Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 229-236

A CROWDED Q-POINT UNDER $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$

ANDRES MILLÁN

ABSTRACT. In this note, we prove that the version $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ of the Covering Property Axiom, which holds in the iterated Sacks model, implies that there exists an ω_1-generated crowded ultrafilter on $\mathbb{Q}$ which is also a Q-point. Since no crowded ultrafilter can be a P-point, this constitutes an interesting example of a Q-point which is not a P-point.

1. Introduction

We will use standard set theoretic notation as in [7]. Let $\mathcal{U}$ be a non-principal ultrafilter on a countable set X. Then, $\mathcal{U}$ is a *P-point* if for every partition $\mathcal{P}$ of X, either $\mathcal{U} \cap \mathcal{P} \neq \emptyset$ or there exists an $X \in \mathcal{U}$ such that $X \cap P$ is finite for each $P \in \mathcal{P}$. $\mathcal{U}$ is a *Q-point* if for every partition $\mathcal{P}$ of X into finite pieces, there exists an $X \in \mathcal{U}$ such that $|X \cap P| \leq 1$ for each $P \in \mathcal{P}$. Given a non-principal ultrafilter $\mathcal{U}$ on X, we say that $\mathcal{B} \subset \mathcal{U}$ is a basis for $\mathcal{U}$ if for every $U \in \mathcal{U}$ there exists a $B \in \mathcal{B}$ such that $B \subset U$. Then, we can define the *character* of $\mathcal{U}$ as $\chi(\mathcal{U}) = \min\{|\mathcal{B}| \colon \mathcal{B} \text{ is a basis for } \mathcal{U}\}$. We say that $\mathcal{U}$ is κ-generated if $\chi(\mathcal{U}) = \kappa$.

Consider $\mathbb{Q}$ with the subspace topology induced by the usual topology on $\mathbb{R}$ and denote by $\mathrm{Perf}(\mathbb{Q})$ the family of its perfect

2000 *Mathematics Subject Classification.* Primary 54D80; Secondary 03E35.
Key words and phrases. Covering Property Axiom CPA, crowded, Q-point, ultrafilter.
This work is a part of the author's Ph.D. thesis written at West Virginia University under the supervision of Krzysztof Ciesielski. The author wishes to thank Professor Ciesielski for his guidance, patience, and encouragement.

subsets. A non-principal filter $\mathcal{U}$ on $\mathbb{Q}$ is *crowded* if the family $\mathrm{Perf}(\mathbb{Q}) \cap \mathcal{U}$ forms a basis for $\mathcal{U}$. The crowded ultrafilters have been studied in connection with the remainder of the Stone-Čech compactification of $\mathbb{Q}$, and their existence follows from the Continuum Hypothesis, Martin's Axiom for countable posets [5], or from the equality $\mathfrak{b} = \mathfrak{c}$ [4].

In [2], K. Ciesielski and J. Pawlikowski showed that a version of their Covering Property Axiom called $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, which holds in the iterated Sacks model, implies that there exists an ω_1-generated crowded ultrafilter on $\mathbb{Q}$, and they noted that no crowded ultrafilter can be a P-point. This result is interesting because CPA implies $\mathfrak{b} < \mathfrak{c}$.

The main result of this paper is that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies the existence of an ω_1-generated crowded ultrafilter on $\mathbb{Q}$ which is also a Q-point[1]. Notice that this contradicts the remark by Ciesielski and Pawlikowski in [2, p. 49] that crowded ultrafilters cannot be Q-points.

It is a result of Arnold W. Miller [9] that there are no Q-points in Richard Laver's model for Borel's Conjecture [8]. Since the equality $\mathfrak{b} = \mathfrak{c}$ holds in Laver's model, it is consistent with ZFC that no crowded ultrafilter on $\mathbb{Q}$ is a Q-point.

2. Preliminaries on $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ and $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$

2.1 Cubes and Prisms

The framework of CPA rests on the concepts of *cube* and *prism.* If $\mathfrak{C}$ denotes the space 2^ω with its usual product topology and $\mathfrak{X}$ is a Polish space, then we define

$$\mathrm{Perf}(\mathfrak{X}) = \{C \subset \mathfrak{X} \colon C \text{ is homeomorphic to } \mathfrak{C}\}.$$

A *perfect cube* in $\mathfrak{C}^\omega$ is any set $C = \prod_{i<\omega} C_i$ where $C_i \in \mathrm{Perf}(\mathfrak{C})$ for every $i < \omega$. If $\mathfrak{X}$ is a Polish space, then a *cube* in $\mathfrak{X}$ is a pair $\langle f, P\rangle$ where $f \colon C \to \mathfrak{X}$ is a continuous injection and $P = f[C]$ for some perfect cube C. The following proposition is one of the principal tools for using CPA, and it is a refinement of a theorem proved independently by H. G. Eggleston [6] and M. L. Brodskiĭ [1].

[1]Recently the author has proven that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies that there is also a crowded Q-point of character $\mathfrak{c}$.

Proposition 1 (Ciesielski and Pawlikowski [3, Claim 1.1.5])**.** *Consider $\mathfrak{C}^\omega$ with its usual topology and its usual product measure. If G is a Borel subset of $\mathfrak{C}^\omega$, which is either of second category or of positive measure, then G contains a perfect cube.*

The notion of *prism* is a generalization of that of a cube. If $\alpha < \omega_1$ is a non-zero countable ordinal, let $\Phi_{\mathrm{prism}}(\alpha)$ be the set of all functions $f\colon \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ with the property that

$$f(x) \restriction \xi = f(y) \restriction \xi \Leftrightarrow x \restriction \xi = y \restriction \xi$$

for all $\xi < \alpha$ and $x, y \in \mathfrak{C}^\alpha$. Put $\mathbb{P}_\alpha = \{\mathrm{range}(f)\colon f \in \Phi_{\mathrm{prism}}(\alpha)\}$ and $\mathbb{P}_{\omega_1} = \bigcup_{0<\alpha<\omega_1} \mathbb{P}_\alpha$. The elements of $\mathbb{P}_{\omega_1}$ are called the *iterated perfect sets*. If $\mathfrak{X}$ is a Polish space, then a *prism* on X is a pair $\langle f, P\rangle$ where $f\colon E \to \mathfrak{X}$ is injective and continuous, $E \in \mathbb{P}_{\omega_1}$, and $P = f[E]$.

It is also immediate to observe that if the pair $\langle f, P\rangle$ is a prism, and $f\colon E \to P$ and $E \in \mathbb{P}_\alpha$, then we can assume that f is defined on the entire $\mathfrak{C}^\alpha$.

It is important to note that the previous definitions imply that perfect cubes are, in particular, iterated perfect sets and therefore, that cubes are prisms. On the other hand, if $\langle g, P\rangle$ is a prism, where $g\colon E \to P$ and $E \in \mathbb{P}_\alpha$, then there exists an $f \in \Phi_{\mathrm{prism}}(\alpha)$ with $E = \mathrm{range}(f)$. In particular, $h = g \circ f\colon \mathfrak{C}^\alpha \to P$ is a continuous injection and the pair $\langle h, P\rangle$ is a cube. Thus, any prism can be thought of as a cube with a different coordinate system imposed on it.

2.2 Subcubes and Subprisms

If $\langle f, P\rangle$ is a cube, then we say that Q is its subcube provided there exists a perfect cube $C \subset \mathrm{dom}(f)$ such that $Q = f[C]$. Subprisms are defined similarly but with replacing the perfect cube C by an iterated perfect set E. Since in the games defined below we will need to consider singletons in the same position as cubes (or prisms) as defined above, in what follows, *singletons will be considered as cubes and prisms.* If P is a singleton in $\mathfrak{X}$, then its only subcube is P itself.

2.3 Games and Strategies

For a Polish space $\mathfrak{X}$, consider the following game $\mathrm{GAME}_{\mathrm{cube}}(\mathfrak{X})$ of length ω_1 played by two players, Player I and Player II. At each stage $\xi < \omega_1$ of the game Player I can play an arbitrary cube P_ξ in $\mathfrak{X}$ (i.e., P_ξ either belongs to $\mathrm{Perf}(\mathfrak{X})$ or is a singleton in $\mathfrak{X}$), and Player II must respond by playing a subcube Q_ξ of P_ξ. The game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ is won by Player I provided that

$$\mathfrak{X} = \bigcup_{\xi<\omega_1} Q_\xi \ ;$$

otherwise, Player II wins.

A strategy for Player II is any function S with the property that $S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi)$ is a subcube of P_ξ for every partial game $\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle$. We say that a game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ is played according to a strategy S for Player II provided that $Q_\xi = S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi)$ for every $\xi < \omega_1$. A strategy S for Player II is a *winning strategy* provided Player II wins any game played according to the strategy S. The corresponding notions of games, strategies, etc., for prisms are defined in a similar way.

2.4 The Axioms

The following principles capture the combinatorial core of the iterated Sacks model.

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$: $\mathfrak{c} = \omega_2$ and for any Polish space $\mathfrak{X}$, Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{cube}}(\mathfrak{X})$.

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$: $\mathfrak{c} = \omega_2$ and for any Polish space $\mathfrak{X}$, Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{prism}}(\mathfrak{X})$.

These axioms are consequences of a more general principle, similar in spirit, called CPA [3]. Their importance comes from the following result.

Proposition 2 (Ciesielski and Pawlikowski, [2, 3])**.** CPA *holds in the iterated perfect set model. In particular,* CPA *is consistent with ZFC set theory.*

3. An ω_1-generated crowded Q-point on $\mathbb{Q}$

If the set $X = [\omega]^{<\omega} \setminus \{\emptyset\}$ has the discrete topology, then the product space $\mathfrak{X} = X^\omega$ is a Polish space, and the family of sets $U_{\langle n,a\rangle} = \{x \in \mathfrak{X} : x(n) = a\}$, where $a \in [\omega]^{<\omega}$ and $n < \omega$, constitutes a subbasis for the product topology. Consider that the set

$$\mathcal{P} = \{x \in \mathfrak{X} : \ \{x(k) : k < \omega\} \text{ is a partition of } \omega\}.$$

It is important to know that

- $\mathcal{P}$ is a G_δ subset of $\mathfrak{X}$. Therefore, $\mathcal{P}$ is a Polish space with the relative topology inherited from $\mathfrak{X}$.

Lemma 1. *Let P be a prism in $\mathcal{P}$ and let $\{A_n : n < \omega\} \subset [\mathbb{Q}]^\omega$ be arbitrary. Then, there exist a subprism Q of P and $B \in [\mathbb{Q}]^\omega$ such that $|B \cap A_n| = \omega$ for every $n < \omega$, and $|x(k) \cap B| \leq 1$ for every $x \in Q$ and $k < \omega$. Moreover, if P is a cube, then Q is a cube as well.*

Proof: Since $|\mathbb{Q}| = \omega$ we can suppose that $\{A_n : n < \omega\} \subset [\omega]^\omega$. Let $\langle R_n : n < \omega\rangle$ be an enumeration of $\{A_n : n < \omega\}$ where each set appears infinitely often.

Case (a): If $P = \{z\}$, then define a sequence $\langle b_n \in \omega : n < \omega\rangle$ such that $b_n \in R_n \setminus \bigcup\{z(k) : k < \omega \ \&\ z(k) \cap \{b_0, \ldots, b_{n-1}\} \neq \emptyset\}$ for every $n < \omega$. It is easy to see that $B = \{b_n : n < \omega\}$ works.

Case (b): If $P \in \mathrm{Perf}(\mathcal{P})$, let f be a witness function for P. By our remarks in section 2, we can assume that f acts from $\mathfrak{C}^\alpha$ onto P. Thus, P is a cube. It is enough to find its subcube with the desired properties.

Let μ be the standard product probability measure on $\mathfrak{C}^\alpha$.

We construct, by induction on $n < \omega$, a sequence $\langle K_n : n < \omega\rangle$ of open subsets of $\mathfrak{C}^\alpha$ and two sequences, $\langle b_n \in R_n : n < \omega\rangle$ and $\langle B_n \in [\omega]^{<\omega} : n < \omega\rangle$, such that for every $n < \omega$:

(i) $b_n > \max\left(\{b_i : i < n\} \cup \bigcup_{j<n} B_j\right)$,
(ii) $\mu(K_n) \geq 1 - 2^{-(n+2)}$, and
(iii) $f(h)(k) \subseteq B_n$ for every $h \in K_n$, $k < \omega$ with $b_n \in f(h)(k)$.

If this construction is possible, put $B = \{b_n : n < \omega\}$. Then, clearly $|B \cap A_n| = \omega$. Condition (ii) implies that $\mu\left(\bigcap_{n<\omega} K_n\right) \geq \frac{1}{2}$. Hence, by Proposition 1, there exists a perfect cube $C \subseteq \bigcap_{n<\omega} K_n$. Then $Q = f[C]$ is a subcube of P and the pair $\langle Q, B\rangle$ is as required. To see this, it is enough to show that $|z(k) \cap B| \leq 1$ for every $z \in Q$ and $k < \omega$. Let $z = f(h)$ for some $h \in C$. By conditions (i) and (iii), for every $b_j \in z(k) = f(h)(k)$ and $n > j$, we have that $b_n \notin z(k)$. Therefore, no two elements of B are in the same $z(k)$ or, in other words, $|z(k) \cap B| \leq 1$ for every $k < \omega$.

Next, we show that the inductive construction is possible. Let $n < \omega$ be such that the appropriate b_i, K_i, and B_i are already constructed for every $i < n$. We will construct b_n, K_n, and B_n satisfying (i)–(iii). We pick b_n as an arbitrary element of R_n, satisfying (i). If $L = \{a \in [\omega]^{<\omega} \colon b_n \in a\}$, then $\left\{f^{-1}\left(U_{\langle m,a\rangle}\right) : \langle m, a\rangle \in \omega \times L\right\}$ is a partition of $\mathfrak{C}^\alpha$ into clopen sets. Thus, we can find a finite set $S \subseteq \omega \times L$ such that $K_n = \bigcup\left\{f^{-1}\left(U_{\langle m,a\rangle}\right) : \langle m, a\rangle \in S\right\}$ satisfies condition (ii). Let $B_n = \bigcup\{a \colon \langle m, a\rangle \in S \text{ for some } m < \omega\}$. Then clearly, B_n is finite. To see that it satisfies (iii), take an $h \in K_n$. Then $f(h) \in U_{\langle m,a\rangle}$ for some $\langle m, a\rangle \in S$. Let $k < \omega$ be such that $b_n \in f(h)(k)$. Since we have also $b_n \in a = f(h)(m)$, we conclude that $k = m$. So, $f(h)(k) = f(h)(m) = a \subseteq B_n$. □

Fix a $p \in \mathbb{R} \setminus \mathbb{Q}$. For $\mathcal{D} \subset [\mathbb{Q}]^\omega$, let $F(\mathcal{D}) = F(p, \mathcal{D})$ be the filter generated by $\mathcal{D} \cup \{I_n \colon n < \omega\}$, where $I_n = [p - 2^{-n}, p + 2^{-n}] \cap \mathbb{Q}$.

Lemma 2 (Ciesielski and Pawlikowski [2, Lemma 4.23])**.** *Suppose that $\mathcal{D} \subset \mathrm{Perf}(\mathbb{Q})$ is a countable family such that $F(\mathcal{D})$ is crowded. Then, for every prism P in $[\mathbb{Q}]^\omega$ there exists a subprism Q of P and a $Z \in \mathrm{Perf}(\mathbb{Q})$ such that $F(\mathcal{D} \cup \{Z\})$ is crowded and either*

(i) *$Z \cap x = \emptyset$ for every $x \in Q$, or else*
(ii) *$Z \subset x$ for every $x \in Q$.*

We will need also the following easy fact.

Lemma 3 (Ciesielski and Pawlikowski [2, Fact 4.21])**.** *Every non-scattered set $B \subset \mathbb{Q}$ contains a subset from* $\mathrm{Perf}(\mathbb{Q})$*.*

Lemma 4. *Let $\mathcal{D} \subset \mathrm{Perf}(\mathbb{Q})$ be a countable family such that $F(\mathcal{D})$ is crowded and let P be a prism in $\mathcal{P}$, then there exists a subprism Q of P and $Z \in \mathrm{Perf}(\mathbb{Q})$ such that $F(\mathcal{D} \cup \{Z\})$ is crowded and $|Z \cap x(k)| \leq 1$ for every $x \in Q$.*

Proof: Observe that since $F(\mathcal{D})$ is crowded, it is possible to find a sequence $\langle D_n \in \mathrm{Perf}(\mathbb{Q}) \colon n < \omega\rangle$ coinitial in $F(\mathcal{D})$ such that $D_{n+1} \subset D_n \subset I_n$ for every $n < \omega$.

Claim. There are sequences $\langle J_k \colon k < \omega\rangle$ of pairwise disjoint intervals in $\mathbb{Q}$ and $\langle S_k \subset J_k \colon k < \omega\rangle$ of perfect subsets of $\mathbb{Q}$ such that if $S = \bigcup_{k<\omega} S_k$ then for every $D \in F(\mathcal{D})$ there exists an $n < \omega$ such that $S \cap I_n \subset D$.

To see it, define sequences $\langle n_k \colon k < \omega\rangle$ and $\langle S_k \in \mathrm{Perf}(\mathbb{Q}) \colon k < \omega\rangle$ such that $S_k \subset D_k \cap I_{n_k} \cap J_k$ where J_k is a clopen interval such that

$p \notin \mathrm{cl}_{\mathbb{R}}(J_k)$. If n_k and S_k are already defined pick $n_{k+1} > n_k$ with $J_k \cap I_{n_{k+1}} = \emptyset$. Since $D_{k+1} \cap I_{n_{k+1}} \in F(\mathcal{D})$ and $F(\mathcal{D})$ is crowded, we can find a clopen interval J_{k+1} such that $p \notin \mathrm{cl}_{\mathbb{R}}(J_{k+1})$ and $J_{k+1} \cap D_{k+1} \cap I_{n_{k+1}} \neq \emptyset$. Define $S_{k+1} = J_{k+1} \cap D_{k+1} \cap I_{n_{k+1}}$. Then, $S_{k+1} \in \mathrm{Perf}(\mathbb{Q})$ and $S_{k+1} \subset D_{k+1} \cap I_{n_{k+1}}$. Now, put $S = \bigcup_{k<\omega} S_k$. Then, $S \in \mathrm{Perf}(\mathbb{Q})$ and $S \cap I_{n_k} = \bigcup_{i \geq k} S_i \cap I_{n_k} = \bigcup_{i \geq k} S_i \subset D_k$. This proves our claim.

Let $\mathcal{B}$ be a countable basis for the topology on $\mathbb{Q}$ consisting of clopen sets and consider the family $\mathcal{B}_0 = \{B \in \mathcal{B} : |B \cap S| = \omega\}$.

If $P \in \mathrm{Perf}(\mathcal{P})$, apply Lemma 1 to P and $\{B \cap S : B \in \mathcal{B}_0\}$ to find a set $T \in [S]^\omega$ and a subprism Q of P such that

(a) $|T \cap (B \cap S)| = \omega$ for every $B \in \mathcal{B}_0$ and
(b) $|T \cap x(k)| \leq 1$ for every $x \in Q$ and $k \in \omega$.

If $P = \{x\}$ is a singleton, we put $Q = P$ and apply Lemma 1 to the family $\{B \cap S : B \in \mathcal{B}_0\}$ and to x to obtain a T satisfying (a) and (b).

In both cases we obtain from (a) that T is dense in S. Since $S_k \in \mathrm{Perf}(\mathbb{Q})$ for every $n < \omega$, we conclude that $T \cap S_k$ is non-scattered and contains a subset Z_k from $\mathrm{Perf}(\mathbb{Q})$ for every $k < \omega$. Hence, if we put $Z = \bigcup_{k<\omega} Z_k$, then $Z \in \mathrm{Perf}(\mathbb{Q})$, $Z \cap I_k \subset D_k$ for every $k < \omega$, and $|Z \cap x(k)| \leq 1$ for every $x \in Q$ and every $k < \omega$. To see that $F(\mathcal{D} \cup \{Z\})$ is crowded, note that $Z \cap D_{n_k} \subset S \cap I_{n_k} \subset D_k$ for every $k < \omega$. □

Theorem 3. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists an ω_1-generated crowded Q-point on $\mathbb{Q}$.*

Proof: For $\mathcal{Y} = [\mathbb{Q}]^\omega \cup \mathcal{P}$, consider the topology τ on $\mathcal{Y}$ whose open sets are those $U \subset \mathcal{Y}$ such that $U \cap [\mathbb{Q}]^\omega$ and $U \cap \mathcal{P}$ are open in $[\mathbb{Q}]^\omega$ and $\mathcal{P}$, respectively. Then $\langle \mathcal{Y}, \tau \rangle$ is a Polish space. Note that $[\mathbb{Q}]^\omega$ and $\mathcal{P}$ are clopen in $\mathcal{Y}$ with this topology. Every prism $P \in \mathrm{Perf}(\mathcal{Y})$ must intersect either $[\mathbb{Q}]^\omega$ or $\mathcal{P}$. Since every non-empty clopen set in a prism is its subprism (see [3], or use Proposition 1), we can suppose, without any loss of generality, that either $P \in \mathrm{Perf}([\mathbb{Q}]^\omega)$ or $P \in \mathrm{Perf}(\mathcal{P})$. Of course, every singleton is in either $[\mathbb{Q}]^\omega$ or $\mathcal{P}$. Therefore, given a prism P in $\mathcal{Y}$ and a countable family $\mathcal{D} \subset \mathrm{Perf}(\mathbb{Q})$ such that $F(\mathcal{D})$ is crowded, we define $Z(\mathcal{D}, P) \in \mathrm{Perf}(\mathbb{Q})$ and a subprism $Q(\mathcal{D}, P)$ of P either as in Lemma 4 if $P \subset [\mathbb{Q}]^\omega$ or as in Lemma 2 if $P \subset \mathcal{P}$.

Consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi) = Q(Z(\{Z_\eta : \eta < \xi\}), P_\xi),$$

where sets Z_η are defined inductively by $Z_\eta = Z(\{Z_\zeta : \zeta < \eta\}, P_\eta)$.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ strategy, S is not a winning strategy for Player II. Hence, there is a game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ played according to S for which Player II loses; so $\mathcal{Y} = \bigcup_{\xi<\omega_1} Q_\xi$.

Let $\mathcal{U} = F(\{Z_\xi : \xi < \omega_1\})$. To see it is an ultrafilter, note that if $x \in [\mathbb{Q}]^\omega$ then there exists a $\xi < \omega_1$ such that $x \in Q_\xi$. But then, either $Z_\xi \subset x$ or $Z_\xi \cap x = \emptyset$. Therefore, either x or its complement is in $\mathcal{U}$. This proves that $\mathcal{U}$ is an ultrafilter and that $\langle Z_\xi : \xi < \omega_1\rangle \subset \mathrm{Perf}(\mathbb{Q})$ is a basis for $\mathcal{U}$. Therefore, $\mathcal{U}$ is crowded. Since no crowded ultrafilter can be principal, it follows that $\mathcal{U}$ is also non-principal. To see that $\mathcal{U}$ is a Q-point, pick an $x \in \mathcal{P}$. Then there exists a $\xi < \omega_1$ such that $x \in Q_\xi$. Thus, $Z_\xi \in \mathcal{U}$ and $|Z_\xi \cap x(k)| \le 1$ for every $k < \omega$. $\square$

References

[1] M. L. Brodskiĭ, "On some properties of sets of positive measure" (Russian), *Uspekhi Matem. Nauk* (N. S.) **4** (1949), no. 3(31), 136–138.

[2] Krzysztof Ciesielski and Janusz Pawlikowski, "Crowded and Selective Ultrafilters under the Covering Property Axiom," *J. Appl. Anal.* **9** (2003), no. 1, 19–55.

[3] ———, *The Covering Property Axiom,* CPA*: A Combinatorial Core of the Iterated Perfect Set Model.* Cambridge Tracts in Mathematics, 164. Cambridge, UK: Cambridge University Press, 2004.

[4] Eva Coplakova and Klaas Pieter Hart, "Crowded rational ultrafilters," *Topology Appl.* **97** (1999), no. 1-2, 79–84.

[5] Eric K. van Douwen, "Better closed ultrafilters on $\mathbb{Q}$," *Topology Appl.* **47** (1992), no. 3, 173–177.

[6] H. G. Eggleston, "Two measure properties of Cartesian product sets," *Quart. J. Math., Oxford Ser. (2)* **5** (1954), 108–115.

[7] Thomas Jech, *Set Theory.* New York-London: Academic Press, 1978.

[8] Richard Laver, "On the Consistency of Borel's Conjecture," *Acta Math.* **137** (1976), no. 3-4, 151–169.

[9] Arnold W. Miller, "There are no Q-points in Laver's model for the Borel conjecture," *Proc. Amer. Math. Soc.* **78** (1980), no. 1, 103–106.

Department of Mathematics; West Virginia University; Morgantown, WV 26506

E-mail address: amillan@math.wvu.edu