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	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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HOMEOMORPHISMS OF $\overline{U} \times R$ AND ROTATION NUMBER

PAUL FABEL

ABSTRACT. Suppose $U \subset R^2$ is bounded open and contractible and $H : \overline{U} \times R \to \overline{U} \times R$ is a homeomorphism leaving invariant $U \times R$. If ∂U is locally connected and not a simple closed curve, H induces a homeomorphism of the solid cylinder leaving invariant sufficiently many vertical lines to determine a rotation number. If ∂U is not locally connected, H admits a natural notion of rotation number despite a general absence of an induced homeomorphism of the solid cylinder.

1. INTRODUCTION

The following theorem, credited to H. D. Ursell and L. C. Young [11], is central to applications ([1],[2],[3],[4],[5],[9]) of prime end theory to dynamical systems:

Theorem 1.1. *Suppose $U \subset R^2$ is bounded open and contractible, $\psi : U \to int(D^2)$ is conformal, and $h : \overline{U} \to \overline{U}$ is a homeomorphism such that $h(U) = U$. Then $\psi h \psi^{-1}$ can be extended to a homeomorphism of D^2.*

In particular if h is orientation preserving then $\overline{\psi h \psi^{-1}}_{S^1} : \partial D^2 \to \partial D^2$ determines a rotation number, measuring in a sense the average rotation by h of ∂U about U.

Specific examples of higher dimensional versions of Theorem 1.1 are in short supply despite a general criteria established in [4].

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For example, if ∂U is not locally connected, there is no corresponding version of Theorem 1.1 for domains $U \times R \subset R^3$ with $\Psi : U \times R \to int(D^2) \times R$ defined via $\psi(u,t) = (\psi(u),t))$. Example 4.1 exhibits a homeomorphism $H : \overline{U} \times R \to \overline{U} \times R$ such that $H(U \times R) = U \times R$ but $\Psi H \Psi^{-1}$ *cannot* be extended to a homeomorphism of $D^2 \times R$.

The main result of this paper (Theorem 8.6) salvages a notion of rotation number for such homeomorphisms $H : \overline{U} \times R \to \overline{U} \times R$. The basic idea is that H preserves the circular order of a certain collection of sets, each of which can be understood as the product of R with an interval of accessible prime ends of U. If ∂U is not locally connected this provides enough information to determine a homeomorphism $g : S^1 \to S^1$ whose rotation number we declare to be that of H.

On the other hand a useful "3 dimensional prime end theory" exists if ∂U is locally connected. Theorem 8.1 shows that Ψ induces a homeomorphism of the (3 cell) two point compactification of $D^2 \times R$. Moreover, if ∂U is not a simple closed curve, the cutpoints of ∂U help to determine a discrete collection of invariant boundary lines which, in turn, determine a rotation number.

Both notions of rotation number are invariant under topological conjugacy and agree with the usual rotation number of h in the special case $H(u,t) = (h(u),t)$ where $h : \overline{U} \to \overline{U}$ is a homeomorphism such that $h(U) = U$.

2. Preliminaries

Throughout this paper, suppose that $U \subset R^2$ is bounded open and contractible, ∂U denotes $\overline{U} \backslash U$, $D^2 \subset R^2$ denotes the closed unit disk, and $\psi : U \to int(D^2)$ is conformal. All function spaces will have the compact open topology.

Define $\Psi : U \times R \to int(D^2) \times R$ via $\Psi(u,t) = (\psi(u),t)$.

Let $D^3 = \{(x,y,z) \in R^3 | \sqrt{x^2+y^2+z^2} \leq 1\}$, the standard 3 cell.

If $J \subset S^1$ is connected let $int(J) = J$ if $J = \{x\}$. Otherwise, let $int(J)$ denote the union of all open intervals contained in J.

If Y is a topological space, attach two points $\{\infty\}$ and $\{-\infty\}$ to $Y \times R$ creating $(Y \times J) \cup \{\infty, -\infty\}$ topologized such that $(y_n,t_n) \to \{-\infty\}$ iff $t_n \to \infty$ and $(y_n,t_n) \to \{\infty\}$ iff $t_n \to -\infty$.

In similar fashion we attach two points $\{\infty\}$ and $\{-\infty\}$ to $Y \times (-1,1)$ and create a new space $\overline{Y \times (-1,1)}$ topologized such that $(y_n, t_n) \to \{-\infty\}$ iff $t_n \to -1$ and $(y_n, t_n) \to \{\infty\}$ iff $t_n \to 1$.

3. The map $\phi : X^* \to D^2$

The following procedure creates a complete metric space X^* whose underlying set can be seen as the union of U and the accessible prime ends of U.

Define a metric $d^* : U \times U \to R$ such that $d^*(x,y) < \varepsilon$ iff there exists a map $f : [0,1] \to U$ such that $f(0) = x, f(1) = y$ and $\forall s,t\ |f(t) - f(s)| < \varepsilon$. Let (X^*, d^*) denote the metric completion of (U, d^*). Let $\partial X^* = X^* \backslash U^*$.

Lemma 3.1. *There exists a map $\phi : X^* \to D^2$ such that ϕ is uniformly continuous, one to one and $\phi(U^*) = int(D^2)$.*

Proof: Let $\overline{id} : X^* \to \overline{U}$ denote the unique extension of the uniformly continuous identity map $id : U^* \to U$. Uniform continuity of $\psi(id)$ is essentially a consequence of Theorem 1.1 and is proved in Theorem 3.1 of [7]. Define $\phi : X^* \to D^2$ to be the unique continuous extension of $\psi(id)$. Suppose $x \neq y$ and $\{x,y\} \subset X^*$. If $\overline{id}(x) \neq \overline{id}(y)$ it follows from [10, Proposition 2.14] that $\psi(\overline{id}(x)) \neq \psi\overline{id}(y)$. If $\overline{id}(x) = \overline{id}(y)$ then $\{\overline{id}(x), \overline{id}(y)\} \subset \partial U$ and $\{x,y\} \subset \partial X^*$. Construct a closed topological disk $E \subset \overline{U}$ such that $\overline{id}(x) \in \partial E$ and $E \backslash \{\overline{id}(x)\} \subset U$. If $d^*(x,y) \neq 0$ then $int(E) \cap \partial U \neq \emptyset$ and $\overline{id}(x)$ and $\overline{id}(y)$ determine distinct prime ends. Let $z \in \partial E \backslash \{\overline{id}(x)\}$. By [10, Theorem 2.15], ψ determines a bijection between the prime ends of U and ∂D^2. In particular, $\partial E \backslash \{z\}$ determines distinct endcuts which map under ψ to distinct points of ∂D^2. Hence, ϕ is one to one. □

Remark 3.2. The injective map $\phi : X^* \to D^2$ need not be an embedding, for example, if $U \subset R^2$ is the region bounded by a "Warsaw circle." The canonical map from $\partial X^* \to \partial D^2$ is a continuous bijection but not a homeomorphism.

4. Failure of homeomorphism extension

Example 4.1. *Suppose $U \subset R^2$ is the interior of the standard Warsaw circle. There there exists $H \in G(U \times R)$ such that $\Psi H \Psi^{-1}$:*

$int(D^2) \times R \to int(D^2) \times R$ is not extendable to a homeomorphism of $D^2 \times R$.

Proof: Let $U \subset R^2$ be the interior of a standardly embedded Warsaw circle such that the impression of the bad prime end (the closed interval "limit bar") is precisely $\partial U \cap (\{0\} \times R)$. Define a homeomorphism $H : U \times R \to U \times R$ via $H(x, y, t) = (x, y, y + t)$. Note $\Psi H \Psi^{-1}$ is not continuously extendable near the bad prime end since H is not level preserving on the impression of the bad prime end. □

5. Homeomorphisms of S^1 and rotation number

The notion of rotation number of a homeomorphism of the unit circle dates back to Poincaré. In Robert L. Devaney's book [6], its properties are derived formally for diffeomorphisms, but the proofs are valid for homeomorphisms. See also [8] for a helpful survey.

Theorem 5.1. *Suppose $g : S^1 \to S^1$ is an orientation preserving homeomorphism, $\Pi : R \to S^1$ is the covering map $\Pi(\theta) = e^{2\pi i\theta}, x \in R$, and $G_1, G_2 : R \to R$ are homeomorphisms such that $\Pi(G_i) = g\Pi$. Then the following limits exist and differ by an integer: $\lim_{n\to\infty} \frac{G_1^n(x)}{n}$ and $\lim_{n\to\infty} \frac{G_2^n(x)}{n}$. This number (mod 1) is the* rotation number *of g and is invariant under the choice of x. If g reverses orientation, then g has two fixed points and we declare g to have rotation number* 0. *The rotation number of g is invariant under topological conjugacy (if $h : S^1 \to S^1$ is a homeomorphism and $\hat{g} = hgh^{-1}$ then $rot(\hat{g}) = rot(g)$). Finally, rot is a continuous function on the space of homeomorphisms of S^1.*

6. Sets with circular order and rotation number

Given four distinct points $\{x_1, x_2, x_3, x_4\} \subset S^1$, declare $x_1 < x_2 < x_3 < x_4$ if there exists a homeomorphism $g : S^1 \to S^1$ such that $g(i^n) = x_n$.

Note it is allowed that g reverses orientation.

Suppose $\{J_1, J_2, J_3, J_4\}$ is a collection of distinct pairwise disjoint nonempty subsets of S^1. Declare $J_1 < J_2 < J_3 < J_4$ if $x_1 < x_2 < x_3 < x_4$ whenever $x_i \in J_i$.

Suppose J is a proper connected subset of S^1. Define $int(J) = J$ if $J = \{x\}$, and otherwise define $int(J)$ to be the largest open interval contained in J.

Definition 6.1. Suppose $\hat{A}$ is a collection of pairwise disjoint subsets of S^1 and $h : \hat{A} \to \hat{A}$ is a bijection. Then h is order preserving if there exists a homeomorphism $g : S^1 \to S^1$ such that $g(J) = h(J)$ for each element $J \in \hat{A}$. The homeomorphism g is said to be *compatible* with h. If $|\hat{A}| \geq 3$, then declare h orientation preserving/reversing iff g is orientation preserving/reversing.

Lemma 6.2. *Suppose $\hat{A}$ is a collection of pairwise disjoint subsets of S^1 such that $|\hat{A}| \geq 4$ and $\cup_{J \in \hat{A}} J$ is dense in S^1. Suppose $J = int(J)$ for each $J \in \hat{A}$. Suppose $h : \hat{A} \to \hat{A}$ is a bijection leaving invariant the set of nontrivial elements of $\hat{A}$. Then h is order preserving if and only if $h(J_3)$ and $h(J_4)$ belong to the same component of $S^1 \backslash \{h(J_1) \cup h(J_2)\}$ whenever $J_1 < J_2 < J_3 < J_4$.*

Proof: Suppose h is order preserving. Let $g : S^1 \to S^1$ be a homeomorphism such that $g(J) = h(J)$ for each $J \in \hat{A}$. Suppose $J_1 < J_2 < J_3 < J_4$. Then J_3 and J_4 belong to the same component of $S^1 \backslash \{J_1 \cup J_2\}$. Since g is a homeomorphism, $g(J_3)$ and $g(J_4)$ belong to the same component of $S^1 \backslash \{g(J_1) \cup g(J_2)\}$. The conclusion follows since $g(J_i) = h(J_i)$. Conversely, suppose $h(J_3)$ and $h(J_4)$ belong to the same component of $S^1 \backslash \{h(J_1) \cup h(J_2)\}$ whenever $J_1 < J_2 < J_3 < J_4$. Suppose $J_1 < J_2 < J_3 < J_4$. Then either $h(J_1) < h(J_2) < h(J_3) < h(J_4)$ or $h(J_1) < h(J_2) < h(J_4) < h(J_3)$. Suppose, in order to obtain a contradiction, that $h(J_1) < h(J_2) < h(J_4) < h(J_3)$. Then $h(J_2)$ and $h(J_3)$ lie in opposite components of $S^1 \backslash \{h(J_1) \cup h(J_4))\}$. On the other hand, by hypothesis, $J_4 < J_1 < J_2 < J_3$ and hence, $h(J_2)$ and $h(J_3)$ lie in the same component of $S^1 \backslash \{h(J_1) \cup h(J_4))\}$ and we have a contradiction. Thus, $h(J_1) < h(J_2) < h(J_3) < h(J_4)$ and globally the bijection h either preserves or reverses orientation. Let $g : S^1 \to S^1$ be the unique homeomorphism mapping J linearly onto $h(J)$ whenever J is a nontrivial component of $\hat{A}$. □

Lemma 6.3. *Suppose $\hat{A}$ is a collection of pairwise disjoint connected subsets of S^1 such that $|\hat{A}| \geq 3$ and $\cup_{J \in \hat{A}} J$ is dense in S^1. Suppose $J = int(J)$ whenever $J \in \hat{A}$ and suppose $h : \hat{A} \to \hat{A}$*

is an order preserving bijection as demonstrated by the compatible orientation preserving homeomorphisms $g_1 : S^1 \to S^1$ *and* $g_2 : S^1 \to S^1$. *Then* g_1 *and* g_2 *have the same rotation number.*

Proof: Let $B \subset S^1$ denote the complement of the union of the open intervals of $A\hat{}$. Then $B \neq \emptyset$, $g_1(B) = B$, and $g_{2|B} = g_{1|B}$. Thus, there exist $b\tilde{} \in R$ and lifts $G_1 : R \to R$ and $G_2 : R \to R$, respectively, of g_1 and g_2 such that $\pi(b\tilde{}) \in B$ and $G_1^n(b\tilde{}) = G_2^n(b\tilde{})$ for every n. Hence, by Theorem 5.1, g_1 and g_2 have the same rotation number. □

7. The order preserving bijection $h : A\hat{} \to A\hat{}$

Define a surjection $p : D^2 \times [-1,1] \to D^3$ via $p(x,y,t) = ((1-|t|)x, (1-|t|)y, t)$. Note p collapses $D^2 \times \{-1\}$ and $D^2 \times \{1\}$ to points and is otherwise injective.

Note the map $\Phi_{X^* \times (-1,1)} : X^* \times (-1,1) \to D^2 \times (-1,1)$, defined such that $\Phi(x,t) = (\phi(x), t)$, can be continuously extended to an injective map $\Phi : \overline{X^* \times (-1,1)} \hookrightarrow \overline{D^2 \times (-1,1)}$ such that $\Phi(\infty) = \infty$ and $\Phi(-\infty) = -\infty$.

Define a metric on $X^* \times R$ via

$$d((x,t),(y,t)) = \max(d^*(x,y), |s-t|).$$

Define a set $A\hat{} \subset 2^{\partial D^2}$such that $\beta \in A\hat{}$ iff $\beta = int(\phi(X))$ for some path component $X \subset \partial X^*$.

Theorem 7.1. *The homeomorphism* $H : \overline{U} \times R \to \overline{U} \times R$ *induces a canonical homeomorphism* $H^* : X^* \times R \to X^* \times R$ *such that* $H^*(\partial X^* \times R) = \partial X^* \times R$ *and such that* H^* *is extendable to a homeomorphism of* $(X^* \times R) \cup \{\infty, -\infty\}$.

Proof: We first obtain, as follows, an induced homeomorphism $H^* : X^* \times R \to X^* \times R$ such that $H^*(U^* \times R) = U^* \times R$ and for each S and T there exists $S\hat{}$ and $T\hat{}$ such that $H^*(X^* \times [S,T]) \cup H^{*-1}(X^* \times [S,T]) \subset X^* \times [S\hat{}, T\hat{}]$. Suppose $\{(z_n, t_n)\}$ is Cauchy in $U^* \times R$. Since $\{t_n\}$ is Cauchy, choose S and T such that $\forall n (z_n, t_n) \in X^* \times [S,T]$. Moreover, for each n, m we may choose a path $\gamma_{nm} \subset U^* \times [S,T]$ such that γ_{nm} connects (z_n, t_n) to (z_m, t_m) and $\mathrm{diam}(\gamma_{nm}) < 2d((z_n,t_n),(z_m,t_m))$. Hence, $\lim_{n,m\to\infty}$

diam$(\gamma_{nm}) = 0$. However, since $\overline{U} \times [S, T]$ is compact, both $H_{\overline{U} \times [S,T]}$ and $H^{-1}_{\overline{U} \times [S,T]}$ are uniformly continuous.

So, $\lim_{n,m\to\infty}$ diam$(H(\gamma_{nm})) = 0 = \lim_{n,m\to\infty} diam(H^{-1}(\gamma_{nm}))$. Hence, H and H^{-1} preserve Cauchy sequences in $U^* \times R$ and thus are extendable to maps $H^* : X^* \times R \to X^* \times R$ and $(H^{-1})^* : X^* \times R \to X^* \times R$ such that $H^*((H^{-1})^*) = (H^{-1})^* H = id$. Since id is a bijection it follows that H^* and $(H^{-1})^*$ are bijections and hence homeomorphisms. By definition, $H(U \times R) = H^{-1}(U \times R) = U \times R$. Since compact subsets of $\overline{U} \times R$ are bounded, we may choose $S\hat{}$ and $T\hat{}$ such that the compactum $H(\overline{U} \times [S, T]) \cup H^{-1}(\overline{U} \times [S, T]) \subset \overline{U} \times [S\hat{}, T\hat{}]$. Note that $\overline{U} \times [S, T]$ (and hence $H(\overline{U} \times [S, T])$) separates $\overline{U} \times R$ into two components. Since $\overline{U} \times (T, \infty)$ is connected, $H(\overline{U} \times (T, \infty))$ intersects and contains exactly one component of $(\overline{U} \times R) \backslash (\overline{U} \times [S\hat{}, T\hat{}])$. Thus if $(z_n, t_n) \to \infty$, then $H(z_n, t_n)$ converges either to ∞ or to $-\infty$. Thus, both H^* and $(H^*)^{-1}$ are extendable to maps of $(X^* \times R) \cup \{\infty, -\infty\}$, and both must be homeomorphisms since $H^*(H^{*-1})$ fixes pointwise a dense set. □

Corollary 7.2. *The induced homeomorphism* $H^* : X^* \times R \to X^* \times R$ *induces a canonical homeomorphism* $H^{**} : \overline{X^* \times (-1, 1)} \to \overline{X^* \times (-1, 1)}$ *and a canonical bijection* $h : A\hat{} \to A\hat{}$.

Proof: Let $H^* : X^* \times R \to X^* \times R$ be induced from $H : \overline{U} \times R \to \overline{U} \times R$ as in Theorem 7.1. Let $T : R \to (-1, 1)$ be any homeomorphism. Define a homeomorphism $H^{**} : \overline{X^* \times (-1, 1)} \to \overline{X^* \times (-1, 1)}$ such that $H^{**}(x, T(t)) = (y, T(s))$ if $H^*(x, t) = (y, s)$. Since $H^{**}(\partial X^* \times (-1, 1)) = \partial X^* \times (-1, 1)$, the homeomorphism H^{**} permutes the path components of $\partial X^* \times (-1, 1)$. Each path component of $\partial X^* \times (-1, 1)$ is of the form $X \times (-1, 1)$ where X is a path component of ∂X^*. By definition, $\beta \in A\hat{}$ iff there exists a path component $X \subset \partial X^*$ such that $\beta = int\phi(X)$. Thus, we define a bijection $h : A\hat{} \to A\hat{}$ satisfying $h(\beta) = \gamma$ if $\beta = int\phi(X)$, and $H^{**}(X \times (-1, 1)) = Y \times (-1, 1)$, and $\gamma = int\phi(Y)$. □

Lemma 7.3. *Suppose* $D \subset D^3$ *is a topological disk such that* $int(D) \subset int(D^3)$, *and* $\partial D \subset \partial D^3$. *Suppose* $\{(0, 0, 1), (0, 0, -1)\} \subset \partial D$ *and suppose* $\alpha : [0, 1] \to D^3 \backslash D$ *satisfies* $\alpha(t) \in int(D^3)$ *iff* $0 < t < 1$. *Suppose* β_1 *and* β_2 *are disjoint nonempty connected subsets of* S^1. *Suppose* $\partial D \subset p((\beta_1 \cup \beta_2) \times [-1, 1])$. *Suppose* $\{\alpha(0), \alpha(1)\} \cap$

$p((\beta_1 \cup \beta_2) \times [-1,1]) = \emptyset$. Then $\pi_1(p^{-1}(\alpha(0), \alpha(1)))$ is contained in a single component of $S^1 \backslash \{\beta_1 \cup \beta_2\}$.

Proof: The topological disk D separates D^3 into two components and ∂D separates ∂D^3 into two components. Moreover, the two components of $\partial D^3 \backslash \partial D$ are contained in distinct components of $D^3 \backslash D$. Hence, since $im(\alpha)$ is connected and $im(\alpha) \cap D = \emptyset$, $\partial\alpha$ belongs to a single component of $\partial D^3 \backslash \partial D$. Moreover, $\partial D^3 \backslash p((\beta_1 \cup \beta_2) \times [-1,1])$ contains at most two components. No two of these components are contained in the same component of $\partial D^3 \backslash \partial D$. Thus, $\{\alpha(0), \alpha(1)\}$ is contained in a single component of $\partial D^3 \backslash p((\beta_1 \cup \beta_2) \times [-1,1])$. Hence, $\pi_1(p^{-1}(\alpha(0), \alpha(1)))$ is contained in a single component of $S^1 \backslash \{\beta_1 \cup \beta_2\}$. □

Corollary 7.4. *If $\left|A^{\hat{}}\right| \geq 4$, the bijection $h : A^{\hat{}} \to A^{\hat{}}$ is order preserving.*

Proof: Suppose $\gamma_1 < \gamma_2 < \gamma_3 < \gamma_4$ and $\gamma_i \in A^{\hat{}}$. Let $int\phi X_i = \gamma_i$. Choose four points $x_i \in X_i$. Let $\lambda \subset X^*$ be an arc connecting x_1 to x_2 such that $int(\lambda) \subset U$. Let $\mathcal{A} \subset X^*$ be an arc connecting x_3 to x_4 such that $int(\mathcal{A}) \subset U$ and such that $\mathcal{A} \cap \lambda = \emptyset$. Let $\Lambda = \{\infty, -\infty\} \cup (\lambda \times (-1,1)) \subset \overline{X^* \times (-1,1)}$. Observe Λ is compact and homeomorphic to D^2. Let $D = p\Phi(H^{**}\Lambda)$. Let $\alpha = p\Phi(H^{**}(\mathcal{A} \times \{0\}))$. Let $\beta_i = h(\gamma_i)$. Apply Lemma 7.3 to conclude β_3 and β_4 belong to the same component of $S^1 \backslash \{\beta_1 \cup \beta_2\}$. Thus, by Lemma 6.2, h is order preserving. □

8. Main results

Define $G(U) = \{H : \overline{U} \to \overline{U} |\ H$ is a homeomorphism and $H(U) = U\}$.

Define $G(U \times R) = \{H : \overline{U} \times R \to \overline{U} \times R |\ H$ is a homeomorphism and $H(U \times R) = U \times R\}$.

Recalling theorems 1.1 and 5.1, define $rot(h) : G(U) \to R$ such that $rot(h)$ is the rotation number of $\overline{\psi h \psi^{-1}}_{S^1}$.

8.1 The case ∂U is not locally connected

Suppose ∂U is not locally connected.

Recall the bijection $h : A^{\hat{}} \to A^{\hat{}}$ from Corollary 7.2. Define a function $Rot : G(U \times R) \to R$ as follows.

If $|A^{\hat{}}| \geq 3$ recall Corollary 7.4, note h is order preserving, and define $Rot(H) = Rot(g)$ where $g : S^1 \to S^1$ is any homeomorphism compatible with h.

If $|A^{\hat{}}| \leq 2$, define $Rot(H) = 0$ if H is orientation reversing and fixes $\{\infty, -\infty\}$ pointwise or if H is orientation preserving and swaps $\{\infty, -\infty\}$. Otherwise, define $Rot(H)$ to be 0 or $\frac{1}{2}$ determined, respectively, by whether $h = ID_{A^{\hat{}}}$ or not.

8.2 The case ∂U is locally connected

The case ∂U is locally connected deserves special treatment since $\overline{U} \times R$ admits a homeomorphism extension theorem analogous to Theorem 1.1 and seems to provide a useful model of "3 dimensional prime end theory" in the sense of [4].

Theorem 8.1. *Suppose ∂U is locally connected and $H \in G(U \times R)$. Then $\Psi H \Psi^{-1} : int(D^2) \times R \to int(D^2) \times R$ can be extended to a canonical homeomorphism of the 3 cell $(D^2 \times R) \cup \{\infty, -\infty\}$.*

Proof: Since ∂U is locally connected, the conformal map $\psi^{-1} : int(D^2) \to U$ can be continuously extended to a surjective map $\overline{\psi^{-1}} : D^2 \to \overline{U}$ [10, Theorem 2.1, p. 20]). To show X^* is compact, it suffices, since X^* is complete, to show that each sequence in X^* has a Cauchy subsequence. Suppose y_n is a sequence in X^*. Let $x_n = \phi(y_n)$. Let $\{x_{n_k}\} \subset D^2$ be a Cauchy subsequence of $\{x_n\}$. Consider the chords $[x_{n_l}, x_{n_k}] \subset D^2$. Since $\overline{\psi^{-1}}$ is uniformly continuous, diam $(\overline{\psi^{-1}}[x_{n_l}, x_{n_k}]) \to 0$. Hence, $d^*(y_{n_l}, y_{n_k}) \to 0$. Consequently, X^* is compact and the injective map from Lemma 3.1, $\phi : X^* \to D^2$, is a homeomorphism. Thus, the homeomorphism from Theorem 7.1, $H^* : X^* \times R \to X^* \times R$, induces a homeomorphism of $D^2 \times R$ leaving invariant the ends $\{\infty\}$ and $\{-\infty\}$.

To see that H is canonical, let $j : U^* \times R \to U \times R$ denote identity. Suppose $H_N \to H$ in $G(U \times R)$. Note $\{j^{-1}H_n j\}$ is uniformly equicontinuous and converges pointwise to $j^{-1}Hj$. Hence, $H_n^* \to H^*$ uniformly. Thus, $\Psi H_n^* \Psi^{-1} \to \Psi H^* \Psi^{-1}$ uniformly. □

To define rotation number when ∂U is locally connected, we seek a nonempty proper totally disconnected set $B^{\hat{}} \subset S^1$ such that $B^{\hat{}} \times R$ is invariant under the induced homeomorphism $\overline{\Psi H \Psi^{-1}} : D^2 \times R \to D^2 \times R$.

The point $x \in \partial U$ is a *cutpoint* if $\partial U \backslash \{x\}$ not connected. Let $f : D^2 \to \overline{U}$ denote the continuous extension of ψ^{-1}. The map f is *light* since the prime ends of U are in bijective correspondence with points of S^1 [10, Theorem 2.15]. Let $B = \{x \in S^1 | f(x)$ is a cutpoint of $\partial U\}$.

Lemma 8.2. *$z \in B$ if and only if $\left|f^{-1}(z)\right| \geq 2$.*

Proof: If $z \in \partial U$ and $\left|f^{-1}(z)\right| = 1$ then $z \notin B$ since $S^1 \backslash f^{-1}(z)$ is connected and hence, $f(S^1 \backslash f^{-1}(z)) = \partial U \backslash z$ is connected. □

Suppose $\left|f^{-1}(z)\right| \geq 2$; choose $x \neq y$ such that $f(y) = f(x) = z$. Let $[x, y] \subset D^2$ denote the chord from x to y. If V is a component of $int(D^2) \backslash [x, y]$, then $f(\partial V) \subset \partial(f(V))$. Thus, since f is light, each complementary domain of the simple closed curve $f([x, y])$ has nonempty intersection with ∂U. Hence, $z \in B$.

Lemma 8.3. *Suppose $x \neq y, \{x, y\} \subset S^1$ and $z = f(x) = f(y)$. Suppose a and b lie in distinct components of $S^1 \backslash \{x, y\}$. Suppose $\{a, b\} \cap f^{-1}(z) = \emptyset$. Then $f(a) \neq f(b)$.*

Proof: The points a and b belong to the boundaries of distinct components of $int(D^2) \backslash [x, y]$. These components in turn map into distinct complementary domains of the simple closed curve $f([x, y])$, since points of $int(D^2)$ can approach the chord $[x, y]$ from distinct sides. By hypothesis, $\{f(a), f(b)\} \cap f([x, y]) = \emptyset$. Thus, $f(a)$ and $f(b)$ belong to distinct components of $f([x, y])$ and in particular $f(a) \neq f(b)$. □

Lemma 8.4. *If ∂U is not a simple closed curve then $B \neq \emptyset$ and $B \neq S^1$.*

Proof: If ∂U is not a simple closed curve then, by Lemma 8.2, f is not one to one and hence, $B \neq \emptyset$. Let $\alpha_0 \subset S^1$ be any closed interval with distinct endpoints x_0 and y_0 such that $z_0 = f(x_0) = f(y_0)$. Proceeding recursively, if possible let $\alpha_{n+1} \subset \alpha_n \backslash f^{-1}(z_n)$ be a closed interval with distinct endpoints x_{n+1} and y_{n+1} such that $f(x_{n+1}) = f(y_{n+1})$ and $|x_{n+1} - y_{n+1}| < \frac{1}{n+1}$. Suppose the process never terminates. Let $a = \lim x_n = \lim y_n$. Suppose $f(b) = f(a)$. By Lemma 8.3, $b \in \cap \alpha_n$. Thus, $b = a$ and by Lemma 8.2, $a \notin B$. If the process terminates, then there must exist a nontrivial open interval $(x, y) \subset \alpha_0$ such that f_J is one to one and $f(x) = f(y)$. Let $a \in (x, y)$, and apply Lemma 8.3 to conclude $a \notin B$. □

Lemma 8.5. *The following are equivalent. 1) $b \notin B$. 2) Suppose $g : S^1 \to \overline{U} \times R$ satisfies $g(1) = (f(b), \hat{t})$ and $g(\theta) \in U \times R$ if $\theta \neq 1$. Then there exists a map $G : D^2 \to \overline{U} \times R$ such that $G(z) \in U \times R$ whenever $z \in int(D^2)$ and $G_{S^1} = g$.*

Proof: Let $\pi_1 : \overline{U} \times R \to \overline{U}$ and $\pi_2 : \overline{U} \times R \to R$ denote the projection maps. Suppose $b \notin B$ and $g : S^1 \to \overline{U} \times R$ satisfies $\pi_1 g(1) = f(b)$ and $g(\theta) \in U \times R$ if $\theta \neq 1$. By Lemma 8.2, $\left|f^{-1}(f(b))\right| = 1$, and hence the formula $\alpha = f^{-1}(\pi_1 g) : S^1 \to D^2$ determines a map. Let $\beta : D^2 \to D^2$ be a continuous extension of α such that $\beta(z) \in int(D^2)$ if $z \in int(D^2)$. Define $\hat{G} : D^2 \to \overline{U}$ via $\hat{G} = f(\beta)$. Since R is contractible, extend $\pi_2(g) : S^1 \to R$ to a map $h : D^2 \to R$. Define $G : D^2 \to \overline{U} \times R$ via $G(z) = (\hat{G}(z), h(z))$.

Conversely, suppose $b \in B$. By Lemma 8.2, choose $\hat{b} \in S^1$ such that $\hat{b} \neq b$ and $f(\hat{b}) = f(b)$. Let $[b, \hat{b}] \subset D^2$ denote the chord. Let $\hat{g}$ map S^1 homeomorphically onto the simple closed curve $f([b, \hat{b}]) \subset \overline{U}$ such that $\hat{g}(1) = f(b) = f(\hat{b})$. Both components of $R^2 \backslash im(\hat{g})$ contain points of ∂U. Hence, $\hat{g}$ is essential in $\{f(b)\} \cup (R^2 \backslash \partial U)$. In particular, G cannot exist since $\pi_1(G)$ would show that $\hat{g}$ is inessential. □

Since condition 2) is a topological property, $B \times R$ maps onto itself under the induced homeomorphism $\overline{\Psi H \Psi^{-1}} : D^2 \times R \to D^2 \times R$. It follows that if $\hat{B} = \{x \in S^1 | \ \{x\}$ is a component of B or x is an endpoint of a nontrivial interval of $B\}$, then $\hat{B} \neq \emptyset$, $\hat{B}$ is totally disconnected, and $\overline{\Psi H \Psi^{-1}}(\hat{B} \times R) = \hat{B} \times R$. If $\left|\hat{B}\right| \neq 2$, let $g : S^1 \to S^1$ be any homeomorphism such that $g(x) = y$ whenever x and y are components of $\hat{B}$ such that $\overline{\Psi H \Psi^{-1}}(\{x\} \times R) = \{y\} \times R$. Define $Rot(H) = rot(g)$. If $\left|\hat{B}\right| = 2$, define $Rot(H) = 0$ if $\overline{\Psi H \Psi^{-1}}$ is orientation reversing and fixes $\{\infty, -\infty\}$ pointwise or if $\overline{\Psi H \Psi^{-1}}$ is orientation preserving and swaps $\{\infty, -\infty\}$. Otherwise, define $Rot(H)$ to be 0 or $\frac{1}{2}$ determined, respectively, by whether $\overline{\Psi H \Psi^{-1}}$ interchanges the lines of $\hat{B} \times R$ or not.

8.3 Main Theorem on rotation number over $U \times R$

Theorem 8.6. *Suppose ∂U is not a simple closed curve. The function $Rot : G(\overline{U} \times R) \to R$ is continuous. If $F, H \in G(\overline{U} \times R)$ then $Rot(H) = Rot(FHF^{-1})$. If there exists $h \in G(\overline{U})$ such that $\forall z, t$ $H(z,t) = (h(z), t)$ then $Rot(H) = rot(h)$.*

Proof: If $\left|A^{\hat{}}\right| \geq 3$, then by Corollary 7.4, *Rot* is well defined. Suppose $g, f : S^1 \to S^1$ are homeomorphisms compatible with H and F, respectively. Then fgf^{-1} is compatible with FGF^{-1}. Hence, $Rot(H) = rot(g) = rot(fgf^{-1}) = Rot(FGF^{-1})$. For continuity of *Rot*, suppose $H_n \to H$. Let $h_n : A^{\hat{}} \to A^{\hat{}}$ and $h : A^{\hat{}} \to A^{\hat{}}$ denote the corresponding order preserving bijections. If J is a nontrivial component of $A^{\hat{}}$, then $h_n(J)$ is eventually constant. Construct compatible homeomorphisms $g_n : S^1 \to S^1$ and $g : S^1 \to S^1$ as in Lemma 6.2. Note that eventually $g_{n|J} = g_{|J}$ and hence $rot(g_n) = rot(g)$ eventually. If each component J of $A^{\hat{}}$ is trivial, then $g_n \to g$ pointwise on a dense set $A^{\hat{}} \subset S^1$ and hence, $g_n \to g$ uniformly. Thus, by Theorem 5.1, $rot(g_n) \to rot(g)$.

If ∂U is not locally connected and $\left|A^{\hat{}}\right| \leq 2$, the theorem follows from the definition of *Rot*.

If ∂U is locally connected, the homeomorphism $\overline{\Psi H \Psi^{-1}}$ varies continuously with H. The theorem follows from the definition of *Rot* and from Theorem 5.1. □

References

[1] Kathleen T. Alligood and James A. Yorke, "Fractal basin boundaries and chaotic attractors," *Chaos and Fractals: The Mathematics Behind the Computer Graphics.* Ed. Robert L. Devaney and Linda Keen. Proceedings of Symposia in Applied Mathematics, 39. Providence, RI: AMS, 1989. 41–55.

[2] Marcy Barge, "Prime End Rotation Numbers Associated with the Hénon Maps." West 15–33.

[3] Marcy Barge and Richard M. Gillette, "Indecomposability and dynamics of invariant plane separating continua," *Continuum Theory and Dynamical Systems (Arcata, CA, 1989).* Ed. Morton Brown. Contemporary Mathematics, 117. Providence, RI: AMS, 1991. 13–38.

[4] Beverly L. Brechner and Joo S. Lee, "A Three-Dimensional Prime End Theory," *Topology Proc.* **20** (1995), 15–47.

[5] M. L. Cartwright and J. E. Littlewood, "Some fixed point theorems," *Ann. of Math. (2)* **54** (1951), 1–37.

[6] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems.* Menlo Park, CA: The Benjamin/Cummings Publishing Co., Inc., 1986.

[7] Paul Fabel,"Self-homeomorphisms of the 2-sphere which fix pointwise a nonseparating continuum," *Fund. Math.* **155** (1998), no. 3, 201–214.

[8] John Franks, "Rotation numbers and instability sets," *Bull. Amer. Math. Soc. (N.S.)* **40** (2003), no. 3, 263–279 (electronic).

[9] John C. Mayer and Lex G. Oversteegen, “Denjoy Meets Rotation on an Indecomposable Cofrontier.” West 183–199.

[10] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps.* Grundlehren der mathematischen Wissenschaften, 299. Berlin: Springer-Verlag, 1992.

[11] H. D. Ursell and L. C. Young, *Remarks on the Theory of Prime Ends.* Memoirs of the American Mathematical Society, no. 3. Providence, RI: AMS, 1951.

[12] Thelma West, ed. *Continuum Theory and Dynamical Systems.* Lecture Notes in Pure and Applied Mathematics, 149. New York: Marcel Dekker, Inc., 1953.

Drawer Ma, Mississippi State University, Ms. State, MS 39762
E-mail address: fabel@ra.msstate.edu