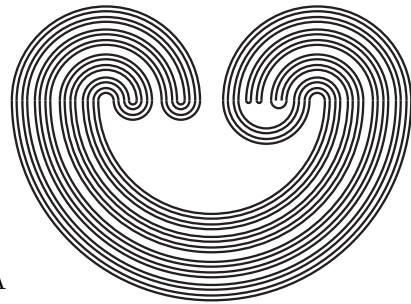


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**PERIODIC-RECURRENT PROPERTY  
FOR A CLASS OF  $\lambda$ -DENDROIDS**

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ABSTRACT. A space  $X$  has the periodic-recurrent property if for each self-mapping the closures of the sets of periodic points and of recurrent points are equal. The property is studied for some continua obtained as compactifications of certain dendrites with a finite number of points deleted. Thus, some new continua with the periodic-recurrent property are presented. Several related questions are asked.

## 1. INTRODUCTION

Let  $X$  be a space, and let  $f : X \rightarrow X$  be a mapping of  $X$  to itself. For any  $n \in \mathbb{N}$  let  $f^n : X \rightarrow X$  denote the  $n$ -th composition of  $f$ . A point  $x$  of  $X$  is said to be

- a *fixed* point of  $f$  if  $f(x) = x$ ;
- a *periodic* point of  $f$  provided that there is  $n \in \mathbb{N}$  such that  $f^n(x) = x$ ; if, moreover,  $f^k(x) \neq x$  for all integers  $k$  with  $1 \leq k < n$ , then  $x$  is called a periodic point of *period*  $n$ ;
- a *recurrent* point of  $f$ , provided that for each open set  $U$  containing  $x$  there is  $n \in \mathbb{N}$  such that  $f^n(x) \in U$ .

For a mapping  $f : X \rightarrow X$ , the sets of fixed points, periodic points, and recurrent points will be denoted by  $F(f)$ ,  $P(f)$ , and  $R(f)$ , respectively. The statement below is a consequence of definitions.

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**Statement 1.1.** *For each compact Hausdorff space  $X$  and each self-mapping  $f : X \rightarrow X$ , we have*

$$(1.1.1) \quad F(f) \subset P(f) \subset R(f), f(P(f)) = P(f), f(R(f)) = R(f),$$

$$(1.1.2) \quad P(f) = P(f^k) \text{ and } R(f) = R(f^k) \text{ for each } k \in \mathbb{N}.$$

**Definition 1.2.** A space  $X$  is said to have the *periodic-recurrent property* (abbreviated *PR-property*) provided that each mapping  $f : X \rightarrow X$  satisfies  $\text{cl}_X(P(f)) = \text{cl}_X(R(f))$ .

Topological dynamics on various spaces started with its study on the interval. (See, for example, an expository paper [20] and references therein.) In particular, it has been shown that the closed unit interval has the PR-property, [8, Theorem 1, p. 316]. This result has been extended to mappings of trees in [21, Theorem 2.6, p. 349] and to ones of some dendrites in [14].

The result saying that trees have the PR-property cannot be extended to all *graphs* (i.e., continua that can be written as the union of finitely many arcs any two of which are either disjoint or intersect only at one or both of their end points) because the unit circle  $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$  does not have the property. Indeed, if  $f : \mathbb{S} \rightarrow \mathbb{S}$  is an irrational rotation (i.e., a rotation by an angle  $\alpha$  such that  $\alpha/\pi$  is irrational), then  $P(f) = \emptyset$ , while  $R(f) = \mathbb{S}$  (compare [4, Remarks 1.2, p. 132]).

However, as it was presented in [4], the PR-property holds for each  $\lambda$ -dendroid which is obtained from a tree  $T$  by inserting (in a special way) finitely many  $\lambda$ -dendroids having the PR-property and satisfying some additional conditions. The aim of the present paper is to show that the same method can be applied to obtain a much larger class of  $\lambda$ -dendroids having the PR-property.

This paper consists of five sections. After Introduction and Preliminaries,  $\lambda$ -dendroids as compactifications are considered in section 3, where auxiliary results are proved, which are used to prove the PR-property for a class  $\Delta$  of  $\lambda$ -dendroids in section 4. Section 5 contains a number of related questions and problems.

## 2. PRELIMINARIES AND AUXILIARY RESULTS

All considered *spaces* are assumed to be metric, and a *mapping* means a continuous function. We denote by  $\mathbb{N}$  the set of all positive

integers and by  $\mathbb{C}$  the set of all complex numbers. For  $A \subset X$  we denote  $\text{cl}_X(A)$  the closure of  $A$  in  $X$ .

A mapping  $r : X \rightarrow Y$  between continua  $X$  and  $Y$  is called a *retraction* if  $Y \subset X$  and the partial mapping  $r|_Y : Y \rightarrow Y$  is the identity. In this case,  $Y$  is called a *retract* of  $X$ .

The following lemma on compositions of mappings is known. (See [4, Lemma 3.1, p. 136] and compare [7, Lemma 2.9].)

**Lemma 2.1.** *Let  $X$  and  $Y$  be spaces with  $Y$  being a closed subset of  $X$ , and let  $g : Y \rightarrow Y$  be a mapping. If  $r : X \rightarrow Y$  is a retraction and  $f = g \circ r : X \rightarrow Y$ , then:*

$$(2.1.1) \quad f^n = g^n \circ r \text{ for each } n \in \mathbb{N};$$

$$(2.1.2) \quad P(f) = P(g);$$

$$(2.1.3) \quad R(f) = R(g).$$

As a consequence of Lemma 2.1, we get a result below (see [4, Proposition 3.2, p. 136]).

**Corollary 2.2.** *The PR-property is preserved under retractions, i.e., if a space  $X$  having the PR-property contains a closed subspace  $Y$  which is a retract of  $X$ , then  $Y$  has the PR-property, too.*

It is quite natural to ask what kinds of mappings (or what kinds of mappings between some special spaces) can be substituted in place of retractions in the above result. This question is discussed, and some results are obtained, in [5].

We use a concept of an *order* of a point  $p$  in a continuum  $X$  (in the sense of Menger-Urysohn), written  $\text{ord}(p, X)$ , as defined in [18, 9.3, p. 141] or [16, §51, I, p. 274].

A point  $p \in X$  is called an *end point* of  $X$  provided that  $\text{ord}(p, X) = 1$ , and it is called a *branch point* of  $X$  provided that  $\text{ord}(p, X) \geq 3$ .

Recall that a continuum  $X$  is said to be

- *hereditarily unicoherent* provided that the intersection of any two subcontinua of  $X$  is connected;
- *hereditarily decomposable* provided that every subcontinuum of  $X$  is the union of two of its proper subcontinua;
- a  *$\lambda$ -dendroid* if it is hereditarily unicoherent and hereditarily decomposable;
- a *dendroid* if it is hereditarily unicoherent and arcwise connected;

- a *dendrite* if it is locally connected and contains no simple closed curve;
- a *tree* if it is a graph containing no simple closed curve.

Thus, we have the following inclusions.

$$\begin{aligned} \{\text{an arc}\} &\subset \{\text{trees}\} \subset \{\text{dendrites}\} \subset \{\text{dendroids}\} \\ &\subset \{\lambda\text{-dendroids}\} \subset \{\text{hereditarily unicoherent continua}\}. \end{aligned}$$

Recall the following characterization of dendrites [11, Theorem, p. 157].

**Proposition 2.3.** *A continuum  $X$  is a dendrite if and only if each subcontinuum of  $X$  is a monotone retract of  $X$ .*

The monotone retraction of Proposition 2.3 is the *first point map* of [18, 10.25 and 10.26, p. 176].

In the sequel, we will use a well known special dendrite, namely the *Gehman dendrite*  $G$ . (See [18, Example 10.39, p. 186]. Note that the infinite binary tree is another name of this dendrite; see e.g., [12, Example 1.6, p. 45].) Recall that  $G$  can be characterized as the only dendrite whose set of end points is homeomorphic to the Cantor set and whose branch points are of order 3 only (see [19, p. 100]).

The following proposition is an immediate consequence of definitions.

**Proposition 2.4.** *Let  $A$  be a subspace of a space  $X$ , and let a mapping  $f : X \rightarrow X$  be such that  $f(A) \subset A$ . Then*

$$(2.4.1) \quad P(f|A) = A \cap P(f);$$

$$(2.4.2) \quad R(f|A) = A \cap R(f).$$

The next result is a reformulation of [6, Proposition 4.1, p. 113]; compare also [2, Proposition 2.2].

**Proposition 2.5.** *Let  $f : X \rightarrow X$  be a mapping of a continuum  $X$  into itself. Then  $\{f^n(X)\}_{n=1}^{\infty}$  is a decreasing sequence of subcontinua of  $X$  and  $M(X, f) = \bigcap \{f^n(X) : n \in \mathbb{N}\}$  is a subcontinuum of  $X$  such that*

$$(2.5.1) \quad M(X, f) = \text{Lim } f^n(X);$$

$$(2.5.2) \quad f|M(X, f) : M(X, f) \rightarrow M(X, f) \text{ is a surjection};$$

$$(2.5.3) \quad P(f) \subset R(f) \subset M(X, f);$$

(2.5.4)  $P(f) = P(f|M(X, f))$  and  $R(f) = R(f|M(X, f))$ ;

(2.5.5)  $M(X, f)$  is a maximal subcontinuum of  $X$  satisfying (2.5.2).

**Remark 2.6.** As it is indicated in [2], Proposition 2.5 shows that, when a mapping  $f$  from a continuum  $X$  into itself is investigated, the whole dynamic for  $f$  is on the subcontinuum  $M(X, f)$  of  $X$ . So, in such a situation, we can replace  $X$  by  $M(X, f)$  and then assume that  $f$  is a surjection. However, we cannot reduce the study of the PR-property of  $X$  to the study of this property on  $M(X, f)$ , since for a given mapping  $f$  from  $X$  into itself, the sets  $X$  and  $M(X, f)$  do not have to be homeomorphic and, in general, a given map  $g$  from  $M(X, f)$  onto itself cannot be extended to a map from  $X$  into itself. In the next example we see that the PR-property on some set  $M(X, f)$  for some mapping  $f$  from  $X$  into itself, does not imply the PR-property of  $X$ .

**Example 2.7.** *The Gehman dendrite  $G$  does not have the PR-property, while for each arc  $A \subset G$  there is a monotone retraction  $f : G \rightarrow A$  for which  $A = M(G, f)$ , so  $M(G, f)$  has the PR-property.*

*Proof:* It is known that  $G$  does not have the PR-property; see [15, Section 2, p. 460], and compare [4, Corollary 3.4, p. 136] and [14, Theorem 2, p. 222]). Take an arc  $A \subset G$ . By Proposition 2.3 there is a monotone retraction  $f : G \rightarrow A$ . Then  $M(G, f) = A$ , so it has the PR-property, as needed.  $\square$

The following question is related to the above example.

**Question 2.8.** *Does there exist a continuum  $X$  having the PR-property, and a mapping  $f$  from  $X$  into itself such that  $M(X, f)$  does not have the PR-property?*

### 3. $\lambda$ -DENDROIDS AS COMPACTIFICATIONS

Given a  $\lambda$ -dendroid  $X$  we denote by  $\mathcal{P}(X)$  the family of all subcontinua  $S$  of  $X$  such that for each finite cover of  $X$  the elements of which are subcontinua of  $X$ , the continuum  $S$  is contained in a member of the cover. Thus, members of  $\mathcal{P}(X)$  are proper subcontinua of  $X$ . A (transfinite) well-ordered sequence (numbered with ordinals  $\alpha$ ) of nondegenerate subcontinua  $X_\alpha$  of a  $\lambda$ -dendroid

$X$  is said to be *normal* provided that the following conditions are satisfied:

$$\begin{aligned} X_1 &= X; \\ X_{\alpha+1} &\in \mathcal{P}(X_\alpha); \\ X_\beta &= \bigcap \{X_\alpha : \alpha < \beta\} \text{ for each limit ordinal } \beta. \end{aligned}$$

The *depth*  $k(X)$  of a  $\lambda$ -dendroid  $X$  is defined as the minimum ordinal number  $\eta$  such that the order type of each normal sequence of subcontinua of  $X$  is not greater than  $\eta$ . The reader is referred to [13] and [17] for additional information related to this concept. The following three assertions concerning the depth are known [13, theorems 1, 2, and 3, pp. 94 and 95].

**Statement 3.1.** *Let  $X$  and  $Y$  be  $\lambda$ -dendroids.*

- (3.1.1) *If  $Y \subset X$ , then  $k(Y) \leq k(X)$ .*
- (3.1.2)  *$X$  is locally connected (i.e., it is a dendrite) if and only if  $k(X) = 1$ .*
- (3.1.3) *If  $Y$  is a continuous image of  $X$ , then  $k(Y) \leq k(X)$ .*

A subcontinuum  $Q$  of a continuum  $X$  is said to be *terminal* provided that for every subcontinuum  $K$  of  $X$  if  $K \cap Q \neq \emptyset$  then either  $K \subset Q$  or  $Q \subset K$ . We need the following result (see [1, Theorem, p. 35] and [6, Theorem 3.1, p. 111]).

**Theorem 3.2.** *If  $M$  is a locally compact, noncompact metric space, then each continuum is a remainder of  $M$  in some compactification of  $M$  as a terminal subcontinuum of the compactification.*

The construction described below is a modification of the one in [6, §3, p. 111].

Let a dendrite  $D$  and points  $q_1, \dots, q_n$  of  $D$  be given for some positive integer  $n$ . Let  $Q_1, \dots, Q_n$  be continua. Choose in  $D$  closed connected and mutually disjoint neighborhoods  $U_1, \dots, U_n$  of points  $q_1, \dots, q_n$ . Then for each  $i \in \{1, \dots, n\}$ , the sets  $U_i \setminus \{q_i\}$  are locally compact and noncompact; thus applying Theorem 3.2 to each of them, we construct in a standard way a compactification

$$(\gamma) \quad \gamma : (D \setminus \{q_1, \dots, q_n\}) \rightarrow \gamma(D \setminus \{q_1, \dots, q_n\})$$

such that

- ( $\gamma$ .1)  $X = \text{cl}_X(\gamma(D \setminus \{q_1, \dots, q_n\}))$  is a continuum;
- ( $\gamma$ .2) the remainder  $X \setminus \gamma(D \setminus \{q_1, \dots, q_n\})$  consists of  $n$  components  $Q_1, \dots, Q_n$ ;
- ( $\gamma$ .3) for each index  $i \in \{1, \dots, n\}$ , the continuum  $Q_i$  is a terminal subcontinuum of  $X$ .

The following proposition is a consequence of the definitions.

**Observation 3.3.** If the inserted continua  $Q_i$  are  $\lambda$ -dendroids, then the resulting continuum  $X$  satisfying ( $\gamma$ .1)-( $\gamma$ .3) is a  $\lambda$ -dendroid, too.

Thus, the concept of the depth  $k(X)$  is well defined for such  $X$  (and for all subcontinua of  $X$ ). We say that the inserted continua  $Q_i$  have *the same finite depth* provided that there is  $d \in \mathbb{N}$  such that for each  $i \in \{1, \dots, n\}$  we have  $k(Q_i) = d$ .

To make formulation of the forthcoming results shorter, accept the following definition.

**Definition 3.4.** Let  $\Delta$  denote the class of all  $\lambda$ -dendroids which are obtained from a dendrite  $D$  by replacing finitely many of its points  $q_1, \dots, q_n$  by  $\lambda$ -dendroids  $Q_1, \dots, Q_n$  of the same finite depth using a compactification  $\gamma$  with ( $\gamma$ .1)-( $\gamma$ .3). For a  $\lambda$ -dendroid  $X \in \Delta$ , we put

$$(3.4.1) \quad A = \bigcup \{Q_i : i \in \{1, \dots, n\}\} \text{ and } H = X \setminus A.$$

**Theorem 3.5.** *Let a  $\lambda$ -dendroid  $X$  belong to the class  $\Delta$ , and let  $Y$  be a nondegenerate subcontinuum of  $X$  such that, for some  $i \in \{1, \dots, n\}$ ,*

$$(3.5.1) \quad Y \cap Q_i \neq \emptyset \neq Y \setminus Q_i.$$

*Then*

$$(3.5.2) \quad k(Y) \geq 1 + k(Q_i).$$

*Proof:* Terminality of the subcontinua  $Q_i$  of  $X$  (see condition ( $\gamma$ .3)) implies by (3.4.1) that  $Q_i \subset Y$ . Therefore, for each finite cover of  $Y$  whose elements are subcontinua of  $Y$ , the continuum  $Q_i$  is contained in some element of the cover. Thus, the family  $\mathcal{P}(Y)$  (see the definition of depth) consists of the continua  $Q_j$  intersecting  $Y$  (thus contained in  $Y$  by their terminality), of their



subcontinua, and of the singletons in  $Y \setminus A$ ; consequently, for each normal sequence in  $Y$  we have  $Y_1 = Y$  and

$$Y_2 \in \bigcup \{C(Q_j) : j \in \{1, \dots, n\} \text{ with } Q_j \cap Y \neq \emptyset\} \cup F_1(Y \setminus A),$$

while each normal sequence in  $Q_i$  starts with  $Q_i$  and has its second term in  $C(Q_i) \setminus \{Q_i\}$ , and thus, the number of its terms is less by one than that of the corresponding normal sequence in  $Y$ . So (3.5.2) follows from finiteness of the depth of each  $Q_i$  by the definition of the depth. The proof is complete.  $\square$

Given a compact space  $X$ , we denote by  $N(X)$  the set of points of  $X$  at which  $X$  is not locally connected. The next statement is known [9, (3), p. 28].

**Statement 3.6.** *If  $f$  is a mapping of a compact space  $X$ , then*

$$N(f(X)) \subset f(N(X)).$$

**Proposition 3.7.** *Let a  $\lambda$ -dendroid  $X$  be in  $\Delta$ , and let  $A$  and  $H$  be defined by (3.4.1). If  $f : X \rightarrow X$  is a surjection, then*

$$(3.7.1) \quad A \subset f(A);$$

$$(3.7.2) \quad \text{there are no indices } i, j \in \{1, \dots, n\} \text{ such that } f(Q_i) \cap Q_j \neq \emptyset \neq f(Q_i) \cap (X \setminus Q_j);$$

$$(3.7.3) \quad \text{there is no index } i \in \{1, \dots, n\} \text{ such that } f(Q_i) \subset H.$$

*Proof:* Since  $N(X) = A$ , Statement 3.6 implies (3.7.1). To see (3.7.2), suppose to the contrary that there are some indices  $i$  and  $j$  with (3.7.2). Then applying Theorem 3.5, we can take  $f(Q_i)$  for  $Y$  in (3.5.1), and we obtain  $k(f(Q_i)) \geq 1 + k(Q_j)$  by (3.5.2). Since  $k(Q_i) \geq k(f(Q_i))$  by (3.1.3) of Statement 3.1, the two inequalities lead to  $d \geq 1 + d$  (where  $d = k(Q_i)$  means the common finite depth of all  $Q_i$ ), a contradiction. To show the rest of the conclusion we again suppose that there is an index  $i$  satisfying (3.7.3), and without loss of generality, we can take  $i = 1$ . Then applying (3.7.1), we have

$$A \subset \bigcup \{f(Q_i) : i \in \{2, \dots, n\}\}.$$

So we see that  $n - 1$  continua  $f(Q_i)$  have to cover  $n$  continua  $Q_j$ , and thus one of the continua  $f(Q_i)$  has to intersect at least two distinct continua  $Q_j$ , which is impossible by (3.7.2). The proof is complete.  $\square$

**Corollary 3.8.** *Let a  $\lambda$ -dendroid  $X$  be in  $\Delta$ . If  $f : X \rightarrow X$  is a surjection, then for each  $i \in \{1, \dots, n\}$  there exists  $j \in \{1, \dots, n\}$  such that  $f(Q_i) = Q_j$ . Furthermore, the correspondence between indices  $i$  and  $j$  is one-to-one and it maps  $\{1, \dots, n\}$  onto itself.*

*Proof:* Take an arbitrary  $f(Q_i)$  and observe that it intersects some  $Q_j$  by (3.7.3) of Proposition 3.7, but it cannot intersect the complement of  $f(Q_j)$  by (3.7.2). Thus,

$$(3.8.1) \quad f(Q_i) \subset Q_j.$$

We claim that

$$(3.8.2) \quad \text{there are no indices } i_1, i_2 \in \{1, \dots, n\} \text{ such that } f(Q_{i_1}) \cap Q_j \neq \emptyset \neq f(Q_{i_2}) \cap Q_j \text{ for some } j \in \{1, \dots, n\}.$$

Indeed, if such indices  $i_1, i_2$ , and  $j$  would exist, then  $f(Q_{i_1}) \cup f(Q_{i_2}) \subset Q_j$  by (3.8.1), and since all  $Q_i$ 's have to be covered by their images according to inclusion (3.7.1), we would have

$$\bigcup \{Q_i : i \in \{1, \dots, n\} \setminus \{j\}\} \subset \bigcup \{f(Q_i) : i \in \{1, \dots, n\} \setminus \{i_1, i_2\}\},$$

so  $n - 1$  continua  $Q_i$  have to be covered by  $n - 2$  continua  $f(Q_i)$ , and consequently, one of  $f(Q_i)$  must intersect at least two distinct continua  $Q_i$ , which again is impossible by (3.7.2). Thus, (3.8.2) is shown. Now (3.8.1) and (3.8.2) lead to  $f(Q_i) = Q_j$  by (3.7.1). So the first part of the conclusion is shown. Finally, (3.7.1) and (3.8.2) imply that the correspondence is both surjective and one-to-one. The proof is complete.  $\square$

Corollary 3.8 can be reformulated in the following form.

**Corollary 3.9.** *Let a  $\lambda$ -dendroid  $X$  be in  $\Delta$ . Then each surjective mapping  $f : X \rightarrow X$  permutes the continua  $Q_i$  for  $i \in \{1, \dots, n\}$ .*

For a  $\lambda$ -dendroid  $X \in \Delta$ , let  $D$  be the dendrite as in Definition 3.4, and let  $p : X \rightarrow D$  be the natural projection, i.e., such a mapping that  $p(Q_i) = \{q_i\}$  for each  $i \in \{1, \dots, n\}$  and that  $p|_H = \gamma^{-1} : H \rightarrow (D \setminus \{q_1, \dots, q_n\})$  is a one-to-one mapping. Thus,  $p|_H$  is a homeomorphism. Let  $f : X \rightarrow X$  be a surjective mapping. Define a mapping  $g : D \rightarrow D$  by

$$(*) \quad g(z) = p(f(p^{-1}(z))) \quad \text{for each point } z \in D.$$

In particular, since  $p^{-1}(q_i) = Q_i$  for each  $i \in \{1, \dots, n\}$ , hence  $g(q_i) = p(f(p^{-1}(q_i)))$ . Therefore,  $g$  is well defined, and its continuity is a consequence of the definition. Further, Statement 3.7

implies a similar statement for the mapping  $g$ . More precisely, we have the following corollary.

**Corollary 3.10.** *Let a  $\lambda$ -dendroid  $X$  be a member of the class  $\Delta$ , a mapping  $f : X \rightarrow X$  be a surjection,  $p : X \rightarrow D$  be the natural projection, and  $g : D \rightarrow D$  be defined by (\*). Then the following assertions hold.*

- (3.10.1)  $p|_H : H \rightarrow D \setminus \{q_1, \dots, q_n\}$  is a homeomorphism;
- (3.10.2)  $g(\{q_1, \dots, q_n\}) = \{q_1, \dots, q_n\}$ .
- (3.10.3)  $g(D \setminus \{q_1, \dots, q_n\}) \supset D \setminus \{q_1, \dots, q_n\}$ .
- (3.10.4) The mapping  $g$  permutes the points  $q_i$  for  $i \in \{1, \dots, n\}$ .
- (3.10.5)  $q_i \in P(g)$  for each  $i \in \{1, \dots, n\}$ .
- (3.10.6)  $p(f(x)) = g(p(x))$  for each point  $x \in X$ ; i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ p \downarrow & & \downarrow p \\ D & \xrightarrow{g} & D \end{array}$$

- (3.10.7)  $p(f^m(x)) = g^m(p(x))$  for each point  $x \in X$  and for each  $m \in \mathbb{N}$ ; i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f^m} & X \\ p \downarrow & & \downarrow p \\ D & \xrightarrow{g^m} & D \end{array}$$

- (3.10.8) for each  $m \in \mathbb{N}$ ,

$$f^m(A) = A.$$

*Proof:* Item (3.10.1) has already been shown when the mapping  $p$  was defined. Items (3.10.2)–(3.10.4) and (3.10.6) are consequences of definitions. (3.10.5) follows from (3.10.4). (3.10.6) implies (using simple induction argument) commutativity of diagram (3.10.7). Finally, (3.10.8) is a consequence of Corollary 3.9.  $\square$

#### 4. PR-PROPERTY FOR THE $\lambda$ -DENDROIDS OF THE CLASS $\Delta$

We apply notation introduced in sections 2 and 3.

**Proposition 4.1.** *For each point  $x \in H$ , we have the following two equivalences:*

(4.1.1)  $x \in P(f)$  is equivalent to  $p(x) \in P(g)$ ;

(4.1.2)  $x \in R(f)$  is equivalent to  $p(x) \in R(g)$ .

*Proof:* The present proof, given here for the sake of completeness and for the reader's convenience only, is a small modification of the proof of [6, Proposition 5.4, p. 116].

We start with showing (4.1.2). Let  $x \in R(f)$ , and let  $U$  be an open neighborhood of  $p(x)$  in  $D$ . Thus, there is  $m \in \mathbb{N}$  such that  $f^m(x) \in p^{-1}(U)$ . So,  $p(f^m(x)) \in p(p^{-1}(U)) = U$ . Since by commutativity of diagram (3.10.7) we have  $p \circ f^m = g^m \circ p$ , it follows that  $g^m(p(x)) \in U$ , which shows one implication. Now let  $p(x) \in R(g)$ , and let  $V$  be an open neighborhood of  $x$  in  $X$ . Since  $H$  is open in  $X$  by its definition, and since the partial mapping  $p|_H : H \rightarrow D \setminus \{q_1, \dots, q_n\}$  is a homeomorphism,  $p(V \cap H)$  is an open neighborhood of  $p(x)$ . Thus by the assumption, there is  $m \in \mathbb{N}$  such that  $g^m(p(x)) \in p(V \cap H)$ . Since  $g^m \circ p = p \circ f^m$  as previously, we have  $p(f^m(x)) \in p(V \cap H)$ , whence  $p^{-1}(p(f^m(x))) \in p^{-1}(p(V \cap H)) = V \cap H \subset V$ , and so  $f^m(x) \in V$ , i.e.,  $x \in R(f)$ . Equivalence (4.1.2) is shown.

To prove (4.1.1), we proceed analogously, omitting the consideration of neighborhoods. Details are left to the reader. The proof is complete.  $\square$

The next theorem corresponds to [6, Theorem 5.7, p. 116], where PR-property was considered for surjective mappings of  $\lambda$ -dendroids belonging to a subclass of the class  $\Delta$ .

**Theorem 4.2.** *Let a  $\lambda$ -dendroid  $X$  be a member of the class  $\Delta$ . If*

(4.2.1) *the dendrite  $D$  has the PR-property, and*

(4.2.2) *the union  $A = Q_1 \cup \dots \cup Q_n$  has the PR-property,*

*then for each surjective mapping  $f : X \rightarrow X$ , the equality  $\text{cl}_X(P(f)) = \text{cl}_X(R(f))$  holds, i. e.,  $X$  has the PR-property for surjections.*

*Proof:* Let a surjective mapping  $f : X \rightarrow X$  be given. According to Definition 1.2 and (1.1.1), it is enough to show that

$$\text{cl}_X(R(f)) \subset \text{cl}_X(P(f)).$$

So, take  $x \in \text{cl}_X(R(f))$  and a sequence of points  $x_k$  of  $R(f)$  tending to  $x$ , and consider three cases.

**Case 1.**  $x \in H$ . Since  $H$  is an open subset of  $X$ , we may assume  $x_k \in H$  for each  $k \in \mathbb{N}$ . Applying equivalence (4.1.2) of Proposition 4.1, we get  $p(x_k) \in R(g)$ . Hence,  $p(x) \in \text{cl}_D(R(g))$ . Since  $D$  has the PR-property by (4.2.1), we have  $p(x) \in \text{cl}_D(P(g))$ . Since  $x \in H$  implies  $p(x) \in \gamma^{-1}(H) = D \setminus \{q_1, \dots, q_n\}$  (which is an open subset of  $D$ ), there is a sequence of points  $t_k \in (D \setminus \{q_1, \dots, q_n\}) \cap P(g)$  tending to  $p(x)$ . Applying equivalence (4.1.1) of Proposition 4.1, we see that  $p^{-1}(t_k) \in P(f)$ . Since  $p|_H$  is a homeomorphism, the sequence  $\{p^{-1}(t_k)\}$  tends to  $p^{-1}(p(x)) = x$ , thereby  $x \in \text{cl}_X(P(f))$ .

**Case 2.**  $x \in Q_i$  and all  $x_k \in Q_i$  for some  $i \in \{1, \dots, n\}$ . By Corollary 3.9 there is  $m \in \mathbb{N}$  such that  $f^m(Q_i) = Q_i$ . Since the condition  $x_k \in R(f)$  is equivalent to the condition  $x_k \in R(f^m)$  for each  $m \in \mathbb{N}$  (see [10, Theorem I, p. 126] and compare (1.1.2)), we get  $x_k \in R(f^m|_{Q_i}) \subset \text{cl}_{Q_i}(R(f^m|_{Q_i}))$ , which implies that  $x_k \in \text{cl}_{Q_i}(P(f^m|_{Q_i}))$ , by assumption (4.2.2). Consequently,

$$x \in \text{cl}_X(P(f^m|_{Q_i})) \subset \text{cl}_X(P(f))$$

according to (2.4.1) of Proposition 2.4.

**Case 3.**  $x \in Q_i$  for some  $i \in \{1, \dots, n\}$ , and all  $x_k \in H$ . As previously, we take  $m \in \mathbb{N}$  such that  $f^m(Q_i) = Q_i$ , and again we have  $x_k \in R(f^m)$  by [10, Theorem I, p. 126], whence we infer by (4.1.2) of Proposition 4.1 that  $p(x_k) \in R(g^m) \subset \text{cl}_D(R(g^m))$ . Applying PR-property for  $D$  according to (4.2.1), we see that  $p(x_k) \in \text{cl}_D(P(g^m))$ . Thus, for each  $k \in \mathbb{N}$  there is a sequence  $\{x_k(r)\}$  of points of  $H$  such that  $x_k = \lim_{r \rightarrow \infty} x_k(r)$  and  $p(x_k(r)) \in P(g^m)$ . By equivalence (4.1.1) of Proposition 4.1, we get  $x_k(r) \in P(f^m)$ . Thus,  $x_k \in \text{cl}_X(P(f^m))$  for each  $k \in \mathbb{N}$ , and consequently,  $x \in \text{cl}_X(P(f^m)) = \text{cl}_X(P(f))$  by (1.1.2).

Therefore, the needed inclusion is shown, and thus  $X$  has the PR-property for surjective mappings. The proof is then complete.  $\square$

As regards assumption (4.2.2) of the above theorem, note the following observation.

**Observation 4.3.** Let continua  $Q_1, \dots, Q_n$  be mutually disjoint. Then the union  $A = Q_1 \cup \dots \cup Q_n$  has the PR-property if and only if each of the continua  $Q_i$  for  $i \in \{1, \dots, n\}$  has the PR-property.

*Proof:* Assume that each continuum  $Q_i$ , where  $i \in \{1, \dots, n\}$ , has the PR-property. Let  $f : A \rightarrow A$ . In the family  $\{Q_1, \dots, Q_n\}$ ,

choose continua  $Q_{i_1}, \dots, Q_{i_m}$  such that  $Q_{i_k} \cap M(A, f) \neq \emptyset$  for each  $k \in \{1, \dots, m\}$ . Since the union  $B = \bigcup\{Q_{i_k} : k \in \{1, \dots, m\}\}$  is open in  $A$ , there exists  $u \in \mathbb{N}$  such that  $f^u(A) \subset B$ . Notice that for each  $k \in \{1, \dots, m\}$  there is  $j \in \{1, \dots, m\}$  such that  $f^u(Q_{i_k}) \subset Q_{i_j}$ . Hence,  $f^u$  induces a permutation of the set  $\{i_1, \dots, i_m\}$ , and thereby there exists  $v \in \mathbb{N}$  such that  $(f^u)^v(Q_{i_k}) \subset Q_{i_k}$  for each  $k \in \{1, \dots, m\}$ . Since by (1.1.2)

$$P(f) = P(f^{uv}) = \bigcup\{P(f^{uv}|_{Q_{i_k}}) : k \in \{1, \dots, m\}\}$$

and

$$R(f) = R(f^{uv}) = \bigcup\{R(f^{uv}|_{Q_{i_k}}) : k \in \{1, \dots, m\}\},$$

and since each continuum  $Q_{i_k}$  has the PR-property, we conclude that  $\text{cl}_A(P(f)) = \text{cl}_A(R(f))$ . Thus, the “if” part is shown.

To see the “only if” part, assume that  $A$  has the PR-property. Then for a fixed  $i_0$  and for a mapping  $f_{i_0} : Q_{i_0} \rightarrow Q_{i_0}$  one can define  $f : A \rightarrow A$  by  $f(x) = f_{i_0}(x)$  for  $x \in Q_{i_0}$ , and  $f(x) = x$  for  $x \in Q_i$  with  $i \neq i_0$ . Then the PR-property for  $Q_{i_0}$  follows again from Proposition 2.4.  $\square$

**Remark 4.4.** Note that the assumption that the continua  $Q_i$  are pairwise disjoint is essential in the above observation, because the circle, which does not have the PR-property, is the union of two arcs having their end points in common only, each of which has the property.

Accept the following definition.

**Definition 4.5.** A compact space  $C$  is said to have the PR-property *hereditarily*, provided that each subcontinuum of  $C$  has the PR-property.

Note that each tree has the PR-property hereditarily. Other known examples of continua having the PR-property have the property hereditarily. Thus, the following question (related to Question 2.8) is of some importance.

**Question 4.6.** *Let a continuum (or a compact space) have the PR-property. Does it follow that it has the PR-property hereditarily?*

In connection with assumption (4.2.1) of Theorem 4.2, recall the following theorem; see [14, Theorem 2, p. 222].

**Theorem 4.7.** *A dendrite  $X$  has the PR-property if and only if  $X$  does not contain any copy of the Gehman dendrite  $G$ .*

Since each subcontinuum of a dendrite  $X$  with no copy of  $G$  in  $X$  also has the same property, we have a corollary.

**Corollary 4.8.** *If a dendrite has the PR-property, then it has the property hereditarily.*

The next theorem is the main result of the paper.

**Theorem 4.9.** *Let a  $\lambda$ -dendroid  $X$  be a member of the class  $\Delta$ . If (4.2.1) the dendrite  $D$  has the PR-property, and (4.9.1) the union  $A = Q_1 \cup \dots \cup Q_n$  has the PR-property hereditarily, then  $X$  has the PR-property.*

*Proof:* Let a mapping  $f : X \rightarrow X$  be given. Observe again (as in the beginning of the proof of Theorem 4.2) that by Definition 1.2 and (1.1.1) it is enough to show that

$$\text{cl}_X(R(f)) \subset \text{cl}_X(P(f)).$$

Let the continuum  $M(X, f)$  be as in Proposition 2.5. Since  $M(X, f)$  is the maximal subcontinuum of  $X$  such that  $f|_{M(X, f)}$  is a surjection by (2.5.2) and (2.5.6), the whole dynamics for  $f$  is on the subcontinuum  $M(X, f)$  of  $X$  (compare Remark 2.6). Thus, the following three cases have to be considered.

**Case 1.**  $M(X, f) \subset H = X \setminus A$ . Since  $H$  is homeomorphic to  $D \setminus \{q_1, \dots, q_n\}$  according to (3.10.1), it follows that  $M(X, f)$  is homeomorphic to a subcontinuum of  $D$ , so it has the PR-property by (4.2.1) and Corollary 4.8; i. e.,  $P((f|_{M(X, f)}) = R(f|_{M(X, f)})$ . Therefore, the conclusion follows from (2.5.4) of Proposition 2.5.

**Case 2.**  $M(X, f) \subset A$ . Then  $M(X, f)$  has the PR-property by (4.2.2), and the conclusion follows again by (2.5.4) of Proposition 2.5.

**Case 3.**  $M(X, f) \cap A \neq \emptyset \neq M(X, f) \cap H$ . Then  $X' = M(X, f)$  is a  $\lambda$ -dendroid which is a member of the class  $\Delta$ , and the partial mapping  $f|_{X'} : X' \rightarrow X'$  is a surjection. Thus,  $X'$  has the PR-property by Theorem 4.2, and the conclusion follows once more by (2.5.4) of Proposition 2.5.

The proof is complete.  $\square$

**Remark 4.10.** Observe that if a  $\lambda$ -dendroid  $X$  is a member of the class  $\Delta$ , then each of subcontinua of  $X$  is also a member of  $\Delta$ . Further, if  $X$  satisfies the assumptions of Theorem 4.9, then each subcontinuum of  $X$  does. Therefore, we have the following corollary.

**Corollary 4.11.** *Let a  $\lambda$ -dendroid  $X$  be a member of the class  $\Delta$ . If (4.2.1) and (4.9.1) hold, then  $X$  has the PR-property hereditarily.*

**Question 4.12.** *Can the assumption in (4.9.1) of Theorem 4.9 that  $A$  has the PR-property hereditarily be relaxed to having the PR-property only, i. e., to the assumption (4.2.2)? (Compare Question 4.6.)*

Observe that if each of the inserted continua  $Q_i$  is a dendrite which does not contain any copy of the Gehman dendrite  $G$ , then assumption (4.9.1) is satisfied, and therefore Theorem 4.9 and Corollary 4.11 imply the next result.

**Corollary 4.13.** *Let a  $\lambda$ -dendroid  $X$  be a member of the class  $\Delta$ . If each of the continua  $Q_i$  is a dendrite which does not contain any copy of the Gehman dendrite  $G$ , then  $X$  has the PR-property hereditarily.*

**Question 4.14.** *Can similar results to Theorem 4.9 be obtained if the number of the inserted continua  $Q_i$  (see §3) is countable and their union is closed in the  $\lambda$ -dendroid  $X$ ?*

## 5. FINAL QUESTIONS

The authors do not know if all the assumptions made in the results proved in the previous sections of the paper are essential, and/or if the results can be extended or generalized in some way. In particular, the following questions related to the subject of the paper seem to be interesting.

**Question 5.1.** *Is the assumption that all inserted continua  $Q_i$  are of the same (finite) depth (see Definition 3.4 of the class  $\Delta$ ) essential in theorems 4.2 and 4.9?*

**Question 5.2.** *Can theorems 4.2 and/or 4.9 be generalized to  $\lambda$ -dendroids  $X$  in which the depth of some (of all) inserted continua  $Q_i$  is infinite?*



**Question 5.3.** *Is the condition that the number  $n$  of the continua  $Q_i$  is finite an essential assumption in theorems 4.2 and/or 4.9? Under what conditions can the result be generalized to  $\lambda$ -dendroids  $X$  in which a) the number of the considered continua  $Q_i$  is countable? b) the union  $A$  of the considered continua  $Q_i$  is closed in  $X$ ?*

As a particular case of the above questions we have the following.

**Question 5.4.** *Can theorems 4.2 and/or 4.9 be extended to some continua  $X$  obtained as compactifications of complements of closed countable subsets of (a) trees (b) dendrites?*

All of the above questions are very particular cases of more general problems, which can be treated as a research program in the area, and which (at the present moment) seem to be rather far from any final solution.

**Problem 5.5.** *What  $\lambda$ -dendroids have the PR-property?*

**Problem 5.6.** *Let an upper semicontinuous decomposition  $\mathcal{D}$  of a continuum  $X$  into continua (possibly degenerate) be given. Consider the following three conditions:*

- (5.6.1) *the continuum  $X$  has the PR-property;*
- (5.6.2) *all (or some) members of the decomposition  $\mathcal{D}$  have the PR-property;*
- (5.6.3) *the decomposition space  $X/\mathcal{D}$  has the PR-property.*

*What are the interrelations among these conditions?*

In particular, the following question related to theorems 4.2 and 4.9 is especially interesting.

**Question 5.7.** *Under what assumptions do conditions (5.6.2) and (5.6.3) imply (5.6.1)?*

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