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WHEN IS A COMPACT SPACE SEQUENTIALLY COMPACT?

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Dedicated to Professor Edison Farah on the occasion of his 90th birthday

Abstract. We obtain some new conditions under which a compact space is sequentially compact.

1. Introduction, Notation and Preliminary Results

Conditions under which compactness implies sequential compactness in Hausdorff spaces have been published by many authors. In 1969, Franklin [5] showed that every compact Hausdorff space of cardinality less than 2^{ω_1} is sequentially compact and in 1973, Malykhin and Šapirovskii [7] showed that under Martin's Axiom each compact Hausdorff space of cardinality less than $2^{\mathfrak{c}}$ is sequentially compact. Fewer results are known relating these concepts in the class of all topological spaces; some, assuming Martin's Axiom may be found in the above mentioned article [7], and in 1976 Levine [6] showed that every compact topological space of cardinality ω_1 is sequentially compact. Some results of this type can also be found in [8], possibly the most important being that if X is countably compact and $w(X) < \mathfrak{s}$ then X is sequentially compact. Our aim in this note is to give some new conditions under which a compact topological space (or a compact T_1-space) is sequentially compact.

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A set X with topology τ will be denoted by (X,τ). Undefined topological notation and terminology can be found in [4], but all separation axioms we use are explicitly stated; specifically, compactness in this paper does not include the Hausdorff axiom. The closure of a set A in a topological space (X,τ) will be denoted by $\mathrm{cl}(A)$. The notation $A \subset^* B$ means that $A \setminus B$ is finite.

A space is *US* if each convergent sequence has a unique limit and is *SC* if it is US and each convergent sequence together with its (unique) limit is closed. Finally, a space is *KC* if every compact subset is closed. Clearly $KC \Rightarrow SC \Rightarrow US \Rightarrow T_1$.

Recall that $\mathfrak{p}$ is the minimal cardinality of a subset of $[\omega]^\omega$ with the strong finite intersection property but with no infinite pseudo-intersection. The splitting number will be denoted by $\mathfrak{s}$, and $\mathfrak{t}$ is the minimum cardinality of a tower in ω, that is to say a subset of $[\omega]^\omega$ which is well-ordered by $\supset^*$ but which has no infinite pseudo-intersection (see [3] for the requisite definitions). It is known that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{s} \leq \mathfrak{c} = 2^\omega$. In the sequel, we identify $\mathfrak{p}$, $\mathfrak{s}$ and $\mathfrak{t}$ with the initial ordinals of cardinality $\mathfrak{p}$,$\mathfrak{s}$ and $\mathfrak{t}$, respectively. The hereditary Lindelöf number of a space X will be denoted by $hL(X)$. Recall that a subset A of a topological space X is *right separated* if there exists a well-order $<$ on A such that every initial segment of $(A,<)$ is open. Note that if X is a T_1-space and A is right separated of order type ω, then it is discrete. Furthermore, it is well known that $hL(X) = \sup\{|A| : A \subseteq X \text{ is right separated}\}$.

2. Compact spaces which are sequentially compact

Each hereditarily Lindelöf compact T_2-space is first countable and hence sequentially compact. We now extend this result to the class of all topological spaces.

Theorem 2.1. *A topological space X with $hL(X) < \mathfrak{t}$ is countably compact if and only if it is sequentially compact.*

Proof. The sufficiency is clear. For the necessity, suppose to the contrary that (X,τ) is a countably compact space with $hL(X) < \mathfrak{t}$ which is not sequentially compact and that S is a sequence in X with no convergent subsequence. Note that by [6], every countable, countably compact topological space is sequentially compact and hence no infinite subset of S is closed (in fact, every infinite subset of S must have uncountably many accumulation points).

As a consequence, for each infinite set $C \subseteq S$, and $x \in \mathrm{cl}(C) \setminus C$, there is an open neighbourhood U_x of x such that $C \setminus U_x$ is infinite, for otherwise, C converges to x contradicting our assumption regarding S. We will construct recursively a strictly increasing nested family of open sets in a subspace of X of length $\mathfrak{t}$, contradicting the fact that $hL(X) < \mathfrak{t}$.

To this end, let $S = S_0$ and choose $x_0 \in \mathrm{cl}(S) \setminus S$ and an open neighbourhood U_0 of x_0 such that $S_1 = S_0 \setminus U_0$ is infinite.

Suppose that for some ordinal $\alpha \in \mathfrak{t}$ we have chosen points $\{x_\beta : \beta \in \alpha\}$, infinite subsets $\{S_\beta : \beta \in \alpha\}$ of S and open sets $\{U_\beta : \beta \in \alpha\}$, such that

i) $x_\beta \in U_\beta$ for all $\beta \in \alpha$,
ii) $x_\beta \in (\mathrm{cl}(S_\beta) \setminus S_\beta) \setminus U_\gamma$ for all $\gamma < \beta < \alpha$,
iii) $S_\beta \subset^* S_\gamma \setminus U_\gamma$ for all $\gamma < \beta < \alpha$, and
iv) $S_\beta \setminus U_\beta$ is infinite for all $\beta \in \alpha$,

we proceed to choose x_α, S_α and U_α as follows:

By *iii)*, $S_\beta \subset^* S_\gamma$ for each $\gamma \in \beta \in \alpha$ and, since $|\alpha| < \mathfrak{t}$, there is some infinite set $S_\alpha \subset^* S_\beta$ for all $\beta \in \alpha$. Again by *iii)*, we have $S_\alpha \subset^* S_\beta \subset^* S_\gamma \setminus U_\gamma$ for all $\gamma \in \beta \in \alpha$ whence it follows that $S_\alpha \subset^* S \setminus U_\gamma$ for all $\gamma \in \alpha$ and hence all accumulation points of S_α lie outside U_γ for each $\gamma \in \alpha$. Choose $x_\alpha \in \mathrm{cl}(S_\alpha) \setminus S_\alpha$ and an open neighbourhood U_α of x_α such that $S_\alpha \setminus U_\alpha$ is infinite. It is clear that $\{x_\beta : \beta \leq \alpha\}$, $\{S_\beta : \beta \leq \alpha\}$ and $\{U_\beta : \beta \leq \alpha\}$ satisfy *i) - iv)* above.

Let $L = \{x_\alpha : \alpha \in \mathfrak{t}\}$; by construction, each $x_\alpha \in L$ has an open neighbourhood $U_\alpha \cap L$ contained in $\{x_\beta : \beta \leq \alpha\}$; that is to say, L is right separated and the result follows. □

Corollary 2.2. *A hereditarily Lindelöf topological space is compact if and only if it is sequentially compact.*

Proof. A hereditarily Lindelöf sequentially compact space is countably compact and hence compact. The converse follows from Theorem 2.1. □

Corollary 2.3. *If S is a sequence in a T_1-space, then either S has a convergent subsequence or S has an infinite discrete subset.*

Proof. Let $S = \{x_n : n \in \omega\}$ be a sequence in a T_1-space (X, τ) which has no convergent subsequence. Since S is countable, $hL(S) = \omega$ and so by the theorem, if S is countably compact then it is sequentially compact, in which case S possesses a convergent subsequence. Alternatively, S is not countably compact in which case it possesses an infinite discrete subspace. □

As we mentioned in Section 1, it was shown in [6], that each compact space of cardinality at most ω_1 is sequentially compact. However, this result cannot be generalized to compact T_1-spaces whose hereditary Lindelöf number, or even whose weight is ω_1, since it is known to be independent of *ZFC* whether $\{0,1\}^{\omega_1}$ is sequentially compact(see [8]).

A result somewhat similar to the next theorem was proved for a more restricted class of spaces in [1]. For completeness, we give a proof here, but first we need a definition and a simple lemma (which may be known).

We say that a point x is a *proper accumulation point* of a set $A \subseteq X$ if every neighbourhood of x meets A in an infinite set. Of course, if X is a T_1-space then every accumulation point is proper.

Lemma 2.4. *If X is an infinite countably compact topological space, then every infinite subset of X has a proper accumulation point.*

Proof. Suppose to the contrary that X is a countably compact space and $S = \{s_n : n \in \omega\} \subseteq X$ is an infinite subset with no proper accumulation point. Clearly we may assume that $\mathrm{cl}(S) = X$. Thus for each point $x \in X$ we may find an open neighbourhood U_x of x such that $U_x \cap S$ is finite and of minimal cardinality. For each $m \in \omega$, let $V_m = \{x \in X : (U_x \cap S) \setminus \{x\} \subseteq \{s_0, \ldots s_m\}\}$; clearly V_m is an open subset of X and $X = \bigcup\{V_m : m \in \omega\}$. Furthermore, since $V_m \subseteq V_{m+1}$ and X is countably compact, there is some $k \in \omega$ such that $V_k = X$ and then the sequence $\{s_n : k < n \in \omega\}$ has no accumulation point, a contradiction. □

Recall that the minimal cardinality of a local base at $x \in X$ is denoted by $\chi(x, X)$.

Theorem 2.5. *A countably compact topological space X such that $\chi(x, X) < \mathfrak{p}$ for all $x \in X$ is sequentially compact.*

Proof. Suppose that S is an injective sequence in X; let a be a proper accumulation point of S and let $\{U_\alpha : \alpha < \lambda\}$ be a local base of open sets at a, where $\lambda < \mathfrak{p}$. Then the family $\{U_\alpha \cap S : \alpha < \lambda\}$ is a family of subsets of S with the strong finite intersection property. By the definition of $\mathfrak{p}$, there is an infinite subset $T \subseteq S$ such that $T \setminus U_\alpha$ is finite for each $\alpha < \lambda$. Clearly then, the countably infinite set T converges to a. □

In light of Lemma 2.4, the following theorem has a proof similar to that of a part of Theorem 6.1 of [3]. We leave the reader to make the necessary changes in the proof found there. We denote by $pw(X)$ the *point weight* of the space X, that is the minimal infinite cardinal κ such that there is a base for the topology of X in which every point lies in at most κ elements of the base.

Theorem 2.6. *Let X be a countably compact topological space with $pw(X) < \mathfrak{s}$, then X is sequentially compact.*

Our next theorem is a simple extension of a theorem of Levine [6].

Theorem 2.7. *Every compact topological space of cardinality no greater than $\mathfrak{t}$ is sequentially compact.*

Proof. We enumerate X as $\{x_\alpha : \alpha < \mathfrak{t}\}$. If $S = \{y_n : n \in \omega\}$ has no convergent subsequences, we will define recursively for each $\beta < \mathfrak{t}$:

(i) Open sets U_β such that $x_\beta \in U_\beta$ and

(ii) Sets $D_\beta \in [\omega]^\omega$ such that $y_n \notin U_\beta$ whenever $n \in D_\beta$ and $D_\beta \subset^* D_\gamma$ whenever $\gamma < \beta$.

Since $S \not\to x_0$, there is an open neighbourhood U_0 of x_0 such that $D_0 = \{n \in \omega : y_n \notin U_0\}$ is infinite.

Suppose now that for some $\alpha < \mathfrak{t}$ we have chosen U_β and D_β satisfying *(i)* and *(ii)* above for all $\beta < \alpha$. We proceed to select U_α and D_α as follows. Since $\alpha < \mathfrak{t}$, there is some infinite set $T_\alpha \subseteq^* D_\beta$ for all $\beta < \alpha$. Since $\{y_n : n \in T_\alpha\} \not\to x_\alpha$, there is some open neighbourhood U_α of x_α such that $\{y_m : m \in T_\alpha\} \setminus U_\alpha$ is infinite. We set $D_\alpha = \{n \in T_\alpha : y_n \notin U_\alpha\}$. Clearly $\{U_\alpha : \alpha \in \mathfrak{t}\}$ is an open cover of X and since this space is compact, there is an irreducible finite subcover $\{U_{\alpha_0}, \dots, U_{\alpha_n}\}$, where we assume $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ and hence $D_{\alpha_n} \subseteq^* D_{\alpha_k}$ for each $0 \leq k < n$.

However, $\{y_m : m \in T_{\alpha_n+1}\} \subseteq^* X \setminus U_\beta$ for each $\beta < \alpha_n + 1$ thus implying that the accumulation points of $\{y_m : m \in T_{\alpha_n+1}\}$ lie outside of $\cup\{U_{\alpha_k} : 0 \leq k \leq n\}$, a contradiction. □

Our final theorem is a generalization of Theorem 6.3 of [3], but first we need a lemma which itself is a generalization of Theorem 1.7 of [1] and which seems to be of some interest in its own right.

If A is a subset of a topological space (X, τ), then A^d will denote the *derived set of* A, that is to say, the set of accumulation points of A.

Lemma 2.8. *Suppose that* X *is a compact KC-space and* A *is a countably infinite subset of* X *with the property that every infinite subset of* A *has infinitely many accumulation points in* X*. Then there are infinite sets* $B, C \subseteq A$ *such that* $B^d \cap C^d = \emptyset$*.*

Proof. Since every infinite KC-space contains an infinite discrete subset, without loss of generality we can assume that A is discrete and we further suppose that $\mathrm{cl}(A) = X$. Given $x \in X \setminus A$, if every neighbourhood U of x is such that $A \setminus U$ is finite, then $A \cup \{x\}$ is compact, hence closed and so x is the unique accumulation point of A, a contradiction. Thus we can choose an open neighbourhood V of x such that $A \setminus V$ is infinite and all its accumulation points lie in $X \setminus V$. Since $X \setminus V$ is closed in X, it is compact and so $A \cup (X \setminus V)$ is Lindelöf; however, this latter set is not closed in X and hence it is not compact, so not countably compact. Thus there is a countably infinite, discrete subset $D \subseteq A \cup (X \setminus V)$ which is closed in $A \cup (X \setminus V)$. Since $X \setminus V$ is compact, $D \cap (X \setminus V)$ must be finite and so $A \cap D$ is infinite and all its accumulation points must lie in V. Thus we have constructed two infinite sets $A \cap D$ and $A \setminus V$ whose derived sets are infinite and disjoint. □

Before applying this result to our final theorem, we make the following observations. First note that the condition of being KC cannot be weakened to US - adding one point ∞ to $\beta\omega$ in such a way that neighbourhoods of ∞ are of the form $U \cup \{\infty\}$ where $\beta\omega \setminus U$ is finite, produces a compact space which is US and satisfies the hypotheses of Lemma 2.8 (with $A = \omega$), but not its conclusion. This raises the question:

Question 2.9. *Is Lemma 2.8 valid with SC replacing KC?*

In this regard, the following example seems illustrative and gives a negative answer to Question 2.13 of [1]. For details of the construction of the Wallman compactification, we refer the reader to [4], page 177.

Example 2.10. *There is a KC-space whose Wallman compactification is not KC.*

Proof. Enumerate the infinite subsets of ω as $[\omega]^\omega = \{A_\alpha : \alpha \in \mathfrak{c}\}$ and let D be a set of cardinality $\mathfrak{c}$ disjoint from ω. Let $p : [\mathfrak{c}]^2 \to D$ be a bijection and for simplicity put $p_{\alpha,\beta} = p(\{\alpha, \beta\})$. For each $\alpha \in \mathfrak{c}$, let $\mathcal{U}_\alpha$ be a free ultrafilter on the countably infinite set A_α and finally let $X = \omega \cup \{p_{\alpha,\beta} : \{\alpha, \beta\} \in [\mathfrak{c}]^2\}$. Define a topology τ on X as follows:

ı) If $W \subseteq \omega$, then $W \in \tau$, and

ıı) If $p_{\alpha,\beta} \in W$, then $W \in \tau$ if and only if $W \supseteq U \cup V$ where $U \in \mathcal{U}_\alpha$ and $V \in \mathcal{U}_\beta$.

First note that (X, τ) is a T_1-space since each point of ω is isolated and each basic open set contains at most one of the points $p_{\alpha,\beta}$. To show that (X, τ) is a KC-space, it clearly suffices to show that each compact subspace of X is finite.

To this end, observe that $X \setminus \omega$ is a closed discrete subspace of X and hence X is scattered with scattering length 2. It follows that a compact subset $C \subseteq X$ can contain only a finite number of the points $p_{\alpha,\beta}$ and so must be countable. If $C \cap \omega$ is infinite, then we enumerate $C \setminus \omega$ as $\{p_{\alpha_k,\beta_k} : 1 \leq k \leq n\}$ and we can then find $A \in [\omega]^\omega$ be such that $A \notin \bigcup\{\mathcal{U}_{\alpha_k} : 1 \leq k \leq n\} \cup \bigcup\{\mathcal{U}_{\beta_k} : 1 \leq k \leq n\}$. Clearly A is an infinite closed discrete subspace of C and hence C is not compact. This proves that the space X is KC.

Finally, the Wallman compactification $W(X)$ of the space X is a compact T_1-space in which every infinite subset of ω has infinitely many accumulation points but which does not satisfy the conclusion of Lemma 2.8; thus $W(X)$ is not KC. $\square$

As mentioned above, the previous example provides a negative answer to Question 2.13 of [1]. However, we conjecture that the compact space $W(X)$ is an SC-space; if true, this would show that Lemma 2.8 cannot be extended to the class of SC-spaces.

In the proof of our final result, we mention only the points of difference with Theorem 6.3 of [3].

Theorem 2.11. *If X is a compact KC-space with $|X| < 2^{\mathfrak{t}}$, then X is sequentially compact.*

Proof. Since X is assumed to be KC, it suffices to show that every countably infinite subset contains an infinite subset with only a finite number of accumulation points. To this end, suppose to the contrary that $A \subseteq X$ is such that every infinite subset of A has an infinite number of accumulation points.

Since X is a T_1-space, it follows that $K^d \subseteq L^d$ whenever $K \subset^* L$ and by the previous lemma, for all infinite $T \subseteq A$ there are infinite $M, N \subseteq T$ such that $M^d \cap N^d = \emptyset$. Using these two facts, as in van Douwen's proof, we can show that for each $\eta \in \mathfrak{t}$ there are subsets $T_f \subseteq X$ for all $f \in {}^{\eta}2$ such that

(a) For all $f \in {}^{\eta}2$ and for all $\xi \in \eta$, $T_f \subset^* T_{f\restriction\xi}$, and

(b) For all $f, g \in {}^{\eta}2$, $T_f^d \cap T_g^d = \emptyset$ whenever $f \neq g$.

A standard tree argument now shows that $|X| \geq 2^{\mathfrak{t}}$. Since X is compact and each T_f^d is closed and infinite, we can define

$\Psi : {}^{\mathfrak{t}}2 \to \mathcal{P}(X) \setminus \{\emptyset\}$ by $\Psi(f) = \bigcap\{T^d_{f\restriction\eta} : \eta \in \mathfrak{t}\}$.

Clearly $\Psi(f) \neq \Psi(g)$ whenever $f \neq g$, implying that $|X| \geq 2^{\mathfrak{t}}$, a contradiction. □

We note that this last result is a theorem in ZFC; however, it is at least consistently true that every compact T_1-space of cardinality less than $2^{\mathfrak{t}}$ is sequentially compact. To see this, we assume that $\mathfrak{t} = \mathfrak{c}$ and $2^{\mathfrak{t}} = \mathfrak{c}^+$ (these conditions are consequences of GCH, for example); the result now follows from Theorem 2.7. Thus we ask:

Question 2.12. *Does Theorem 2.11 hold in ZFC for compact spaces which are SC (respectively, US, T_1)?*

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