Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No. 1, 2005
Pages 385-388

A SHORT PROOF OF A THEOREM OF MORTON BROWN ON CHAINS OF CELLS

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Abstract. Suppose that a topological space X is the union of an increasing sequence of open subsets each of which is homeomorphic to the Euclidean space $\mathbf{R}^n$. Then X itself is homeomorphic to $\mathbf{R}^n$. This is an old theorem of Morton Brown. We observe that this theorem is an immediate consequence of two other theorems of Morton Brown concerning near homeomorphisms and cellular sets.

1. Introduction

Consider the following theorem due to Morton Brown [4]:

Theorem 1.1. *Suppose that a topological space $X = \cup_{i=0}^{\infty} U_i$ is the union of an increasing sequence of open subsets U_i each of which is homeomorphic to the Euclidean space $\mathbf{R}^n$. Then X is homeomorphic to $\mathbf{R}^n$.*

The aim of this paper is to give a very short proof of this theorem, based on two other theorems by Morton Brown concerning near homeomorphisms and cellular sets. These theorems read:

Theorem 1.2 ([3], [1], [6, Theorem 6.7.4])**.** *Let (X_n) be an inverse sequence of compact metric spaces with limit X_∞. If all bonding maps $X_k \to X_n$ are near homeomorphisms, then so are the limit projections $X_\infty \to X_n$.*

2000 *Mathematics Subject Classification.* Primary: 57N60. Secondary: 54B15, 54E45, 57N50.

Key words and phrases. Bing's shrinking criterion, cellular sets, near homeomorphisms.

Theorem 1.3 ([2], [5, Theorem 5.2, propositions 6.2 and 6.5])**.** *Let F be a closed subset of the n-sphere S^n. The following conditions are equivalent:*

(1) *F is cellular;*
(2) *the quotient map $S^n \to S^n/F$ (which collapses F to a point) is a near homeomorphism;*
(3) *the quotient space S^n/F is homeomorphic to S^n.*

Corollary 1.4. *Let $f : S^n \to S^n$ be a map of the n-sphere onto itself such that only one point-inverse of f has more than one point. Then f is a near homeomorphism.*

Let us explain the notions used in these theorems. A map $X \to Y$ between compact spaces is a *near homeomorphism* if it is in the closure of the set of all homeomorphisms from X onto Y, with respect to the compact-open topology on the space $C(X, Y)$ of all maps from X to Y. A (closed) *n-cell* is a space homeomorphic to the closed n-cube $[0, 1]^n$. A compact subset C of a Hausdorff n-manifold M is *cellular* if it has a base of open neighborhoods in M homeomorphic to $\mathbf{R}^n$, or, equivalently, if it is the intersection of a decreasing sequence (B_k) of closed n-cells such that each B_{k+1} lies in the interior of B_k.

Cellular sets were used in the beautiful paper [2] to prove the Generalized Schoenflies Theorem [5, Theorem 6.6]. For that, a stronger version of Corollary 1.4 was needed: every onto self-map of S^n with two non-trivial point-inverses is a near homeomorphism. This requires a little more effort. For our purposes, the elementary Theorem 1.3 suffices. To make the paper less dependent on external sources, we show in section 3 that Theorem 1.3 readily follows from Bing's Shrinking Criterion.

2. A short proof of Theorem 1.1

The proof can be made one line: consider one point compactifications, and apply Corollary 1.4 and Theorem 1.2. We now elaborate.

Let $X = \cup_{i=0}^{\infty} U_i$ be the union of an increasing sequence of open subsets U_i each of which is homeomorphic to the Euclidean space $\mathbf{R}^n$. Note that X must be Hausdorff: any two points $x, y \in X$ lie in a Hausdorff open subspace U_k. Let $X_\infty = X \cup \{\infty\}$ be the one point compactification of X. Let F_i be the complement of U_i

in X_∞. Let $X_i = X_\infty/F_i$ be the space obtained by collapsing the closed set F_i to a point. Then X_i is a one-point compactification of U_i and hence homeomorphic to the n-sphere S^n.

Since the sequence (F_i) is decreasing, there are natural maps $p_i^j : X_j \to X_i$ for $j > i$, and we get an inverse sequence (X_i) of n-spheres. Since the quotient maps $p_i^\infty : X_\infty \to X_i$ separate points of X_∞, the limit of this sequence can be identified with X_∞.

The maps $p_i^j : X_j \to X_i$ have at most one non-trivial point-inverse. According to Corollary 1.4, they are near homeomorphisms. In virtue of Theorem 1.2, so is the map $p_0^\infty : X_\infty \to X_0$. It follows that X_∞ is homeomorphic to S^n. Hence, X is homeomorphic to $\mathbf{R}^n$.

3. Shrinkable decompositions and cellular sets

To make the paper more self-contained, we show how to deduce Theorem 1.3 from Bing's Shrinking Criterion.

A *decomposition* of a set is a cover by disjoint subsets. If G is a decomposition of X, a subset of X is *G-saturated* if it is the union of some elements of G.

A decomposition G of a compact Hausdorff space X is *upper semicontinuous* if one of the following equivalent conditions holds: (1) there exists a compact Hausdorff space Y and a continuous map $f : X \to Y$ such that $G = \{f^{-1}(y) : y \in Y\}$; (2) the set $\bigcup\{g \times g : g \in G\}$ is closed in $X \times X$; (3) for every closed subset F of X its *G-saturation* $\bigcup\{g \in G : g \text{ meets } F\}$ is closed. An upper semicontinuous decomposition G of a compact metric space X is *shrinkable* if for every $\epsilon > 0$ and every cover $\mathcal{U}$ of X by G-saturated open sets there exists a homeomorphism h of X onto itself such that: (1) for every $g \in G$ the set $h(g)$ has diameter $< \epsilon$; (2) for every $x \in X$ there exists $U \in \mathcal{U}$ such that $x \in U$ and $h(x) \in U$.

Bing's Shrinking Criterion ([5, Theorem 5.2], [6, Theorem 6.1.8]). *An onto map $f : X \to Y$ between compact metric spaces is a near homeomorphism if and only if the decomposition $\{f^{-1}(y) : y \in Y\}$ of X is shrinkable.*

Proof of Theorem 1.3: (1) $\Rightarrow$ (2). If F is a cellular set in a compact n-manifold M, the decomposition G_F of M whose only non-singleton element is F is shrinkable. This easily follows from the fact that for every $\epsilon > 0$ there exists a homeomorphism of the n-cube $[0,1]^n$ onto itself which is identity on the boundary and shrinks the subcube $[\epsilon, 1-\epsilon]^n$ to a set of small diameter. Bing's Shrinking Criterion implies that the quotient map $M \to M/F$ is a near homeomorphism.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Suppose S^n/F is homeomorphic to S^n. We want to prove that F is cellular. Let U be an open neighborhood of F. Denote the quotient map $S^n \to S^n/F$ by p. Let $a \in S^n/F$ be the point onto which F collapses, $p(F) = \{a\}$. Then $p(U)$ is an open neighborhood of a. Since S^n/F topologically is a sphere, there exists a neighborhood V of a such that $V \subset p(U)$ and the complement C of V in S^n/F is cellular. Then $p^{-1}(C)$ is cellular in S^n (note that p restricted to $S^n \setminus F$ is a homeomorphism).

From the first part of the proof (implication (1) $\Rightarrow$ (2)), it follows that the complement of any cellular subset of S^n is homeomorphic to $\mathbf{R}^n$. Thus, $S^n \setminus p^{-1}(C) = p^{-1}(V)$ is homeomorphic to $\mathbf{R}^n$, and it is an open neighborhood of F which is contained in U. Since U was arbitrary, F is cellular. □

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