n$. Thus, $y \in A \setminus \cup_n A_n$. □

A slight generalization of the above result that does not require the IO property nor the intersection property for the pre-wrapping ξ may be found in [4]. Indeed, Proposition 3.8 is a slight generalization of Theorem 2.2 of [6, p. 383].

A nonempty subset A of a *normed* space X is called *approximatively compact* if given $x \in X$ and a sequence $\{a_n\}$ in A such that $\lim_n \|x - a_n\| = \inf \{\|x - a\| ; a \in A\}$, then $\{a_n\}$ contains a convergent subsequence with limit in A. As a direct consequence of the definition, it is clear that any nonempty approximatively compact subset of a normed space is proximinal with respect to the pre-wrapping considered in Remark 2.7. Let us show how Theorem 3.7 led us to give a generalization of the previous statement.

Definition 3.9. Let ξ be a pre-wrapping for X. A set $A \in \mathcal{P}(X)$ is said to be *approximatively compact* if given $x \in X$ and a sequence $\{a_n\}$ in A such that $a_n \in A \cap \xi(x, d_\xi(x, A) + 1/n)$, then $\{a_n\}$ contains a convergent subsequence with τ_ξ-limit in A.

Proposition 3.10. (See [4].) *Let ξ be an IO pre-wrapping for X with the intersection property and let $A \in \mathcal{P}(X)$. If A is approximatively compact, then A is proximinal.*

Proof: We will prove that A has the ξ covering property. Fix $x \in X$, an increasing sequence $\{A_n\}$ of nonempty (ξ, A)-open subsets of A, and $\varepsilon_n \downarrow \varepsilon$ such that $\xi(x, \varepsilon_n) \cap A_n = \emptyset$ for all n. We consider only the case $\varepsilon = d_\xi(x, A)$. (The other cases can be treated as above.) Passing to a subsequence, if necessary, we can assume that $d_\xi(x, A) < \varepsilon_n \leq d_\xi(x, A) + 1/n$. For each n, fix a point $a_n \in \xi(x, \varepsilon_n) \cap A$. Let $a \in A$ be a τ_ϵ-limit point of a subsequence $\{a_{n_k}\}$ of $\{a_n\}$. First, note that $a \in \xi(x, \varepsilon_n)$ for all n. Indeed, suppose that $a \notin \xi(x, \varepsilon_n)$ for all $n \geq m$. Since ξ has the outer property, we can pick $\delta > 0$ such that $\xi(a, \delta) \cap \xi(x, \varepsilon_m) = \emptyset$. This also implies that $\xi(a, \delta) \cap \xi(x, \varepsilon_n) = \emptyset$ for all $n \geq m$. However, by the definition of τ_ξ-limit, there exists a natural n_0 such that $a_{n_k} \in \xi(a, \delta)$ for all $k \geq n_0$. This contradicts the fact $\xi(a, \delta) \cap \xi(x, \varepsilon_n) = \emptyset$ for all $n \geq m$. So we conclude that $a \in A \setminus \cup_n A_n$ and hence, A has the ξ-covering property. $\square$

We recall that a subset $A \in \mathcal{P}(X)$ is called a Chebyshev set if for each $x \in X$ the set of best approximants for x by elements of A has, at most, one element. For complete metric spaces, we have the following well known existence and uniqueness result. (In metric spaces, we have that $\xi(x, r)$ is the closed ball of radius r and center x; hence, $d_\xi(x, A)$ is the metric distance from x to A.)

Theorem 3.11. *Let (X, d) be a complete metric space and let A be a nonemtpy closed subset of X. For each $x \in X$ and each positive integer n, we denote*

$$A_n(x) = A \cap \xi\Big(x, d_\xi(x, A) + \tfrac{1}{n}\Big)$$

and

$$\delta_n(x) = \delta(A_n(x)) = \sup_{a,b \in A_n(x)} d(a, b).$$

If $\lim_{n\to\infty} \delta_n(x) = 0$ for each $x \in X$, then A is proximinal and Chebyshev.

Proof: For $x \in X$, it is clear that $\{A_n(x)\}_{n=1}^\infty$ is a non-increasing sequence of nonempty closed subsets of X, whose diameters $\delta_n(x)$

tend to 0. By the Cantor principle, we have that the intersection of the family has exactly one element. Since

$$\begin{aligned}\bigcap_{n=1}^{\infty} A_n(x) &= A \cap \left(\bigcap_{n=1}^{\infty} \xi\Big(x, d_\xi(x,A) + \frac{1}{n}\Big)\right) \\ &= A \cap \xi(x, d_\xi(x,A)) = P_\xi(x,A),\end{aligned}$$

we have that A is proximinal and Chebyshev. □

As an example of the above result, we can state that nonempty closed convex subsets of uniformly convex spaces are proximinal and Chebyshev. A Banach space $(X, \|.\|)$ is called uniformly convex if, for each positive $\varepsilon \le 2$, there exists $\delta > 0$ such that for $u, v \in X$, if $\|u\| \le 1$, $\|v\| \le 1$, and $\|u - v\| \ge \varepsilon$, then $\left\|\frac{1}{2}(u+v)\right\| \le 1 - \delta$. Let A be a nonempty closed convex subset of an uniformly convex Banach space $(X, \|\|)$. It is easy to state that

$$\lim_{n\to\infty} \delta_n(x) = 0.$$

If contrary, there exists $d > 0$ and a subsequence $\{A_{n_j}(x)\}$ such that $\delta_{n_j}(x) \ge d$. Hence, there exist $a_j, b_j \in A_{n_j}(x)$ fulfilling $\|a_j - b_j\| > \frac{d}{2}$. Consider

$$u = \frac{a_j - x}{d_\xi(x,A) + \frac{1}{n_j}}, \quad v = \frac{b_j - x}{d_\xi(x,A) + \frac{1}{n_j}}$$

and

$$\varepsilon = \min\Big(2, \frac{d/2}{d_\xi(x,A) + 1}\Big).$$

Then $\|u\| \le 1$, $\|v\| \le 1$, and $\|u - v\| > \varepsilon$, and by the uniform convexity of X

$$\begin{aligned}\left(d_\xi(x,A) + \frac{1}{n_j}\right)\left\|\frac{1}{2}(u+v)\right\| &= \left\|\frac{a_j + b_j}{2} - x\right\| \\ &\le \left(d_\xi(x,A) + \frac{1}{n_j}\right)(1-\delta),\end{aligned}$$

but for $n_j > (1-\delta)/(\delta d_\xi(x,A))$, this implies that $(a_j + b_j)/2$ is not in A. This is absurd, since A is convex.

4. Semicontinuity

Recall that, for a topological space X, a function $f : X \to \mathbb{R}$ is *lower* (*upper*) *semicontinuous* if, for each $b \in \mathbb{R}$ the set $f^{-1}((b,\infty))$ ($f^{-1}((-\infty,b))$) is open. Moreover, f is continuous if and only if it is upper and lower semicontinuous (see [1, p. 61]). As above we consider on X the topology τ_ξ induced by the pre-wrapping ξ

One of the main gaps in the definition of a pre-wrapping is that it contains no relations between the sets $\xi(x,r)$ and $\xi(y,s)$ for $x \neq y$ and $r,s > 0$. Such information is needed (as least locally) in studying continuity. Even more, the notion of pre-wrapping with the inner property does not provide enough arguments. The reason is that we do not have a measure of the size of t in the assertion: if $y \in V_\xi(x,r)$, there exists $t > 0$ such that $\xi(y,t) \subset V_\xi(x,r)$. (See the next section.) Propositions 4.1. and 4.2 provide characterizations of the upper and lower semicontinuity, respectively. Actually, they can be seen as transcriptions of the definitions. Condition (ii) in propositions 4.1 and 4.2 is not of a pure geometrical nature because it refers indirectly to the function $d_\xi(\cdot,A)$. In spite of this, Proposition 4.1 can be used to obtain more natural results related with upper semicontinuity (under additional assumptions). For the case of lower semicontinuity, we will need a new definition.

To simplify the exposition, for a fixed $A \in \mathcal{P}(X)$ the function $d_\xi(\cdot,A) : X \to [0,\infty)$ will be denoted by $f_A(\cdot)$. In particular, if A is a singleton, say $A = \{a\}$, we will write $f_a(\cdot) = f_{\{a\}}(\cdot)$.

Proposition 4.1. *Let ξ be an IO pre-wrapping for X and let $A \in \mathcal{P}(X)$. The following assertions are equivalent:*

(i) *The function f_A is upper semicontinuous.*

(ii) *For each $x \in X$ and all $r,\varepsilon > 0$, if $\xi(x,r) \cap A \neq \emptyset$, there exists $\delta > 0$ such that, for every $y \in \xi(x,\delta)$, $\xi(y,r+\varepsilon) \cap A \neq \emptyset$.*

Proof: For $b \in \mathbb{R}$, denote $C(A,b) = \{x \in X : f_A(x) < b\}$. Notice that, for $b \le 0$, $C(A,b) = \emptyset$.

First, assume that assertion (ii) holds. If $b > 0$ and $x \in C(A,b)$, fix $s > 0$ such that $f_A(x) + s < b$. Choose $\delta > 0$ as in (ii) with $r = f_A(x) + s/2$ and $\varepsilon = s/2$. It is sufficient to prove that $\xi(x,\delta) \subset C(A,b)$. If $y \in \xi(x,\delta)$, then $\xi(y, f_A(x) + s) \cap A \neq \emptyset$. Therefore, $f_A(y) \le f_A(x) + s < b$. That is, $\xi(x,\delta) \subset C(A,b)$.

Now suppose that for each $b \in \mathbb{R}$ the set $C(A, b)$ is τ_ξ-open and fix $x \in X$. We should verify condition (ii) for $r \geq f_A(x)$. In fact, it is sufficient to verify (ii) for $r = f_A(x)$. Fix $\varepsilon > 0$. Since the set $W = C(A, f_A(x) + \varepsilon)$ is τ_ξ-open and $x \in W$, there exists $\delta > 0$ such that $\xi(x, \delta) \subset W$. If $y \in \xi(x, \delta) \subset W$, then $f_A(y) < f_A(x) + \varepsilon$. It follows that $\xi(y, f_A(x) + \varepsilon) \cap A \neq \emptyset$. □

The following assertion can be proved in the same fashion.

Proposition 4.2. *Let ξ be an IO pre-wrapping for X and let $A \in \mathcal{P}(X)$. The following assertions are equivalent:*

(i) *The function f_A is lower semicontinuous.*

(ii) *For each $x \in X$ and all $r > 0$, if $\xi(x, r) \cap A = \emptyset$ and $\varepsilon \in (0, r)$, there exists $\delta > 0$ such that, for every $y \in \xi(x, \delta)$, $\xi(y, r - \varepsilon) \cap A = \emptyset$.*

Let us present a sufficient condition for the upper semicontinuity of f_A.

Proposition 4.3. *Let ξ be an IO pre-wrapping for X and let $A \in \mathcal{P}(X)$. If for each $a \in A$ the function f_a is upper semicontinuous, then the function f_A is upper semicontinuous.*

Proof: We will show that for all $b > 0$ the set $C(A, b)$ is τ_ξ-open. Given $x \in C(A, b)$, choose $\varepsilon > 0$ such that $f_A(x) + \varepsilon < b$, and let $a \in A \cap \xi(x, f_A(x) + \varepsilon)$. Since f_a is upper semicontinuous, the set $G = \{z \in X : f_a(z) < f_A(x) + \varepsilon\}$ is τ_ξ-open. As $x \in G$, there exists a neighborhood U of x such that, if $y \in U$, then $f_a(y) < f_A(x) + \varepsilon < b$. Now it is sufficient to note that $f_A \leq f_a$ to conclude that $U \subset C(A, b)$. □

In order to present a sufficient condition for the lower semicontinuity of f_A, we need the next definition.

Definition 4.4. Let ξ be a pre-wrapping for X. We say that ξ has the *weak triangular property* if for all $x \in X$, $r > 0$, and $\varepsilon \in (0, r)$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $y \in \xi(x, \delta)$, then $\xi(y, r - \varepsilon) \subset \xi(x, r)$.

Some facts related with the weak triangular property will be presented in the last section. Now we will state that if a pre-wrapping ξ has the weak triangular property, then it has the inner one.

Proposition 4.5. *If a pre-wrapping ξ has the weak triangular property, then it has the inner one.*

Proof: Fix $x \in X$ and $r > 0$. If $\varepsilon \in (0, r)$, by the weak triangular property there exists $\delta > 0$ such that, for $y \in \xi(x, \delta)$, $\xi(y, r - \varepsilon) \subset \xi(x, r)$. Fix $q \in (0, r - \varepsilon)$. Again by the weak triangular property, there exists $t \in (0, \delta)$ such that for $u \in \xi(y, t)$, $\xi(u, r - \varepsilon - q) \subset \xi(y, r - \varepsilon) \subset \xi(x, r)$. Then, if $0 < s \leq \min\{t, r - \varepsilon - q\}$, we have that $\xi(y, s) \subset V_\xi(x, r)$. □

Notice that in the following proposition no conditions on A are supposed.

Proposition 4.6. *If ξ is a pre-wrapping for X with the outer and the weak triangular properties, then, for each $A \in \mathcal{P}(X)$, the function f_A is lower semicontinuous.*

Proof: Fix $b \in \mathbb{R}$ and denote $B = \{x \in X : f_A(x) > b\}$. Of course, if $b < 0$, then $B = X$ and B is τ_ξ-open. Thus, in what follows we assume that $b \geq 0$. If $B = \emptyset$ then it is τ_ξ-closed, so we will suppose $B \neq \emptyset$. Fix $x \in B$ and $s \in (0, f_A(x) - b)$. By the weak triangular property (with $\varepsilon = s/2$ and $r = f_A(x) - s/2$), there exists $\delta > 0$ such that for $y \in \xi(x, \delta)$,

$$\xi(y, f_A(x) - s) \subset \xi\Big(x, f_A(x) - \frac{s}{2}\Big).$$

By definition of f_A, we have $f_A(x) - s < f_A(y)$. That is,

$$b = f_A(x) - (f_A(x) - b) < f_A(x) - s < f_A(y).$$

We have proved that the τ_ξ-neighborhood $\xi(x, \delta)$ of x is contained in B. □

From propositions 4.3, 4.5, and 4.6 we obtain the following.

Theorem 4.7. *Let ξ be a pre-wrapping for X with the outer and the weak triangular properties, and let $A \in \mathcal{P}(X)$. If for each $a \in A$ the function f_a is continuous, then the function f_A is continuous.*

Following Casimir Kuratowski [3] (see also [1, p. 63] and [6]), given two topological spaces X and Y, we will say that a set-valued function $f : Y \to 2^X$ is *lower* (*upper*) *semicontinuous* if for every open set $U \subset X$ the set

$$\{y \in Y : f(y) \cap U \neq \emptyset\} \ \ (\{y \in Y : f(y) \subset U\})$$

is open in Y. Of course, $f : Y \to 2^X$ is *lower* (*upper*) *semi-continuous* if and only if for every closed set $K \subset X$ the set $\{y \in Y : f(y) \subset K\}$ ($\{y \in Y : f(y) \cap K\}$) is closed in Y.

Given $x \in X$, if the set $P_\xi(x, A)$ is a singleton, the only point in this set will be denoted by $P^*_\xi(x, A)$. If $A \subset X$ is Chebyshev, we can define a function $P^*_\xi : X \to A$ by the equation $P^*_\xi(x) = P^*_\xi(x, A)$. The following assertion is easy to verify.

Proposition 4.8. *Let ξ be an IO pre-wrapping for X and let A be a Chebyshev subset of X. Then the function $P_\xi(\cdot, A) : X \to 2^A$ is upper semicontinuous if and only if the function $P^*_\xi : X \to A$ is continuous.* (*We consider A as a topological subspace of (X, τ_ξ)*).

We now show an example of a proximinal subset A for which the function of best approximation $P_\xi(\cdot, A)$ is not upper semicontinuous.

Example 4.9. We consider the space $\mathbb{R}^2$ endowed with the Euclidean norm. For $r > 0$ we define the wrapping by the family of ellipses

$$\xi(x, r) = \begin{cases} \left\{y = (y_1, y_2) \in \mathbb{R}^2 : \frac{y_1^2}{r^2} + \frac{y_2^2}{(\frac{r}{2})^2} \leq 1\right\} \text{ if } x = (0, 0), \\ \left\{y \in \mathbb{R}^2 : \|y - x\|_2 = r\right\} \text{ if } x \neq (0, 0), \end{cases}$$

and $\xi(x, 0) = \{x\}$ for all $x \in \mathbb{R}^2$.

The subset $A = \left\{y \in \mathbb{R}^2 : \|y\|_2 = 1\right\}$ is ξ-proximinal (since it is compact), and for all $x \in \mathbb{R}^2$ we have

$$d_\xi(x, A) = \|x - A\|_2,$$

and this assures that the real valued function $d_\xi(\cdot, A)$ is uniformly continuous.

On the other hand, if x_0 stands for the point $(0, 1)$, the set $F = \{x_0\}$ is closed but

$$\left\{x \in \mathbb{R}^2 : P_\xi(x, A) \cap F \neq \emptyset\right\} = \{tx_0 : t > 0\}$$

is not. Hence, the function $P_\xi(\cdot, A)$ is not upper semicontinuous.

Our next result generalizes and improves a result of Ivan Singer [5, Theorem 1] for the case of metric space.

Theorem 4.10. *Let ξ be a pre-wrapping for X with the intersection, the outer, and the weak triangular properties, and let $A \in$*

$\mathcal{P}(X)$ be a proximinal set. If for each $a \in A$ the function f_a is upper semicontinuous, then $P_\xi(\cdot, A)$ is upper semicontinuous.

Proof: By the intersection property, we have

$$P_\xi(x, A) = A \cap \xi(x, f_A(x)),$$

for all $x \in X$. We will prove that for each nonempty closed set $F \subset X$, the set

$$B_F = \{x \in X : \ P_\xi(x, A) \cap F \neq \emptyset\}$$

is also closed. Suppose that B_F is not a closed set (so it is nonempty). Since (X, τ_ξ) is a first countable space, there exists a sequence $\{x_n\}$ in B_F which has a limit point $x \in X \setminus B_F$, the convergence being in the $\tau - \xi$ topology. For each n, fix a point $a_n \in P_\xi(x_n, A) \cap F$.

Since $\{x_n\}$ converges to x, and since by propositions 4.3 and 4.5, the function f_A is upper semicontinuous, for each positive integer p there exists $N = N(p)$ such that, for $n > N$, $x_n \in V_\xi(x, 1/2p)$ and $f_A(x_n) \leq f_A(x) + 1/4p$. It follows from the weak triangular property (with $r = f_A(x) + 1/p$ and $\varepsilon = 1/(2p)$) that there exists $\delta > 0$ such that, for $y \in \xi(x, \delta)$,

$$\xi(y, f_A(x) + \tfrac{1}{2p}) \subset \xi(x, f_A(x) + \tfrac{1}{p}).$$

Fix $M > N$ such that, for $n > M$, $x_n \in \xi(x, \delta)$. For $n > M$,

$$a_n \in \xi\Big(x_n, f_A(x_n) + \tfrac{1}{4p}\Big) \subset \xi\Big(x_n, f_A(x) + \tfrac{1}{2p}\Big) \subset \xi\Big(x, f_A(x) + \tfrac{1}{p}\Big).$$

Let us denote $C = \{a_n : \ n \in N\}$ and let C^* be the set of all limit points of C.

Our proof concludes by proving the following three facts:

Claim 1. The set C^* is nonempty.

Claim 2. $C^* \subset P_\xi(x, A)$.

Claim 3. $C^* \subset F$.

In fact, if these assertions hold, we have a contradiction since $P_\xi(x, A) \cap F \neq \emptyset$ and $x \notin B_F$.

Proof of Claim 1: The set C has a limit point on A. In fact, suppose that this is not the case. We have that C^* is closed in the τ_ξ topology. (Recall that by Proposition 3.4, A is τ_ξ-closed.) Denote $B = C^* \cup P_\xi(x, A)$. Now, we have that B is τ_ξ-closed, $P_\xi(x, A) = P_\xi(x, B)$, and $f_A(x) = f_B(x)$. By Theorem 3.7, B has the ξ covering property for x. Thus, if for each p we define

$B_p = \{a_n : a_n \in \xi(x, f_A(x) + 1/p)\}$ (a τ_ξ-closed set) and $A_p = B \setminus (B_p \cup P_\xi(x, A))$, we obtain an increasing sequence of (ξ, B)-open sets. Without loss of generality we assume that for each p, $B_p \neq \emptyset \neq A_p$. Moreover, $\xi(x, f_B(x) + 1/p) \cap A_p = \emptyset$. It follows from the ξ covering property that $B \neq \cup_p A_p$ or there exists an M such that, for $p > M$, $\xi(x, f_B(x) + 1/p) \cap B = \emptyset$. The last case is not possible, since $P_\xi(x, B) \subset \xi(x, f_B(x) + 1/p) \cap B$. But, if $B \neq \cup_p A_p$, then

$$\emptyset \neq B \setminus \cup_p A_p = \cap_p (B \setminus A_p) = \cap_p B_p = \emptyset,$$

and we have a contradiction. Our claim is proved.

Proof of Claim 2: If $a \in A$ is a limit point of the set C, then $a \in P_\xi(x, A)$. In fact, if the sequence $\{B_n\}$ is defined as in Claim 1, then by the intersection property,

$$a \in \bigcap_p B_p \subset \left(A \cap \bigcap_{t > f_B(x)} \xi(x, t) \right) = (A \cap \xi(x, f_B(x))) = P_\xi(x, A).$$

Thus, Claim 2 is proved. □

Proof of Claim 3: It follows from the fact that $C \subset F$ and F is closed. □

In closing this section, we give an example that shows that the hypothesis of intersection property of Theorem 4.10 cannot be removed.

Example 4.11. For a real number r, $]r[$ stands for the bigger of the integers strictly lower than r; it is straightforward to check that

$$]r[+]s[\leq]r + s[\tag{4.1}$$

for any real numbers r, s. Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be defined by

$$f(r) = \begin{cases} 0 & \text{if } r = 0 \\ r +]r[& \text{if } r > 0 \end{cases}$$

Then $f(r) = 0$ if and only if $r = 0$, f is non decreasing and $\lim_{r \to \infty} f(r) = \infty$. For each $x \in \mathbb{R}^2$ and $r \geq 0$, we consider

$$\xi(x, r) = B(x, f(r)),$$

where $B(x, s)$ denotes the Euclidean closed ball of center x and radius s. Then ξ is an IO pre-wrapping for $\mathbb{R}^2$. Also, taking into account (4.1), it is easy to check that ξ satisfies the weak triangular property and for each $x \in \mathbb{R}^2$, the function $d_\xi(\cdot, \{x\})$ is continuous at x. Nevertheless, this wrapping fails to fulfill the intersection property because f is not right continuous at the natural numbers.

Let $A = \left\{x \in \mathbb{R}^2 : \|x\|_2 \geq 2\right\}$. Note that A is a proximinal set. However, if we consider the closed subset $F = \{x_0\}$, where $x_0 = (0, 2) \in A$, we get

$$B_F \equiv \left\{x \in \mathbb{R}^2 : P_\xi(x, A) \cap F \neq \emptyset\right\} = \\ \left\{t\, x_0 : 1 \leq t \leq \tfrac{1}{2}\right\} \cup (B(x_0, 2) \cap S(0, 1)),$$

where $S(0, 1)$ denotes the open unit ball in $\mathbb{R}^2$ centered at the origin. Since B_F is not closed, the function $P_\xi(\cdot, A)$ is not upper semicontinuous.

5. Additional properties for Pre-Wrappings

We need a definition from [4].

Definition 5.1. Let ξ be a pre-wrapping for X. We say that ξ has the *triangular property* if for all $x \in X$, $r, s > 0$, and $y \in \xi(x, r)$, $\xi(y, s) \subset \xi(x, r + s)$.

Some of the main results in [4] are obtained under the assumption that the triangular property holds. In this section, we analyze how this property is related with the conditions used in previous sections.

For $x \in X$, $r > 0$ and $y \in V_\xi(x, r)$, define

$$\delta_x(y, r) = \sup\{\delta > 0 : \ \xi(y, \delta) \subset \xi(x, r)\}.$$

Definition 5.2. We say that a pre-wrapping ξ for X satisfies the *reverse Lipschitz condition* if, for all $x \in X$, $r > 0$, and $\varepsilon \in (0, r)$, if $y \in V_\xi(x, r) \cap \xi(x, \varepsilon)$, then $r - \varepsilon \leq \delta_x(y, r)$.

We will show that the reverse Lipschitz condition is something between the triangular property and the weak triangular property.

Proposition 5.3. *Let ξ be a pre-wrapping for X. If ξ has the triangular property, then it satisfies the reverse Lipschitz condition.*

Proof: Suppose that ξ has the triangular property. Fix $x \in X$ and $r > 0$.

If $\varepsilon > 0$ and $y \in V_\xi(x, r) \cap \xi(x, \varepsilon)$, then $\xi(y, r - \varepsilon) \subset \xi(x, r)$. Thus, by definition, $\delta_x(y, r) \geq r - \varepsilon$. □

Proposition 5.4. *If a pre-wrapping ξ for X satisfies the reverse Lipschitz condition, then it has the weak triangular property.*

Proof: Fix $x \in X$, $r > 0$, $\varepsilon \in (0, r)$, and $s \in (0, \varepsilon)$. Since ξ fulfills the reverse Lipschitz condition, there exists $\delta \in (0, \varepsilon - s)$ such that $\xi(x, \delta) \subset V_\xi(x, r)$. If $y \in \xi(x, \delta) \subset V_\xi(x, r) \cap \xi(x, \varepsilon - s)$, then $r - \varepsilon + s \leq \delta_x(y, r)$. Therefore, $\xi(y, r - \varepsilon) \subset \xi(x, r)$. □

Taking into account the previous result and Proposition 4.5, we get the following.

Corollary 5.5. *If ξ is a pre-wrapping for X having the reverse Lipschitz property, then it has the inner property.*

Remark 5.6. It was noticed in [4] that if ξ has the triangular property, the function $\rho_\xi : X \times X \to [0, \infty)$ defined by

$$\rho_\xi(x, y) = d_\xi(x, \{y\}) + d_\xi(y, \{x\})$$

is a pseudometric in X. It is easy to verify that the topology associated with the pseudometric coincides with τ_ξ.

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