Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 207-215

NOTES ON g-METRIZABLE SPACES

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Abstract. A space is called a g-metrizable space if it is a regular space with a σ-locally finite weak base (see F. Siwiec, "On Defining a Space by a Weak Base"). In this paper, we discuss spaces with a σ-HCP (wHCP) weak base and give answers or partial answers to questions posed by A. V. Arhangel'skii during a seminar in 2004 at Ohio University, by S. Lin in a personal communication, and by Y. Tanaka in "σ-Hereditarily Closure Preserving k-Networks and g-Metrizability."

1. Introduction

Weak base was introduced by A. V. Arhangel'skii [1] in 1966. Frank Siwiec [14] defined g-metrizable spaces as a spaces with a σ-locally finite weak base. Yoshio Tanaka [16], L. Foged [4], Shou Lin [8], and Chuan Liu and Mu Min Dai [11] have made much contribution on this field. We discuss topological spaces with a σ-HCP (wHCP)weak base and give answers or partial answers to Lin's, Arhangel'skii's and Tanaka's questions.

In this paper all spaces are regular and T_1; all mappings are continuous and onto. $\mathbb{N}$ denotes the natural numbers. Readers may refer to [6] for unstated definitions.

2000 *Mathematics Subject Classification.* Primary 54E40; Secondary 54D55.

Key words and phrases. g-metrizable space, g-second countable space; weak base, sn-network .

2. g-metrizable spaces

Definition 2.1. Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space X such that for each $x \in X$,

(1) $\mathcal{P}_x$ is a network of x in X;
(2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is called a *weak base* [1] for X if whenever $G \subset X$ satisfying for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$, then G is open in X; $\mathcal{P}$ is called an *sn-network* [9] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X (i.e., every convergent sequence with the limit point x is eventually in the element) for each $x \in X$.

A space X is called a *g-metrizable space* [14] (resp., an *sn-metrizable space* [5]) if it has a σ-locally finite weak base (resp., *sn*-network), and a space is *g-first countable* [1], (resp., sn-first countable) if each $\mathcal{P}_x$ is countable. A space is called *g-second countable* [14] if it has a countable weak base.

A collection $\{C_\alpha : \alpha \in I\}$ of subsets of X is called *hereditarily closure preserving, HCP (weakly hereditarily closure preserving, wHCP)* [3] if for any $J \subset I$, $\{B_\alpha : B_\alpha \subset C_\alpha, \alpha \in J\}$ is closure preserving ($\{x_\alpha : x_\alpha \in C_\alpha, \alpha \in J\}$ is closed discrete).

Tanaka [16] asked the following.

Question 2.2. If X has a σ-HCP weak base, is X g-metrizable?

By slightly modifying Liang-Xue Peng's proof in [13], we can obtain the following.

Lemma 2.3. *Suppose that X has a σ-closure preserving weak base, then X is hereditary meta-Lindelöf.*[1]

Tanaka [16] proved that a Lindelöf space with a σ-HCP weak base is g-second countable and that the following proposition holds under (CH). By using Lemma 2.3, we may omit (CH).

Proposition 2.4. *X is g-second countable if and only if X is a separable space with a σ-HCP weak base.*

X has property (**) if for any non-isolated point x of X, there is countable subset $D \subset X$ so that $x \in cl(D \backslash \{x\})$. A space X has property (*) [10] if for any non-isolated point x of X, there is

[1] Lin informed the author that he also obtained this result.

a non-trivial sequence converging to x. Obviously, a space having countable tightness[2] or property (*) has property (**).

Lemma 2.5. *Let X have a σ-HCP weak base. Then X has property (*) if it has property (**).*

Proof: For a non-isolated point $x \in X$, there is a countable subset $D \subset X$ with $x \in cl(D \backslash \{x\})$. By Lemma 2.3, $cl(D)$ is a Lindelöf space. Tanaka [16] proved that every ω_1-compact space with a σ-HCP weak base is g-first countable. $cl(D \backslash \{x\})$ is a sequential space; X has property (*). $\square$

The author [10] proved that X is a g-metrizable space if and only if X has a σ-HCP weak base and property (*); by Lemma 2.5, we have

Theorem 2.6. *X is g-metrizable if and only if X has a σ-HCP weak base and property (**).*

Corollary 2.7. *A space X is g-metrizable if and only if X has a σ-HCP weak base and has countable tightness.*

Next, we discuss spaces with a σ-wHCP weak base. In [3], it was proved that a k-space with a σ-wHCP base is metrizable, but not every (paracompact) space with a σ-wHCP base is metrizable. Thus, every space with a σ-wHCP weak base need not to be g-metrizable. We don't know if a k-space with a σ-wHCP weak base is g-metrizable or not. But we have following.

Theorem 2.8. *(CH) Suppose X is a separable space with a σ-wHCP weak base, then X is g-second countable, hence g-metrizable.*

First, let us prove a lemma.

Lemma 2.9. *Suppose that X has a σ-wHCP weak base, then X has a σ-wHCP k-network[3] and a σ-compact-finite k-network.*

Proof: Let $\mathcal{B} = \cup_{n \in \mathbb{N}} \mathcal{B}_n$ is a σ-wHCP weak base. We may assume $\mathcal{B}_n \subset \mathcal{B}_{n+1}$. For each $n \in \mathbb{N}$, let $D_n = \{x \in X : \mathcal{B}_n$ is not point-finite at $x\}$. Let $\mathcal{B}'_n = \{B \backslash D_n : B \in \mathcal{B}_n\} \cup \{\{x\} : x \in D_n\}$.

[2] A space X has countable tightness if whenever $x \in cl(A)$ for $x \in X$ and a subset A of X, there is a countable subset $C \subset A$ such that $x \in cl(C)$.

[3] A cover $\mathcal{P}$ of subsets of X is a k-network if, whenever $K \subset U$ with K compact and U open in X, there is a finite subfamily $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \cup\mathcal{F} \subset U$.

Then $\mathcal{B}'_n$ is compact-finite for each n. In fact, let K be a compact subset of X; it is easy to see that $K \cap D_n$ is finite. Notice that $\{B \backslash D_n : B \in \mathcal{B}_n\}$ is wHCP and point-finite, and K meets at most finitely many elements of $\{B \backslash D_n : B \in \mathcal{B}_n\}$. Hence, $\mathcal{B}' = \cup_{n \in \mathbb{N}} \mathcal{B}'_n$ is a σ-compact-finite network. Any compact subset of X has a countable network; hence, it is metrizable. By Proposition A(3) in [17], $\mathcal{B}$ is a σ-wHCP k-network. Now we prove that $\mathcal{B}'$ is a k-network. Let $K \subset U$ with K compact and U open; there are $m \in \mathbb{N}$ and a finite subfamily $\mathcal{P}$ of $\mathcal{B}_m$ such that $K \subset \cup \mathcal{P} \subset U$. Let $\mathcal{F} = \{P \backslash D_m : P \in \mathcal{P}\} \cup \{\{x\} : x \in K \cap D_m\}$, then $\mathcal{F} \subset \mathcal{B}'_m \subset \mathcal{B}'$ and $K \subset \cup \mathcal{F} \subset U$. □

Now we give a proof of Theorem 2.8.

Since X is separable, by (CH), the character of X, $\chi(X) \leq \omega_1$. Let $\mathcal{B} = \cup_{n \in \mathbb{N}} \mathcal{B}_n = \cup\{\mathcal{P}_x : x \in X\}$ is a σ-wHCP weak base. We may assume $\mathcal{B}_n \subset \mathcal{B}_{n+1}$. First, we prove that X is g-first countable.

For $x \in X$, if $\{x\}$ is open, then X is g-first countable at x. If $\{x\}$ is not open, $\mathcal{B}_n \cap \mathcal{P}_x$ is locally countable at x for $n \in \mathbb{N}$. Suppose not. Let $\{V_\alpha : \alpha < \omega_1\}$ be the local base at x. Notice that for any neighborhood V of x, $V \cap (P \backslash \{x\}) \neq \emptyset$ for $P \in \mathcal{P}_x$. Then, by induction, there are a subset $S = \{x_\alpha : \alpha < \omega_1\}$ of X and a subcollection $\{B_\alpha : \alpha < \omega_1\}$ of $\mathcal{B}_n \cap \mathcal{P}_x$ such that $x_\alpha \in V_\alpha \cap B_\alpha$, where $x_\alpha \neq x$, and the B_α's are distinct. x is an accumulation of S, so S is not closed. Since $\mathcal{B}_n$ is wHCP, S is a closed discrete subset; this is a contradiction. Hence, X is g-first countable. By Lemma 2.9, X has a σ-compact-finite k-network. Under (CH), a separable, sequential space with a σ-compact-finite k-network is an $\aleph_0$-space[4] [12]. X is a g-first countable, $\aleph_0$-space; hence, X is g-second countable [14].

We don't know if we can omit (CH) or not in the above theorem.

Question 2.10. Is a separable space with a σ-wHCP weak base g-second countable?

We define iterates of the operator seq cl inductively for a space X as follows:

(1) seq $cl^0(S) = S$;
(2) seq $cl(S) = \{x : x$ is a limit point of $S\}$;

[4] A space with a countable k-network.

(3) if α is an ordinal, let seq $cl^{\alpha+1}(S) =$ seq $cl($seq $cl^{\alpha}(S))$;
(4) if α is a limit ordinal, let seq $cl^{\alpha} = \cup_{\beta<\alpha}$seq $cl^{\beta}(S)$.

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Lemma 2.11. *Let X have a σ-wHCP weak base. If $A \subset X$ is ω_1-compact, then seq $cl(A)$ is ω_1-compact.*

Proof: Assume to the contrary that there is a discrete subset $\{x_\alpha : \alpha < \omega_1\}$ in seq $cl(A)\backslash A$. For $\alpha < \omega_1$, let $\{x_n(\alpha)\} \subset A$ be a sequence converging to x_α. Let $\mathcal{B} = \cup_{n\in\mathbb{N}}\mathcal{B}_n$ be a σ-wHCP weak base of X. We may assume $\mathcal{B}_n \subset \mathcal{B}_{n+1}$. For each α, there is $B_\alpha \in \mathcal{B}$ such that $x_\alpha \in B_\alpha$, B_α contains a tail of $\{x_n(\alpha)\}$, and all $B_\alpha \cap \{x_\beta : \beta \neq \alpha\} = \emptyset$. Without loss of generality, we assume B_α contains $\{x_n(\alpha)\}$ and $B_\alpha \in \mathcal{B}_n$ for some n.

Case 1. $|\{x_n(\alpha) : n \in \mathbb{N}, \alpha < \omega_1\}| = \omega_1$.

By induction, there is an uncountable subset $S = \{x_\beta : \beta < \omega_1\}$ of $\{x_n(\alpha) : n \in \mathbb{N}, \alpha < \omega_1\}$ such that $x_\beta \in B_\beta$ and $B_\beta \neq B_\gamma$ if $\beta \neq \gamma$. Since $\mathcal{B}_n$ is wHCP, $S \subset A$ is closed discrete; this is a contradiction.

Case 2. $|\{x_n(\alpha) : n \in \mathbb{N}, \alpha < \omega_1\}| \neq \omega_1$.

There exists an α_0 such that infinitely many B_α's contain a subsequence of $\{x_n(\alpha_0)\}$. Suppose not. For every $\alpha < \omega_1$, there is $m(\alpha)$ such that $\{B_\alpha : \alpha < \omega_1\}$ is point-finite at $x_{m(\alpha)}(\alpha)$. Since $|\{x_{m(\alpha)}(\alpha) : n \in \mathbb{N}, \alpha < \omega_1\}| \neq \omega_1$ and $x_{m(\alpha)}(\alpha) \in B_\alpha$, $\{B_\alpha : \alpha < \omega_1\}$ is not point-finite at some $x_{m(\alpha)}(\alpha)$; this is a contradiction. Thus, infinitely many B_α's contain a subsequence of $\{x_n(\alpha_0)\}$, then $\{x_n(\alpha_0)\}$ has a subsequence that is discrete, a contradiction. Hence, seq $cl(A)$ is ω_1-compact. □

Theorem 2.12. *Let X be a separable space with a σ-wHCP weak base. If the sequential order of X is countable, then X is g-second countable.*

Proof: Let $D \subset X$ with $|D| = \omega$, $cl(D) = X$. Since the sequential order of X is countable, $X = \cup_{n\in\mathbb{N}}\text{seq } cl^n(D)$. D is ω_1-compact, and seq $cl^n(D)$ is ω_1-compact for each n by Lemma 2.11; hence, X is ω_1-compact. Let $\mathcal{B} = \cup_{n\in\mathbb{N}}\mathcal{B}_n$ be a σ-wHCP weak base of X. For $x \in X$, if x is not an isolated point, then $\mathcal{B}\cap\mathcal{P}_x$ is locally countable at x. Suppose not. There is $n \in \mathbb{N}$ such that $\mathcal{B}_n \cap \mathcal{P}_x$ is not locally countable at x. By induction, we can select an uncountable subset $\{x_\alpha : \alpha < \omega_1\}$ and an uncountable subfamily $\{B_\alpha : \alpha < \omega_1\} \subset \mathcal{B}_n$ such that $\{x, x_\alpha\} \subset B_\alpha$, $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$. $\{x_\alpha : \alpha < \omega\}$ is discrete; this is a contradiction because X is ω_1-compact. Hence, X is g-first countable.

X is ω_1-compact and has a σ-wHCP k-network by Lemma 2.9; hence, X is an $\aleph_0$-space [7]. Thus, X is g-second countable [14]. $\square$

It is well known that g-metrizable spaces are not preserved by perfect mappings. Arhangel'skii [2], in a topology seminar at Ohio University, asked the following question:

Question 2.13. Let X be a topological space; if every perfect image of X is g-metrizable, is X metrizable?

We shall give an affirmative answer to this question; in fact, we may prove a slightly stronger version.

The sequential fan S_ω[5] is a perfect image of the Arens' space S_2[6]. It is well known that S_ω is not g-first countable.

Theorem 2.14. *Let X be a g-metrizable space. If every perfect image of X has a σ-wHCP weak base, then X is metrizable.*

Proof: First, we prove that a space Y with a σ-wHCP weak base $\mathcal{B} = \cup_{n\in\mathbb{N}}\mathcal{B}_n$ does not contain a copy of S_ω. Suppose not. There is a non-trivial sequence $\{y_n\}$ converging to a point $y \in Y$ that is not an isolated point and Y is not g-first countable at y. There is $n \in \mathbb{N}$ and infinitely many weak neighborhoods of y in $\mathcal{B}_n$, each containing a tail of $\{x_n\}$. Since $\mathcal{B}_n$ is wHCP, there is a subsequence L of $\{x_n\}$ such that L is discrete. This is a contradiction.

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[5] S_ω is a space obtained from the topological sum of ω many convergent sequences by identifying all limit points to a single point.

[6] $S_2 = (\mathbb{N} \times \mathbb{N}) \cup \mathbb{N} \cup \{\infty\}$ is the space with each point of $\mathbb{N} \times \mathbb{N}$ isolated. A basic neighborhood of $n \in \mathbb{N}$ consists of all sets of the form $\{n\} \cup \{(m, n) : m \geq k\}$. And U is a neighborhood of ∞ if and only if $\infty \in U$ and U is a neighborhood of all but finitely many $n \in \mathbb{N}$.

Since no perfect image of X contains a copy S_ω, then X contains no copy of S_2. X is a sequential space and every point is a G_δ-set; hence, X is a Fréchet-Urysohn space [15]. Since every Fréchet-Urysohn, g-metrizable space is metrizable [14], then X is metrizable. $\square$

Lin [9] introduced sn-networks to generalize weak bases (it is easy to see that a weak base is an sn-network). An sn-network is a weak base if the topological space is a sequential space. Many results on g-metrizable spaces can be generalized in terms of sn-networks [5]. Dai and the author [11] proved the following.

Theorem 2.15. *Let X be a k-space with a σ-HCP k-network, then X is g-metrizable if X contains no copy of S_ω.*

In a personal communication with the author, Lin asked if we can generalize the above theorem as follows:

Question 2.16. Let X have a σ-HCP k-network. Does X have a σ-locally finite sn-network if it contains no copy of S_ω?

We give a negative answer to the question.

Example 2.17. *There is an $\aleph_0$-space that contains no copy of S_ω, but it is not sn-first countable.*

Proof: Let $X = \{\infty\} \cup \{x_i(n) : i \in \mathbb{N}, n \in \mathbb{N}\}$, and let $\{f_\alpha : \alpha < 2^\omega\}$ be all maps from $\mathbb{N}$ to $\mathbb{N}$. Endow X with topology as follows: each $\{x_i(n)\}$ is open for $i \in \mathbb{N}, n \in \mathbb{N}$; the neighborhood of ∞ is $X \setminus \cup \{x_{f(n)}(n) : f \in \mathcal{F} \in \{f_\alpha : \alpha < 2^\omega\}^{<\omega}\}$. It is easy to see that $x_i(n) \to \infty$ for each $n \in \mathbb{N}$ and any convergent sequence in X is contained in the finite union of $\{\{x_i(n)\} : n \in \mathbb{N}\}$.

(1) X has a countable k-network. $\{\{x_i(n)\} : i \in \mathbb{N}, n \in \mathbb{N}\} \cup \{\{x_i(n) : i > m\} \cup \{\infty\} : m \in \mathbb{N}, n \in \mathbb{N}\}$ is a countable cs*-network[7] for X. Since X is countable, then each point of X is a G_δ-set; hence, X has a countable k-network by Proposition B(1) in [17].

(2) X contains no copy of S_ω. Let $g : \mathbb{N} \to \mathbb{N}$ be a surjection. It is obvious that $\{x_i(n) : i \le g(n), n \in \mathbb{N}\}$ is not closed in X; therefore, X contains no copy of S_ω.

[7]A cover $\mathcal{P}$ is a cs*-network of X if whenever σ is a sequence converging to a point x and $x \in U$ with U open, then for some $P \in \mathcal{P}$, $x \in P \subset U$, and P contains a subsequence of σ.

(3) X is not sn-first countable. Suppose not. Let $\{P_n : n \in \mathbb{N}\}$ be a decreasing countable sn-network at ∞. For each $n \in \mathbb{N}$, $x_i(n) \to \infty$, pick $x_{i_n}(n) \in P_n$, then $x_{i_n}(n) \to \infty$. This is a contradiction because $X \backslash \{x_{i_n}(n)\}$ is an open neighborhood of ∞. $\square$

Acknowledgment. The author would like to thank the referee for many valuable comments.

References

[1] A. V. Arhangel'skii, "Mappings and spaces," *Russian Math. Surveys* **21** (1966), 115–162.

[2] ———, Seminar on Topology. Ohio University, 2004.

[3] D. Burke, R. Engelking, and D. Lutzer, "Hereditarily closure-preserving collections and metrization," *Proc. Amer. Math. Soc.* **51** (1975), 483–488.

[4] L. Foged, "On g-metrizability," *Pacific J. Math.* **98** (1982), no. 2, 327–332.

[5] Yin Ge, "On spaces with a σ-locally-finite universal *cs*-network," *Questions Answers Gen. Topology* **18** (2000), no. 1, 93–96.

[6] G. Gruenhage, "Generalized Metric Spaces." *Handbook of Set-Theoretic Topology.* Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 423–501.

[7] Shou Lin, "A study of pseudobases," *Questions Answers Gen. Topology* **6** (1988), no. 1, 81–97.

[8] ———, "On g-Metrizable Spaces," (Chinese) *Chinese Ann. Math. Ser. A* **13** (1992), no. 3, 403–409.

[9] ———, "A note on the Arens' space and sequential fan," *Topology Appl.* **81** (1997), no. 3, 185–196.

[10] Chuan Liu, "Notes on weak bases," *Questions Answers Gen. Topology* **22** (2004), 39–42.

[11] Chuan Liu and Mu Min Dai, "g-metrizability and S_ω," *Topology Appl.* **60** (1994), no. 2, 185–189.

[12] Chuan Liu and Yoshio Tanaka, "Spaces Having σ-Compact-Finite k-Networks, and Related Matters," *Topology Proc.* **21** (1996), 173–200.

[13] Liang-Xue Peng, "The Hereditarily MetaLindelöf Property of k-Spaces with σ-HCP Closed k-Network," *Topology Proc.* **26** (2001-2002), no. 1, 307–310.

[14] Frank Siwiec, "On defining a space by a weak base," *Pacific J. Math.* **52** (1974), 233–245.

[15] Yoshio Tanaka, "Metrizability of certain quotient spaces," *Fund. Math.* **119** (1983), no. 2, 157–168.

[16] ———, "σ-hereditarily closure-preserving k-networks and g-metrizability," *Proc. Amer. Math. Soc.* **112** (1991), no. 1, 283–290.

[17] ———, "Theory of k-networks," *Questions Answers Gen. Topology* **12** (1994), no. 2, 139–164.

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