Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 267-276

ABSOLUTE CONES

SAM B. NADLER, JR.

Abstract. An *absolute cone* is a continuum X such that for each point $p \in X$, there is a compactum Y_p such that the pairs (X, p) and $(Cone(Y_p), v_{Y_p})$ are homeomorphic. J. de Groot conjectured that the finite-dimensional absolute cones are the finite-dimensional cells. Here the conjecture is verified for dimensions 1 and 2.

1. Introduction

All spaces are metric. A *compactum* is a nonempty compact space, and a *continuum* is a connected compactum.

We denote the cone over a space Y by $Cone(Y)$ and its vertex by v_Y.

We use $X \approx Y$ to denote that the spaces X and Y are homeomorphic, and we use $(X, p) \approx (Cone(Y), v_Y)$ to mean that there is a homeomorphism of X onto $Cone(Y)$ that takes the point $p \in X$ to the vertex v_Y.

J. de Groot [3, p. 158] defined a continuum X to be a *cone in every point*, which we call an *absolute cone*, provided that for each point $p \in X$, there is a compactum Y_p such that

$$(X, p) \approx (Cone(Y_p), v_{Y_p}).$$

2000 *Mathematics Subject Classification.* Primary 54B17, 54F15.

Key words and phrases. absolute cone, absolute hyperspace, absolute (neighborhood) retract, absolute suspension, arc, book with three pages, cone, continuum, dimension, n-cell, n-sphere, Peano continuum, simple triod, simple closed curve.

De Groot conjectured that the only n-dimensional absolute cone is an n-cell (i.e., the n-fold Cartesian product of the interval $[0,1]$ with itself).

Our main purpose here is to show that de Groot's conjecture about absolute cones is true in dimension 2; we also include a verification of the conjecture in dimension 1. Our results are in section 4 (Theorem 4.1 and Theorem 4.5).

We comment on the fact that our results are only for the two lowest dimensions. In particular, we compare our results with work on a similar problem concerning suspensions. We denote the suspension over Y by $Sus(Y)$ and its vertices by v_Y^- and v_Y^+.

De Groot [3, p. 158] defined a continuum X to be a *suspension in every pair of points* (an *absolute suspension*) provided that for each two points $p, q \in X$, there is a compactum $Y_{p,q}$ such that (here, $\approx$ means a homeomorphism of triples)

$$(X, p, q) \approx (Sus(Y_{p,q}), v_{Y_{p,q}}^-, v_{Y_{p,q}}^+).$$

De Groot conjectured that the only n-dimensional absolute suspension is an n-sphere. He reduced the verification of the conjecture to proving that finite-dimensional absolute suspensions are manifolds [3, p. 157, Theorem 2]. Andrzej Szymański [11] showed the conjecture is true in dimensions ≤ 3. The conjecture remains unresolved in higher dimensions (a partial answer is in [8]). Szymański's proof depended significantly on the fact that absolute suspensions are homogeneous. (Homogeneity follows by noting that any suspension admits a self-homeomorphism that interchanges its vertices.) On the other hand, absolute cones obviously need not be homogeneous; thus, we need methods different from Szymański's methods. The main ideas we use do not seem to generalize, even to dimension 3, although some of our ideas may be useful in higher dimensions.

2. Further Notation and Terminology

I denotes the interval $[0,1]$, and an *arc* is a space $\approx I$; S^1 denotes the unit circle in the Euclidean plane $\mathbb{R}^2$, and a *simple closed curve* is a space $\approx S^1$; dim stands for topological dimension [5]; $A \times B$ denotes Cartesian product; ∂X denotes the manifold boundary of a manifold X.

We often assume without saying so that $Cone(Y)$ is specifically the quotient space $Y \times I \diagup_{Y \times \{1\}}$ [9, p. 41, 3.15], in which case we consider $Y \times [0, 1)$ as a subspace of $Cone(Y)$. With this in mind, we write points in $Cone(Y)$ that are not the vertex v_Y as ordered pairs (y, t); however, we identify Y with the base of $Cone(Y)$, so we write points $(y, 0)$ in the base of $Cone(Y)$ simply as y, which will not cause confusion.

A *free arc in a space* X is an arc A such that $A - \partial A$ is open in X.

An *end point of a space* X is a point e of X such that e has arbitrarily small neighborhoods in X with one-point boundaries (i.e., $ord_p(X) = 1$) [9, p. 99, 6.25].

A *Peano continuum* is a locally connected continuum.

A *simple triod* is the cone over a 3-point (discrete) space. The arcs from the base of the cone to the vertex are called *legs of the simple triod*.

The cone over a simple triod is called a *book with three pages*; the cone over each leg of the simple triod is called a *page of the book*, and the cone over the vertex of the simple triod is called the *seam of the book*.

We use AR and ANR to stand for absolute retract and absolute neighborhood retract (compact or not), respectively.

The term *nondegenerate* refers to a space that contains more than one point.

We use the following well-known fact several times (a proof is in [7, p. 438, Theorem 1]):

Lemma 2.1. *Let Z be a Peano continuum that is acyclic in dimension* 1 *(i.e., contractible with respect to the circle S^1). If a subset K of Z separates two points in Z, then some subcontinuum of K separates the two points in Z.*

3. A General Lemma for Absolute Cones

The lemma is for any finite-dimensional absolute cone and, thus, is relevant to the general problem of determining absolute cones of all finite dimensions.

Lemma 3.1. *Let X be an n-dimensional absolute cone, $n < \infty$. For each point $p \in X$, let Y_p be a compactum such that*

$$(X, p) \approx (Cone(Y_p), v_{Y_p}).$$

Then X is an AR and each Y_p is an $(n-1)$-dimensional ANR; furthermore, if $n \geq 2$, each Y_p is a continuum.

Proof: We prove the parts of the lemma, one at a time, in Claims 1-3 and Claim 5. We use Claim 4 in the proof of Claim 5.

Claim 1. *X is an AR.* Each $Cone(Y_p)$ is contractible and is locally contractible at v_p; thus, X is contractible and X is locally contractible at each point p. Therefore, since X is compact and $\dim(X) < \infty$, X is an AR [2, p. 122, 10.5].

Claim 2. *Each Y_p is an ANR.* Since X is a compact finite-dimensional AR (by Claim 1), $Cone(Y_p)$ is an AR. Hence, $Y_p \times [0, 1)$ is an ANR [2, p. 96, 10.1]. Therefore, since Y_p is an r-image of $Y_p \times [0, 1)$, Y_p is an ANR [2, p. 87, 3.2].

Claim 3. $\dim(Y_{p_0}) = n - 1$ *for each Y_p.* Since $\dim(Cone(Y_p)) = n$, we have $\dim(Y_p \times [0, 1)) = n$ [5, p. 32, Corollary 2]. Therefore, since the dimension of a Cartesian product of a nonempty compact space and a 1-dimensional space is the sum of the dimensions of the two spaces [4], we have $\dim(Y_p) = n - 1$.

Claim 4. *If $n \geq 2$, then all components of each Y_p are non-degenerate.* Assume that some Y_{p_0} has a degenerate component $\{q\}$. Then, since Y_{p_0} is an ANR (by Claim 2), $\{q\}$ is open in Y_{p_0} [2, p. 101, 2.7]. Thus, since $X \approx Cone(Y_{p_0})$, X contains a free arc F. Let $r \in F - \partial F$. Then, since $(X, r) \approx (Cone(Y_r), v_{Y_r})$ and F is a neighborhood of r in X, v_{Y_r} has a neighborhood in $Cone(Y_r)$ that is an arc. Thus, since any neighborhood of the vertex of a cone contains a copy of the cone, $Cone(Y_r)$ itself is an arc. Hence, X is an arc. Therefore, $n = 1$.

Claim 5. *If $n \geq 2$, each Y_p is a continuum.* Suppose by way of contradiction that some Y_{p_0} is not connected. Then $v_{Y_{p_0}}$ is a separating point of $Cone(Y_{p_0})$. Hence, p_0 is a separating point of X. On the other hand, let $x_0 \in X - \{p_0\}$ and let h be a homeomorphism of (X, x_0) onto $(Cone(Y_{x_0}), v_{Y_{x_0}})$. Then $h(p_0) \in Cone(Y_{x_0}) - \{v_{Y_{x_0}}\}$; hence, for some (unique) component K of Y_{x_0}, $h(p_0) \in Cone(K) - \{v_{Y_{x_0}}\}$. (We consider $Cone(K)$ in the natural way as a subspace of $Cone(Y_{x_0})$, so the vertex of $Cone(K)$ is $v_{Y_{x_0}}$.) By Claim 4, K is a nondegenerate continuum. Thus, since the cone over a nondegenerate continuum has no separating point, $h(p_0)$ is not a separating point of $Cone(K)$. Therefore, since $h(p_0) \neq v_{Y_{x_0}}$, it follows that $h(p_0)$ is not a separating point of $Cone(Y_{x_0})$. Hence, p_0 is not a separating point of X. Therefore, we have a contradiction. □

4. Absolute Cones in Dimensions 1 and 2

We prove theorems 4.1 and 4.5.

Theorem 4.1. *A 1-dimensional continuum X is an absolute cone if and only if X is an arc.*

Proof: An arc is an absolute cone since

$$(I,p) \approx (Cone(\{0\}), v_{\{0\}}), \text{ if } p = 0 \text{ or } 1$$

and

$$(I,p) \approx (Cone(\{0,1\}), v_{\{0,1\}}), \text{ if } 0 < p < 1.$$

Conversely, assume that X is a 1-dimensional absolute cone. For each point $p \in X$, let Y_p be a compactum such that

$$(X,p) \approx (Cone(Y_p), v_{Y_p}).$$

Fix $p_0 \in X$. By Lemma 3.1, Y_{p_0} is a 0-dimensional ANR; hence, Y_{p_0} is a finite set [2, p. 101, 2.7]. Therefore, since $X \approx Cone(Y_{p_0})$, there is a point $p_1 \in X$ such that p_1 has a neighborhood in X that is an arc. Thus, since

$$(X,p_1) \approx (Cone(Y_{p_1}), v_{Y_{p_1}}),$$

$v_{Y_{p_1}}$ has a neighborhood N in $Cone(Y_{p_1})$ that is an arc. Now, since N contains a copy of $Cone(Y_{p_1})$, we have that $Cone(Y_{p_1})$ is an arc. Therefore, X is an arc. □

We proceed to prove our theorem about 2-dimensional absolute cones. We prove three lemmas; we use the third lemma in the proof of our theorem. The definitions of a book with three pages and its seam are in section 2.

Lemma 4.2. *If B is a book with three pages and seam S, then no two points of S can be separated in B by an arc.*

Proof: Let Q_1, Q_2, and Q_3 be the three 2-cells such that $B = \cup_{i=1}^3 Q_i$ and $Q_i \cap Q_j = S$ for $i \neq j$. Let $p, q \in S$. Assume that S is ordered by $<$ so that $p < q$.

It suffices to prove that no arc in $B - \{p, q\}$ that intersects S in only finitely many points separates p and q in B.

So, for the proof, let A be an arc in $B - \{p, q\}$ such that $A \cap S$ is finite. Let $s_1, s_2, ..., s_n$ be the points of $A \cap S$ that lie between p and q. Assume that the indexing for the points is such that

$$p < s_1 < s_2 < \cdots < s_n < q.$$

Choose points $r_k, t_k \in S$ for each $k = 1, 2, ..., n$ such that

$$p < r_1 < s_1 < t_1 < r_2 < s_2 < t_2 < \cdots < r_n < s_n < t_n < q.$$

Fix one of the points s_k. Then there is an open neighborhood U_k of s_k in B such that $U_k \cap A$ is contained in the union of at most two of the pages Q_1, Q_2, and Q_3 of B. Hence,

$$(U_k \cap A) \cap Q_{j_k} = \{s_k\} \text{ for some } j_k = 1, 2, \text{ or } 3.$$

Thus, there is an arc α_k in Q_{j_k} joining r_k and t_k such that $\alpha_k \cap A = \emptyset$.

Now, using interval notation for subarcs of S, let

$$C = [p, r_1] \cup \alpha_1 \cup [t_1, r_2] \cup \alpha_2 \cup \cdots \cup [t_{n-1}, r_n] \cup \alpha_n \cup [t_n, q].$$

It follows easily that C is a connected subset of $B - A$ such that $p, q \in C$. Therefore, A does not separate p and q in B. □

Lemma 4.3. *Let Y be a continuum, and let e be an end point of Y. Then, for any $t \in I$, (e, t) does not lie in the seam of a book with three pages in $Y \times I$.*

Proof: Let π_1 and π_2 denote the projections of $Y \times I$ onto Y and I, respectively.

Suppose by way of contradiction that for some $t \in I$, $(e, t) \in S$ where S is the seam of a book B with three pages in $Y \times I$. Let Q_1, Q_2, and Q_3 be the three 2-cells such that $B = \cup_{i=1}^3 Q_i$ and $Q_i \cap Q_j = S$ for $i \neq j$.

Assume first, as leads to a contradiction, that $\pi_1(S) = \{e\}$ (i.e., S is an arc in $\pi_1^{-1}(e)$). Let (e, s) be a separating point of S, and let p and q be the end points of S. Then $\pi_2^{-1}(s) \cap B$ separates p and q in B into the points in B with second coordinate $< s$ and the points in B with second coordinate $> s$. Clearly, then, $\pi_2^{-1}(s) \cap Q_i$ separates p and q in Q_i for each i. Thus, by Lemma 2.1, some subcontinuum C_i of $\pi_2^{-1}(s) \cap Q_i$ separates p and q in Q_i for each i. Since $Q_1 \cap Q_2 = S$, we have that

$$C_1 \cap C_2 = \{(e, s)\}.$$

Hence, $C_1 \cup C_2$ is a subcontinuum of $\pi_2^{-1}(s)$, and (e, s) is a separating point of $C_1 \cup C_2$. Therefore, (e, s) is not an end point of $\pi_2^{-1}(s)$; however, since (by the vertical projection $\pi_1|\pi_2^{-1}(s)$)

$$\big(\pi_2^{-1}(s), (e, s)\big) \approx \big(Y, e\big),$$

(e, s) is an end point of $\pi_2^{-1}(s)$. This contradiction proves that

$$\pi_1(S) \neq \{e\}.$$

Therefore, since $e \in \pi_1(S)$ (because $(e, t) \in S$), $\pi_1(S)$ is a nondegenerate subcontinuum of Y. Thus, since e is an end point of Y, there is a point $y_0 \in \pi_1(S)$ such that y_0 separates two points of $\pi_1(S)$ in Y. Hence, the arc $\pi_1^{-1}(y_0)$ separates two points p' and q' of S in $Y \times I$, which implies that $\pi_1^{-1}(y_0) \cap B$ separates p' and q' in B. Therefore, by Lemma 2.1, some arc in $\pi_1^{-1}(y_0) \cap B$ separates p' and q' in B. This contradicts Lemma 4.2. $\square$

Lemma 4.4. *If X is a 2-dimensional absolute cone, then X contains a 2-cell with nonempty interior in X.*

Proof: For each point $p \in X$, let Y_p be a compactum such that

$$(\#) \qquad (X, p) \approx (Cone(Y_p), v_{Y_p}).$$

We prove that some Y_p contains a free arc, from which our lemma follows immediately.

Suppose by way of contradiction that no Y_p contains a free arc. We will obtain a contradiction by applying Corollary 5.2 of [1, p. 105]. We first need to prove that the assumptions of the corollary are satisfied.

Since $\dim(X) = 2$, each Y_p is a nondegenerate Peano continuum by Lemma 3.1. Hence, by our assumption that no Y_p contains a free arc, each Y_p contains a simple triod [9, p. 135, 8.40(b)]. Therefore, it follows from (#) that

(1) each point of X lies in the seam of a book with three pages in X.

Fix $p_0 \in X$. Then, since $X \approx Cone(Y_{p_0})$, we have by (1) that each point of $Cone(Y_{p_0})$ lies in the seam of a book with three pages in $Cone(Y_{p_0})$. It follows that each point of $Y_{p_0} \times I$ lies in the seam of a book with three pages in $Y_{p_0} \times I$. (The homeomorphism h of $Y_{p_0} \times I$ onto itself given by $h(y,t) = (y, 1-t)$ shows this for the points of the form $(y,1)$.) Hence, by Lemma 4.3, no point of Y_{p_0} is an end point of Y_{p_0}. Therefore, by [7, p. 320, Theorem 15],

(2) each point y of Y_{p_0} lies in the manifold interior of an arc A_y in Y_{p_0}.

Let

$$M = Cone(Y_{p_0}) - (Y_{p_0} \cup \{v_{Y_{p_0}}\})$$

and let (where A_y is as in (2))

$$D_y = A_y \times [\tfrac{t}{2}, \tfrac{t+1}{2}] \text{ for each } (y,t) \in M.$$

The following statements verify that $Cone(Y_{p_0})$ satisfies the assumptions of Corollary 5.2 of [1, p. 105]: (a) *$Cone(Y_{p_0})$ is an AR* (by Lemma 3.1); (b) *M is of the second category of Baire in $Cone(Y_{p_0})$* (by the Baire Theorem [6, p. 414]); (c) *each point (y,t) of M lies in the manifold interior of the 2-cell D_y in $Cone(Y_{p_0})$* (by (2)); (d) *none of the 2-cells D_y are neighborhoods of (y,t) in $Cone(Y_{p_0})$* (by (1) since $X \approx Cone(Y_{p_0})$).

Therefore, $\dim(Cone(Y_{p_0})) > 2$ [1, p. 105, Corollary 5.2]. This contradicts the assumption in our lemma that $\dim(X) = 2$. □

Theorem 4.5. *A 2-dimensional continuum X is an absolute cone if and only if X is a 2-cell.*

Proof: A 2-cell is an absolute cone since (I^2 denotes $I \times I$)

$$(I^2, p) \approx (Cone(I), v_I), \text{ if } p \in \partial I^2$$

and

$$(I^2, p) \approx (Cone(S^1), v_{S^1}), \text{ if } p \in I^2 - \partial I^2.$$

Conversely, assume that X is a 2-dimensional absolute cone. Then, by Lemma 4.4, there is a 2-cell neighborhood of some point p_0 in X. Hence, letting Y_{p_0} be a compactum such that

$$(X, p_0) \approx (Cone(Y_{p_0}), v_{Y_{p_0}}),$$

the vertex $v_{Y_{p_0}}$ has a 2-cell neighborhood D in $Cone(Y_{p_0})$.

Suppose that Y_{p_0} contains a simple triod. Then, since D contains a copy of $Cone(Y_{p_0})$, D contains the cone over a simple triod; however, this is impossible by the Brouwer Invariance of Domain Theorem [5, p. 95, Theorem VI 9]. Hence, Y_{p_0} does not contain a simple triod.

Furthermore, Y_{p_0} is a locally connected continuum by Lemma 3.1. Hence, Y_{p_0} must be an arc or a simple closed curve [9, p. 135, 8.40(b)]. Therefore, X is a 2-cell. $\square$

In [10], the author determines when various types of hyperspaces are absolute cones or absolute suspensions. In the same paper, the author introduces the notion of absolute hyperspaces and determines the continua that are certain types of absolute hyperspaces. As noted in section 1 of [10], the question of when the hyperspace $C_n(X)$ is an absolute cone remains unanswered for $n \geq 2$.

Added in proof: Recently, in a preprint entitled "A solution to de Groot's absolute cone conjecture," Professor C. R. Guilbault has solved the absolute cone problem. He has shown that de Groot's conjecture is true in dimensions < 4 (using techniques different from mine) and false in dimensions > 4; he has also shown that in dimension 4, the conjecture is true if and only if the 3-dimensional Poincaré Conjecture is true.

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Department of Mathematics; West Virginia University; P. O. Box 6310; Morgantown, WV 26506-6310

E-mail address: nadler@math.wvu.edu