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SOME THEOREMS ON BASE-NORMALITY

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Abstract. Corresponding to base-paracompactness due to John E. Porter ["Base-paracompact spaces," *Topology Appl.* **128** (2003), 145–156], we previously introduced base-normality of a space in "Base-normality and product spaces," [*Topology Appl.* **148** (2005), 123–142]. In this paper, we prove the following theorems. (1) For a base-normal space X and an F_σ-set A of X, if $w(A) = w(X)$, then A is base-normal. (2) Bing's examples G and H are base-normal. (3) For the σ-product X of countably many spaces X_i, $i \in \mathbb{N}$, satisfying that finite subproducts $\Pi_{i \leq n} X_i$, $n \in \mathbb{N}$, are base-normal, X is normal if and only if X is base-normal.

1. Introduction

Throughout this paper, all spaces are assumed to be T_1 topological spaces. The symbol $\mathbb{N}$ denotes the set of all natural numbers. Let κ denote an infinite cardinal and ω the first infinite cardinal. The cardinality of a set X is denoted by $|X|$. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Bases and neighborhood bases mean open bases and open neighborhood bases, respectively. For a space X, $w(X)$ stands for the weight of X. For a space X and a subspace A of X, the closure of A in X is denoted by $\overline{A}$. For a collection $\mathcal{A}$ of subspaces of a space X, $\{\overline{A} : A \in \mathcal{A}\}$ is denoted by $\overline{\mathcal{A}}$.

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In [11], John E. Porter introduced the notion of base-paracompactness and proved some fundamental theorems. A space X is said to be *base-paracompact* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X has a locally finite refinement by members of $\mathcal{B}$. In a previous paper [15], we introduced the notion of base-normality and studied base-normality of products with a metric factor, countable products, and Σ-products. A space X is said to be *base-normal* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every binary (= two-element) open cover $\{U_0, U_1\}$ of X admits a locally finite cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that $\overline{\mathcal{B}'}$ refines $\{U_0, U_1\}$. For a Hausdorff space X, X is base-normal and paracompact if and only if X is base-paracompact [15]; this fact will be used without reference throughout the paper.

In this paper, we first show that an F_σ-set A of a base-normal space X with $w(A) = w(X)$ is base-normal. This result corresponds to a theorem on base-paracompact spaces X due to Porter [11, section 2]. Next, we give some basic examples of base-normal spaces; in particular, we show that every normal almost compact space and every ordinal space are base-normal. We also show that Bing's examples G and H are base-normal. Finally, we study base-normality from viewpoints of the σ-product of countably many spaces. Some open questions are also given.

2. F_σ-sets of base-normal spaces

In this section, we show that the theorems on base-paracompactness due to Porter in [11, section 2] can be extended to those on base-normality. Recall the following:

Theorem 2.1 (Porter [11]). *Let X be base-paracompact. If $A \subset X$ is an F_σ-set with $w(A) = w(X)$, then A is base-paracompact.*

It is unknown whether the condition "$w(A) = w(X)$" in Theorem 2.1 can be removed or not [11]. In fact, to show that the hereditarity of F_σ-sets is equivalent is to solve positively the open problem posed by Porter [11]: "Is any paracompact space base-paracompact?"

For base-normal spaces, we have the following Theorem 2.2. As was shown in [15], it is consistent with ZFC that the fact "base-normal spaces are not hereditary to clopen subsets" holds. So, the condition "$w(A) = w(X)$" in Theorem 2.2 cannot be removed.

Theorem 2.2. *Let X be base-normal. If $A \subset X$ is an F_σ-set with $w(A) = w(X)$, then A is base-normal.*

Since every F_σ-set of a paracompact space is paracompact, for a Hausdorff space X, Theorem 2.2 refines Theorem 2.1.

In [11], Theorem 2.1 is observed as a corollary to a more general theorem as follows:

Theorem 2.3 (Porter [11])**.** *If X is paracompact and the countable union of closed base-paracompact sets relative to X, then X is base-paracompact.*

Here, a subspace A of a space X is said to be *base-paracompact relative to* X if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open (in X) cover of A has a locally finite (in X) partial refinement $\mathcal{B}' \subset \mathcal{B}$ such that $A \subset \bigcup \mathcal{B}'$ [11].

Similarly, we prove Theorem 2.2 by giving Theorem 2.5. Our proof is based on that of Porter [11, section 2], with a slight modification. We call a subspace A of a space X *base-normal relative to* X if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that for every binary open (in X) cover $\{U_0, U_1\}$ of A there is a locally finite (in X) family $\mathcal{B}' \subset \mathcal{B}$ such that $\overline{\mathcal{B}'}$ is a partial refinement of $\{U_0, U_1\}$ and $A \subset \bigcup \mathcal{B}'$.

Note that X is base-normal if and only if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every locally finite open cover $\mathcal{U}$ of X admits a locally finite cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that $\overline{\mathcal{B}'}$ refines $\mathcal{U}$. The proof of the following lemma is straightforward and left to the reader.

Lemma 2.4. *The following statements hold.*

(1) *If X is a base-normal space and F is a closed subspace of X, then F is base-normal relative to X.*

(2) *Let X be a normal space and F a closed subspace of X. Then, F is base-normal relative to X if and only if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that for every locally finite open* (*in X*) *cover $\mathcal{U}$ of F there is a locally finite* (*in X*) *family $\mathcal{B}' \subset \mathcal{B}$ such that $\overline{\mathcal{B}'}$ is a partial refinement of $\mathcal{U}$ and $F \subset \bigcup \mathcal{B}'$.*

Theorem 2.5. *If X is normal and the countable union of closed base-normal sets relative to X, then X is base-normal.*

Proof: The proof is based on that of [11, Theorem 2.4].

Let X be a normal space and $X = \bigcup_{i<\omega} F_i$, where each F_i is closed base-normal relative to X. By Lemma 2.4(2), we can take a base $\mathcal{B}_i$ for X with $|\mathcal{B}_i| = w(X)$ such that every locally finite open cover $\mathcal{U}$ (in X) of F_i admits a locally finite (in X) family $\mathcal{B}' \subset \mathcal{B}_i$ such that $\overline{\mathcal{B}'}$ is a partial refinement of $\mathcal{U}$, and $F_i \subset \bigcup \mathcal{B}'$. Since $\bigcup_{i<\omega} \mathcal{B}_i$ is a base for X with $|\bigcup_{i<\omega} \mathcal{B}_i| = w(X)$, it suffices to show that $\bigcup_{i<\omega} \mathcal{B}_i$ witnesses base-normality of X. To prove this, let $\mathcal{U}$ be a binary open cover of X. There is a locally finite (in X) family $\mathcal{A}_0 \subset \mathcal{B}_0$ such that $\overline{\mathcal{A}_0}^X$ is a partial refinement of $\mathcal{U}$, and $F_0 \subset \bigcup \mathcal{A}_0$. Since $\mathcal{A}_0 \cup \{X - F_0\}$ is a locally finite open cover of a normal space X, there is a locally finite open star-refinement of $\mathcal{A}_0 \cup \{X - F_0\}$ [6, 5.1.14]. Repeating this process, we can inductively define locally finite (in X) families $\mathcal{A}_n \subset \mathcal{B}_n$, $n < \omega$, satisfying that $\mathcal{A}_n$ is a partial refinement of $\mathcal{A}_i^*$ for each $i < n$, and $\overline{\mathcal{A}_n}^X$ is a partial refinement of $\mathcal{U}$, and we can define locally finite open covers $\mathcal{A}_n^*$, $n < \omega$, of X such that $\mathcal{A}_n^*$ is a star-refinement of $\mathcal{A}_n \cup \{X - F_n\}$ and $\mathcal{A}_n^*$ refines $\mathcal{A}_i^*$ for each $i < n$. For each $j < \omega$, set $\mathcal{V}_j = \{V \in \mathcal{A}_j : V \not\subset U$ for every $U \in \bigcup_{i<j} \mathcal{A}_i\}$. Define $\mathcal{V} = \bigcup_{j<\omega} \mathcal{V}_j$. Then, $\mathcal{V}$ is a locally finite cover of X by members of $\bigcup_{i<\omega} \mathcal{B}_i$, and $\overline{\mathcal{V}}$ refines $\mathcal{U}$. Thus, X is base-normal. This completes the proof. □

Proof of Theorem 2.2: Let X be base-normal and A an F_σ-set of X with $w(A) = w(X)$. Let $A = \bigcup_{n<\omega} F_n$, where F_n, $n < \omega$, are closed in X. Fix $n < \omega$. By Lemma 2.4(1), F_n is base-normal relative to X. Now, we can show that F_n is base-normal relative to A. Since normality is hereditary with respect to F_σ-sets, it follows from Theorem 2.5 that A is base-normal. This completes the proof. □

3. Examples of base-normal spaces

A Tychonoff space X is said to be *almost compact* if $|\beta X - X| \leq 1$, where βX is the Stone-Čech compactification of X.

Proposition 3.1. *Every normal almost compact space is base-normal.*

Proof: Let X be a normal almost compact space and $\mathcal{B}$ a base for X with $|\mathcal{B}| = w(X)$. Put

$$\mathcal{B}^* = \mathcal{B} \cup \Big\{ X - \bigcup_{i \le n} \overline{B_i} \ : \ B_0, \cdots, B_n \in \mathcal{B}, \ n < \omega \Big\}.$$

Clearly, $|\mathcal{B}^*| = w(X)$ and $\mathcal{B}^*$ is a base for X. It is not difficult to show that $\mathcal{B}^*$ witnesses base-normality of X, because either one of any two disjoint closed sets is compact. This completes the proof. □

As another example of base-normal spaces, we have the following result. Every ordinal is equipped with the usual order topology. As usual, $cf(\kappa)$ stands for the cofinality of an ordinal κ.

Proposition 3.2. *Every ordinal is base-normal.*

Proof: Let κ be an ordinal. If $cf(\kappa) \le \omega$, then κ is regular Lindelöf, and it follows from [11, Theorem 3.5] that κ is base-paracompact; therefore, κ is base-normal. Assume $cf(\kappa) > \omega$. In this case, κ is almost compact. Indeed, every real-valued continuous function $f : \kappa \to \mathbb{R}$ is constant on a tail. Hence, either one of any two disjoint closed subsets of κ is bounded, that is, compact. So, it follows from Proposition 3.1 that κ is base-normal. This completes the proof. □

It is unknown whether there exist ZFC examples of normal non-base-normal spaces or not (Question 5.2: see footnote, page 12). Related to this problem, we show that some classical examples of R. H. Bing [2] are base-normal. To prove this, we first give a technical lemma. For a topological space X and a subspace A of X, X_A is the set X equipped with the topology $\{U \cup V : U$ is open in X, and $V \subset X - A\}$. For $x \in X$, $\chi(x)$ is the *character* of x in X, that is, the smallest cardinal number of the form $|\mathcal{B}_x|$, where $\mathcal{B}_x$ is a neighborhood base of x in X.

Lemma 3.3. *Let X be a topological space, A a subspace of X, and assume $w(X_A) = \max\{2^{|A|} \cdot \sup_{x \in A} \chi(x), \ |X - A|\}$. If X_A is normal, then X_A is base-normal. Therefore, if X_A is paracompact Hausdorff, then X_A is base-paracompact.*

Proof: Assume X_A is normal. For $x \in A$, we denote by $\mathcal{B}_x$ a neighborhood base of x in X with $|\mathcal{B}_x| = \chi(x)$. Set

$$\mathcal{B} = \Big\{ \bigcup_{x \in A'} B_x : \ A' \subset A, \ B_x \in \mathcal{B}_x, \ x \in A' \Big\} \cup \Big\{ \{z\} : z \in X - A \Big\}.$$

Then, $\mathcal{B}$ is a base for X_A, and it follows from the assumption that $|\mathcal{B}| = w(X_A)$. To see that $\mathcal{B}$ witnesses base-normality for X_A, let F_0 and F_1 be disjoint closed subsets of X_A. Since X_A is normal, there is an open cover $\{U_0, U_1\}$ of X_A such that $\overline{U_0}^{X_A} \cap F_1 = \emptyset$ and $\overline{U_1}^{X_A} \cap F_0 = \emptyset$. For every $a \in U_0 \cap A$, take $B_a \in \mathcal{B}_a$ such that $B_a \subset U_0$. Similarly, for every $b \in U_1 \cap A$, take $B_b \in \mathcal{B}_b$ such that $B_b \subset U_1$. Set $V_0 = \bigcup\{B_a : a \in U_0 \cap A\}$ and $V_1 = \bigcup\{B_b : b \in U_1 \cap A\}$. Then, $\mathcal{B}' = \{V_0, V_1\} \cup \{\{z\} : z \in X - V_0 \cup V_1\}$ is a locally finite cover of X_A consisting of elements of $\mathcal{B}$, and for every $W \in \mathcal{B}'$ we have $\overline{W}^{X_A} \cap F_0 = \emptyset$ or $\overline{W}^{X_A} \cap F_1 = \emptyset$. Hence, it follows that X_A is base-normal, and this completes the proof. □

Remark 3.4. It follows from Lemma 3.3 that the Michael line $\mathbb{R}_\mathbb{Q}$ is base-paracompact, which had been noted in [10].

Let κ be an uncountable cardinal, $D_s = \{0, 1\}$ for each $s \in 2^\kappa$. Let $X = \Pi_{s \in 2^\kappa} D_s$ and $A = \{f_\alpha : \alpha \in \kappa\}$, where $f_\alpha \in X$ is defined by $f_\alpha(s) = 1$ if $\alpha \in s$, and $f_\alpha(s) = 0$ otherwise, for $s \in 2^\kappa$. Then, the space X_A is *Bing's example G* ([2]; see also [6, 5.1.23]). *Bing's example H* ([2]; see also [6, 5.5.3(a)]) is constructed as follows : Consider the set $Z = (A \times \{0\}) \cup \bigcup_{i \in \mathbb{N}} (X \times \{1/i\})$, and generate a topology on Z taking as a base at a point $(x, 0)$ the sets $\{(x, 0)\} \cup \bigcup_{i=k}^{\infty} (U \times \{1/i\})$, where U is a neighborhood of x in X_A and $k \in \mathbb{N}$, and letting all the remaining points be isolated. The *metacompact version of Bing's example due to Michael* is the space X_0 constructed as follows ([8]; see also [6, 5.5.3.(c)]): let S be the subspace of the space X_A, where X_A is Bing's example G consisting of all points of X which have at most finitely many coordinates distinct from zero, and consider the space $X_0 = S \cup A$ with the subspace topology of X_A. These examples are known as normal and non-collectionwise Hausdorff spaces.

Theorem 3.5. *Bing's examples G and H and the metacompact version of Bing's example due to Michael are base-normal.*

Proof: Let X_A be Bing's example G constructed as above. Then, we can see that $|A| = \kappa$, $\sup_{x\in A}\chi(x) = 2^\kappa$, $|X - A| = 2^{2^\kappa}$, and $w(X_A) = 2^{2^\kappa}$. Hence, it follows from Lemma 3.3 that G is base-normal. A similar argument works for Bing's example H. Let $X_0 = S \cup A$ be the metacompact version of Bing's example due to Michael constructed as above. Then, $|A| = \kappa$, $\sup_{x\in A}\chi(x) = 2^\kappa$, $|X_0 - A| = |S| = 2^\kappa$, and $w(X_0) = 2^\kappa$. Apply Lemma 3.3. □

4. Base-normality and σ-products

Previously, in [15], we proved theorems on base-normality of products with a metric factor, countable products, and Σ-products. In this section, we give a theorem on the σ-product of countably many spaces by applying the method given in [15].

Let us recall the definition of σ-products from [5]. Let Ω be a set with $|\Omega| \geq \omega$, $X = \Pi_{\alpha\in\Omega}X_\alpha$ a product space, and $p = (p_\alpha)$ a fixed point of X. The subspace $\sigma = \{x = (x_\alpha) \in X : |\{\alpha \in \Omega : x_\alpha \neq p_\alpha\}| < \omega\}$ of X is called the *σ-product of spaces X_α, $\alpha \in \Omega$, (about p)*. For collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of a space X, we set $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A},\ B \in \mathcal{B}\}$ and $\bigwedge \mathcal{A} = \{\bigcap_{i=1}^n A_i : A_i \in \mathcal{A},\ i = 1, \cdots, n,\ n \in \mathbb{N}\}$.

The "only if" part of the following theorem was proven by Hui Teng [13], which had been given by Keiko Chiba (see [3]) assuming further countable paracompactness of finite subproducts $\Pi_{i\leq n}X_i$, $n \in \mathbb{N}$, and the "if" part was independently proven by Chiba in [4] and Teng in [13].

Theorem 4.1 (Chiba [3], [4]; Teng [13]). *Let X be the σ-product of countably many spaces X_i, $i \in \mathbb{N}$, and assume finite subproducts $\Pi_{i\leq n}X_i$, $n \in \mathbb{N}$, are normal. Then, X is normal if and only if X is countably paracompact.*

Note that it is unknown whether the following holds or not: For the σ-product X of spaces X_α, $\alpha \in \Omega$, satisfying that all finite subproducts $\Pi_{\alpha\in\delta}X_\alpha$, $\delta \in [\Omega]^{<\omega}$, are normal, X is normal if and only if X is countably paracompact.

Theorem 4.2. *Let X be the σ-product of countably many spaces X_i, $i \in \mathbb{N}$, and assume finite subproducts $\Pi_{i\leq n}X_i$, $n \in \mathbb{N}$, are base-normal. Then, X is normal if and only if X is base-normal.*

To prove this, we need the following:

Lemma 4.3 ([15, Lemma 3.1]). *Let X be a base-normal space, and $\mathcal{B}_X$ a base which witnesses base-normality for X. Let $\mathcal{U}$ be a locally finite open cover of X, and R^0, R^1 and K closed subsets of X such that $R \cap K = \emptyset$, where $R = R^0 \cap R^1$. Then, there is a locally finite cover $\mathcal{B}$ of X by members of $\mathcal{B}_X \wedge \mathcal{U}$ satisfying the following conditions: for every $B \in \mathcal{B}$,*

(*a*) $B \cap R = \emptyset \implies \overline{B} \cap R^0 = \emptyset$ *or* $\overline{B} \cap R^1 = \emptyset$;
(*b*) $\overline{B} \cap R \neq \emptyset \implies \overline{B} \cap K = \emptyset$.

Proof of Theorem 4.2: The proof is based on that of [15, Theorem 4.1]. Let X be the σ-product of countably many spaces X_i, $i \in \mathbb{N}$, and assume finite subproducts $\Pi_{i \leq n} X_i$, $n \in \mathbb{N}$, are base-normal. We may assume $|X_i| \geq 2$ for every $i \in \mathbb{N}$. Assume X is normal. We set $X^{(n)} = \Pi_{i \leq n} X_i$ for each $n \in \mathbb{N}$. Let $p_n : X \to X^{(n)}$ be the restriction of the natural projection $\pi_n : \Pi_{i \in \mathbb{N}} X_i \to X^{(n)}$ to X. Note that p_n is open and onto for $n \in \mathbb{N}$. The map $\pi^i_j : X^{(i)} \to X^{(j)}$ stands for the natural projection for $j \leq i$ and $i, j \in \mathbb{N}$.

For every $n \in \mathbb{N}$, let $\mathcal{G}_n$ be a base which witnesses base-normality for $X^{(n)}$. For later use, for $n \in \mathbb{N}$, we set

$$\mathcal{G}'_n = \mathcal{G}_n \cup \left(\pi^n_{n-1}\right)^{-1}\left(\mathcal{G}_{n-1}\right) \cup \left(\pi^n_{n-2}\right)^{-1}\left(\mathcal{G}_{n-2}\right) \cup \cdots \cup \left(\pi^n_1\right)^{-1}\left(\mathcal{G}_1\right),$$

and

$$\mathcal{G}^*_n = \bigwedge \mathcal{G}'_n.$$

Define $\mathcal{G} = \bigcup_{n \in \mathbb{N}} p_n^{-1}\left(\mathcal{G}^*_n\right)$. Since $|\mathcal{G}| \leq \sup_{n \in \mathbb{N}} w(X^{(n)}) = w(X)$ and $\mathcal{G}$ is a base for X, we shall show that $\mathcal{G}$ witnesses base-normality for X.

To prove this, let $\{U_0, U_1\}$ be a binary open cover of X. As usual, for every $n \in \mathbb{N}$ and every $i = 0, 1$, set

$$W^i(n) = \bigcup \{W : \ W \text{ is open in } X^{(n)}, \ p_n^{-1}(W) \subset U_i\}$$

and $W(n) = W^0(n) \cup W^1(n)$. Then, as in a proof in [4] and [13], $\{p_n^{-1}(W(n)) : n \in \mathbb{N}\}$ is an increasing open cover of X. It follows from Theorem 4.1 that X is countably paracompact. Hence, take an increasing open cover $\{G_n : n \in \mathbb{N}\}$ of X such that $\overline{G_n}^X \subset p_n^{-1}(W(n))$ for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, define

$$H_n = \bigcup \{H : H \text{ is open in } X^{(n)},\ p_n^{-1}(H) \subset G_n\}.$$

Then, $\{p_n^{-1}(H_n) : n \in \mathbb{N}\}$ is an increasing open cover of X satisfying that $\overline{H_n}^X \subset W(n)$.

Claim 1. There are locally finite covers $\mathcal{B}(n)$, $n \in \mathbb{N}$, of $X^{(n)}$, where each $\mathcal{B}(n)$ consists of members of $\mathcal{G}_n^*$, such that the following conditions are satisfied:

(a) for $n \in \mathbb{N}$ with $n > 1$, $p_n^{-1}(\mathcal{B}(n))$ refines $p_{n-1}^{-1}(\mathcal{B}(n-1))$;

(b) for $n \in \mathbb{N}$ and $B \in \mathcal{B}(n)$,

$$B \subset W(n) \implies \overline{B}^{X^{(n)}} \subset W^0(n) \ \text{ or } \ \overline{B}^{X^{(n)}} \subset W^1(n);$$

(c) for $n \in \mathbb{N}$ and $B \in \mathcal{B}(n)$,

$$\overline{B}^{X^{(n)}} \not\subset W(n) \implies \overline{B}^{X^{(n)}} \cap \overline{H_n}^{X^{(n)}} = \emptyset.$$

Proof of Claim 1: Put $X = X_1$,

$$\mathcal{B}_X = \mathcal{G}_1, \quad \mathcal{U} = \{X_1\}, \quad R^0 = X_1 - W^0(1),$$

$$R^1 = X_1 - W^1(1), \quad K = \overline{H_1}^{X^{(1)}}.$$

(Hence, we have $R = X_1 - W(1)$.) Apply Lemma 4.3 and define the resulting cover $\mathcal{B}$ by $\mathcal{B}(1)$.

Next, assume $\mathcal{B}(n)$ has been constructed so as to satisfy conditions (a), (b), and (c) above. To apply Lemma 4.3, we put

$$X = X^{(n+1)}, \quad \mathcal{B}_X = \mathcal{G}_{n+1}, \quad \mathcal{U} = \left(\pi_n^{n+1}\right)^{-1}\left(\mathcal{B}(n)\right),$$
$$R^0 = X^{(n+1)} - W^0(n+1), \quad R^1 = X^{(n+1)} - W^1(n+1),$$
$$K = \overline{H_{n+1}}^{X^{(n+1)}}.$$

(Hence, we have $R = X^{(n+1)} - W(n+1)$.) Define the resulting cover $\mathcal{B}$ by $\mathcal{B}(n+1)$. This completes the proof of Claim 1. □

For every $n \in \mathbb{N}$, set

$$\mathcal{A}(n) = \{B \in \mathcal{B}(n) : B \subset W(n)\} \ \text{ and } \ \mathcal{A}'(n) = \mathcal{B}(n) - \mathcal{A}(n).$$

Then, for every $n \in \mathbb{N}$, we have

$$p_{n+1}^{-1}\left(\bigcup \mathcal{A}'(n+1)\right) \subset p_n^{-1}\left(\bigcup \mathcal{A}'(n)\right). \tag{1}$$

To show this, let $x \in p_{n+1}^{-1}(\bigcup \mathcal{A}'(n+1))$. Take $B \in \mathcal{A}'(n+1)$ such that $x \in p_{n+1}^{-1}(B)$. By (a) of Claim 1, there is $B' \in \mathcal{B}(n)$ such that $p_{n+1}^{-1}(B) \subset p_n^{-1}(B')$. By the definition, we have $B \not\subset$

$W(n+1)$; hence, $p_{n+1}^{-1}(B) \not\subset p_{n+1}^{-1}(W(n+1))$. Since $\{p_n^{-1}(W(n)) : n \in \mathbb{N}\}$ is increasing, it follows that $p_n^{-1}(B') \not\subset p_n^{-1}(W(n))$. Hence, $B' \not\subset W(n)$, that is, $B' \in \mathcal{A}'(n)$. Thus, $x \in p_n^{-1}(\bigcup \mathcal{A}'(n))$. This completes the proof of (1).

Define

$$\mathcal{L}_0 = p_1^{-1}\big(\mathcal{A}(1)\big), \text{ and } \mathcal{L}_n = p_n^{-1}\big(\mathcal{A}'(n)\big) \ \wedge \ p_{n+1}^{-1}\big(\mathcal{A}(n+1)\big)$$

for $n \in \mathbb{N}$. Set $\mathcal{L} = \bigcup_{n \geq 0} \mathcal{L}_n$. Since each member of $\mathcal{L}$ is that of $\mathcal{G}$, to complete the proof, it suffices to show the following claims:

Claim 2. For every $L \in \mathcal{L}$, either $\overline{L}^X \subset U_0$ or $\overline{L}^X \subset U_1$ holds.

Proof of Claim 2: Let $L \in \mathcal{L}$. Then, $L \subset p_n^{-1}(A)$ for some $n \in \mathbb{N}$ and $A \in \mathcal{A}(n)$. Since $A \in \mathcal{B}(n)$ and $A \subset W(n)$, it follows from (b) of Claim 1 that $\overline{A}^{X^{(n)}} \subset W^0(n)$ or $\overline{A}^{X^{(n)}} \subset W^1(n)$ holds. Now, we can show that

$$\overline{L}^X \subset \overline{p_n^{-1}(A)}^X \subset p_n^{-1}\big(\overline{A}^{X^{(n)}}\big) \subset p_n^{-1}(W^i(n)) \subset U_i \ \text{ for } \ i = 0 \ \text{ or } \ 1.$$

This completes the proof of Claim 2. □

Claim 3. $\mathcal{L}$ is a cover of X.

Proof of Claim 3: Let $x \in X$. Since $\{p_n^{-1}(H_n) : n \in \mathbb{N}\}$ is a cover of X, take $n \in \mathbb{N}$ such that $x \in p_n^{-1}(H_n)$. Since $\mathcal{B}(n)$ is a cover of $X^{(n)}$, there is $B \in \mathcal{B}(n)$ such that $p_n(x) \in B$. As $B \cap H_n \neq \emptyset$, it follows from (c) of Claim 1 that $B \subset W(n)$; hence, $B \in \mathcal{A}(n)$. Thus, $x \in p_n^{-1}(\bigcup \mathcal{A}(n))$. Now, let m be the minimum m $(\leq n)$ such that $p_m(x) \in \bigcup \mathcal{A}(m)$. Then, either $x \in \bigcup p_1^{-1}(\mathcal{A}(1))$ or $x \in \bigcup p_m^{-1}(\mathcal{A}(m)) - \bigcup p_{m-1}^{-1}(\mathcal{A}(m-1))$ holds. If $x \in \bigcup p_1^{-1}(\mathcal{A}(1))$, we clearly have $x \in \bigcup \mathcal{L}_0$. So, we may assume $x \in \bigcup p_m^{-1}(\mathcal{A}(m)) - \bigcup p_{m-1}^{-1}(\mathcal{A}(m-1))$. It follows from $X = \big(\bigcup p_{m-1}^{-1}(\mathcal{A}(m-1))\big) \cup \big(\bigcup p_{m-1}^{-1}(\mathcal{A}'(m-1))\big)$ that $x \in \bigcup p_{m-1}^{-1}(\mathcal{A}'(m-1))$. Take $B' \in \mathcal{A}'(m-1)$ and $B \in \mathcal{A}(m)$ such that $x \in p_{m-1}^{-1}(B') \cap p_m^{-1}(B)$. Hence, we have $x \in \bigcup \mathcal{L}_{m-1}$. This completes the proof of Claim 3. □

Claim 4. $\mathcal{L}$ is locally finite in X.

Proof of Claim 4: Let $x \in X$. Since $\{p_n^{-1}(H_n) : n \in \mathbb{N}\}$ is an increasing open cover, take $m \in \mathbb{N}$ and a neighborhood U of $p_m(x)$ in $X^{(m)}$ such that $p_m^{-1}(U) \subset p_m^{-1}(H_m)$. Since each $\mathcal{L}_i$ is locally finite in X, it suffices to show that $\bigcup_{j \geq m} \mathcal{L}_j$ is locally finite at x. The following holds.

$$U \cap \big(\bigcup \mathcal{A}'(m)\big) = \emptyset. \tag{2}$$

To prove (2), first prove $\big(\bigcup \mathcal{A}'(m)\big) \cap H_m = \emptyset$ by using (c) of Claim 1. Since $U \subset H_m$, we have $U \cap \big(\bigcup \mathcal{A}'(m)\big) = \emptyset$. This completes the proof of (2).

To finish the proof of Claim 4, it suffices to show that

$$p_m^{-1}(U) \cap L = \emptyset \quad \text{for every} \quad L \in \bigcup_{j \geq m} \mathcal{L}_j. \tag{3}$$

To prove (3), assume to the contrary that $p_m^{-1}(U) \cap L \neq \emptyset$ for some $L \in \mathcal{L}_j$ with $j \geq m$. Then, L is expressed as $L = p_j^{-1}(B') \cap p_{j+1}^{-1}(B)$ for some $B' \in \mathcal{A}'(j)$ and $B \in \mathcal{A}(j+1)$. It follows from (1) that

$$\begin{aligned} \emptyset &\neq p_m^{-1}(U) \cap p_j^{-1}(B') \cap p_{j+1}^{-1}(B) \subset p_m^{-1}(U) \cap p_j^{-1}\big(\bigcup \mathcal{A}'(j)\big) \\ &\subset p_m^{-1}(U) \cap p_m^{-1}\big(\bigcup \mathcal{A}'(m)\big). \end{aligned}$$

Hence, $U \cap \big(\bigcup \mathcal{A}'(m)\big) \neq \emptyset$, which contradicts (2). Thus, (3) holds. This completes the proof of Claim 4. □

Claims 2, 3, and 4 complete the proof of Theorem 4.2. □

A space X is said to be *base-κ-paracompact* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X of cardinality at most κ has a locally finite refinement by members of $\mathcal{B}$ [15]. In particular, a space X is said to be *base-countably paracompact* if X is base-ω-paracompact. The example of a normal non-base-normal space, introduced in [15], is countably paracompact but not base-countably paracompact. For a space X is normal and base-countably paracompact if and only if X is base-normal and countably paracompact [15, Proposition 2.2].

By Theorems 4.1 and 4.2, we have:

Corollary 4.4. *Let X be the σ-product of countably many spaces X_i, $i \in \mathbb{N}$, and assume finite subproducts $\Pi_{i \leq n} X_i$, $n \in \mathbb{N}$, are base-normal. Then, the following statements are equivalent.*

(1) *X is normal*;
(2) *X is countably paracompact*;
(3) *X is base-normal*;
(4) *X is base-countably paracompact.*

5. Problems and related comments

In this section, we introduce some open problems. In [11, Theorem 3.6 and Corollary 3.8], Porter proved that base-paracompactness is an inverse invariant of perfect mappings. Hence, paracompact M-spaces (= paracompact p-spaces) are base-paracompact. The proofs of the above facts in [11] essentially show that if $f : X \to Y$ be a perfect map onto a base-κ-paracompact space Y and $w(X) \leq \kappa$, then X is base-(κ)-paracompact. Hence, we have the following:

Question 5.1 ([15])**.** Is base-κ-paracompactness an inverse invariant of perfect mappings?

The (necessarily consistent) example of normal non-base-normal spaces introduced in [15] is normal non-collectionwise Hausdorff. Hence, it is natural to ask the following:

Question 5.2.[1] Are there ZFC examples of normal non-base-normal spaces?

Question 5.3. Is any normal countably compact space base-normal?

Proposition 3.1 can be seen as a partial answer to Question 5.3.

Porter [11, Question 4.2] asked if any paracompact GO-space is base-paracompact. For our case, we ask

Question 5.4. Is any GO-space base-normal?

A version of the Morita-Rudin-Starbird Theorem is obtained in [15, Theorem 1.1]: For a base-normal space X and a metrizable space Y, if the product space $X \times Y$ is normal, then $X \times Y$ is base-normal. Recently, Yukinobu Yajima [14, Corollary 6.5] proved the following: For a base-paracompact space X and a base-paracompact σ-space Y, if the product space $X \times Y$ is paracompact and rectangular, then $X \times Y$ is base-paracompact. This is obtained from the following important result [14, Theorem 4.1]: For a space X, a paracompact σ-space Y, and a normal open cover $\mathcal{O}$ of $X \times Y$,

[1]Gary Gruenhage has constructed a ZFC example of a countably compact LOTS which is not base-normal. Hence, his example answers Question 5.2 affirmatively, and questions 5.3 and 5.4 negatively.
(See http://web6.duc.auburn.edu/~gruengf/preprints/basepara.pdf.)

$\mathcal{O}$ has a σ-locally finite refinement consisting of cozero-rectangles if and only if $\mathcal{O}$ has a locally finite (and σ-discrete) refinement, consisting of cozero-rectangles, which has a shrinking consisting of zero-rectangles. A product space $X \times Y$ of spaces X and Y is said to be *rectangular* if every finite cozero-set cover of $X \times Y$ has a σ-locally finite refinement consisting of cozero-rectangles of $X \times Y$ [9], and a subspace A of the product space $X \times Y$ is said to be a *cozero-rectangle* (*zero-rectangle*, resp.) if A is denoted by $A = B \times C$ by using cozero-sets (zero-sets, resp.) B and C of X and Y, respectively. Hence, it should be noted that by using [14, Theorem 4.1], we can show: For a base-normal space X and a base-paracompact σ-space Y, if the product space $X \times Y$ is normal and rectangular, then $X \times Y$ is base-normal. This result, together with the fact "every normal product $X \times Y$ of a normal space X and a metrizable space Y is rectangular" due to B. A. Pasynkov [9], also implies [15, Theorem 1.1]. On the other hand, it is unknown whether a normal product $X \times Y$ of a normal space X and a paracompact σ-space Y (more generally, a Lašnev space Y) is rectangular or not. Hence, a natural question arises:

Question 5.5. Let X be a base-normal space. Let Y be a base-paracompact σ-space, in particular, a base-paracompact Lašnev space. Assume $X \times Y$ is normal. Is $X \times Y$ base-normal?

By using [11, Theorem 3.6], we can extend [15, Corollary 6.6] as follows: For a base-paracompact space X and a paracompact M-space (= a paracompact p-space) Y, the product space $X \times Y$ is normal if and only if $X \times Y$ is base-paracompact. Being motivated by this fact, we have another question.

Question 5.6. Let X be a base-normal space. Let Y be a paracompact M-space, in particular, a compact space. Assume $X \times Y$ is normal. Is $X \times Y$ base-normal?

Remark 5.7. If Question 5.1 is affirmative for $\kappa = \omega$, then Question 5.6 is also affirmative. To show this, let X be a base-normal space and Y a paracompact M-space, and assume $X \times Y$ is normal. Let $f : Y \to Z$ be a perfect map onto a metrizable space Z. First assume Z is non-discrete. Then, X is countably paracompact. It follows from [15, Theorem 6.1] that $X \times Z$ is base-countably paracompact. Hence, if Question 5.1 is affirmative for $\kappa = \omega$, it follows that $X \times Y$ is base-countably paracompact. By [15, Proposition 2.2], we have $X \times Y$ is base-normal. Next, we assume Z is discrete and $f^{-1}(z)$ is non-discrete for some $z \in Z$. Since $f^{-1}(z)$ is non-discrete compact and $X \times f^{-1}(z)$ is normal, it follows from [12, Theorem 3.12] that X is countably paracompact. In the case where Z is non-discrete, we have $X \times Y$ is base-normal. Finally, we assume Z is discrete and $f^{-1}(z)$ is discrete for every $z \in Z$. In this case, since Y is discrete, $X \times Y$ is base-normal.

As a version of the Gul'ko-Rudin Theorem, we obtained in [15] the following: Every Σ-product of metric spaces is base-normal. In view of A. P. Kombarov's theorem in [7], we have a natural question.

Question 5.8. Let Σ be a Σ-product of paracompact M-spaces, and assume Σ is normal. Is Σ base-normal?

We conclude this paper by asking:

Question 5.9 ([15])**.** In the definition of base-normality, is it possible to replace "a locally finite cover $\mathcal{B}'$" by "a σ-locally finite cover $\mathcal{B}'$"? (Note that, for base-countably paracompact spaces, such a replacement is possible.)

If Question 5.9 has an affirmative answer, the proofs of many results related to base-paracompactness and base-normality could be simplified.

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