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QUOTIENTS OF TEXTURES AND OF DITOPOLOGICAL TEXTURE SPACES

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Abstract. The author considers equivalence direlations and the corresponding quotients of textures, using the interior relation on the texture as a tool in this investigation. A close relationship with surjective difunctions is shown, and the notions of quotient ditopology and quotient difunction are defined and studied. Finally, the existence of a T_0 reflection on the category **dfDitop** is established.

1. Introduction

In this section we recall some basic notions regarding textures and ditopological texture spaces, which should make the article accessible to the casual reader. Full details, motivation and background material may be obtained from [2–9, 17].

Textures

Let S be a set. We work within a subset $\mathcal{S}$ of the power set $\mathcal{P}(S)$ called a *texturing*. A texturing is a point-separating, complete, completely distributive lattice with respect to inclusion, which contains S and $\emptyset$, and for which arbitrary meets coincide with intersections, and finite joins with unions.

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If $\mathcal{S}$ is a texturing of S the pair $(S, \mathcal{S})$ is called a *texture* [2].

The internal definition of textural concepts are expressed using the *p–sets* and *q–sets*, namely, for $s \in S$, the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}, \quad Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}.$$

Examples 1.1.

(1) The *discrete texture* $(X, \mathcal{P}(X))$ on the set X. For $x \in X$, $P_x = \{x\}$, $Q_x = X \setminus \{x\}$.
(2) The texture $(L, \mathcal{L})$, where $L = (0, 1]$ and $\mathcal{L} = \{(0, r] \mid 0 \le r \le 1\}$. Here, for $r \in L$, $P_r = Q_r = (0, r]$.
(3) The *unit interval texture* $(\mathbb{I}, \mathcal{I})$, $\mathbb{I} = [0, 1]$, $\mathcal{I} = \{[0, r) \mid r \in \mathbb{I}\} \cup \{[0, r] \mid r \in \mathbb{I}\}$. Here, for $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$.
(4) The *product texture* $(S \times T, \mathcal{S} \otimes \mathcal{T})$ of textures $(S, \mathcal{S})$ and $(T, \mathcal{T})$. Here the product texturing $\mathcal{S} \otimes \mathcal{T}$ of $S \times T$ consists of arbitrary intersections of sets of the form

$$(A \times T) \cup (S \times B),\ A \in \mathcal{S} \text{ and } B \in \mathcal{T}.$$

For $(s, t) \in S \times T$, $P_{(s,t)} = P_s \times P_t$ and $Q_{(s,t)} = (Q_s \times T) \cup (S \times Q_t)$.

For $A \in \mathcal{S}$, the set

$$A^\flat = \bigcap\left\{\bigcup\{A_j \mid j \in J\} \mid \{A_j \mid j \in J\} \subseteq \mathcal{S},\ A = \bigvee\{A_j \mid j \in J\}\right\},$$

called the *core* of A is also of importance, although in general it need not belong to $\mathcal{S}$. We note the following fundamental properties of p–sets and q–sets. These expose a form of duality which is useful in defining pairs of dual properties.

Lemma 1.2. [6, Theorem 1.2]

(1) *$s \notin A \implies A \subseteq Q_s \implies s \notin A^\flat$ for all $s \in S$, $A \in \mathcal{S}$.*
(2) *For $A \in \mathcal{S}$, $A^\flat = \{s \in S \mid A \not\subseteq Q_s\}$.*
(3) *For $A \in \mathcal{S}$, A is the smallest element of $\mathcal{S}$ containing $A^\flat$.*
(4) *For $A, B \in \mathcal{S}$, if $A \not\subseteq B$ then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.*
(5) *$A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ for all $A \in \mathcal{S}$.*
(6) *$A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.*

Ditopology: Since a texturing $\mathcal{S}$ of S need not be closed under set complementation we must forego the traditional relationship between open and closed sets. Hence we define a *dichotomous topology*, or *ditopology* for short, as a pair (τ, κ) of generally unrelated subsets τ, κ of $\mathcal{S}$ satisfying

$$\begin{array}{ll} S, \emptyset \in \tau, & S, \emptyset \in \kappa, \\ G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau, & K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa, \\ G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau. & K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa. \end{array}$$

The elements of τ are called *open* and those of κ *closed*. We refer to τ as the *topology* and to κ as the *cotopology* of (τ, κ). Any bitopology (u, v) on X gives rise to the ditopology (u, v^c) on the discrete texture $(X, \mathcal{P}(X))$, while $\tau = \{[0, r) \mid 0 \leq r \leq 1\} \cup \{\mathbb{I}\}$, $\kappa = \{[0, r] \mid 0 \leq r \leq 1\} \cup \{\emptyset\}$ is a natural ditopology on the unit interval texture $(\mathbb{I}, \mathcal{I})$.

Textures and ditopological texture spaces were first conceived as a means of representing fuzzy sets and topologies in a point-based setting, and this aspect of the theory continues to be of interest [4–8]. However, much of the theory has been developed quite independently of this context, and has concentrated on developing concepts appropriate to a complement-free setting which enable the expression of powerful results in the context of often quite minimal structures. As a case in point the unit interval texture with its natural ditopology is seen to play a role analogous to that of the unit interval in classical topology, despite the fact that we work within a very small subset of $\mathcal{P}(\mathbb{I})$. These aspects of the theory are likely to find applications in the modelling of negation-free logics, and in the development of structures for computation.

In this paper we give a systematic method of constructing and studying quotients of textures and of ditopological texture spaces by using the notion of direlation [6]. The reader may refer to [5] for some early ideas concerning subtextures and quotient textures. At the suggestion of the referee we characterize the basic notions relating to direlations and difunctions in terms of the interior relation, thereby making the interior relation available as a tool in this investigation.

For terms from lattice theory not defined here the reader may consult [14], while we follow [1] for terms from category theory.

2. Textures and the Interior Relation

It is clear that if $\mathcal{S}$ is a texturing of S then $\mathcal{S}^c = \{A^c \mid A \in \mathcal{S}\}$ is a T_0 topology on S, that is $\mathcal{S}$ may be regarded as the set of closed sets of the T_0 topological space $(S, \mathcal{S}^c)$. Viewed in this light, textures are just the closed set version of the *locally supercompact spaces*, *core spaces* or *C-spaces* studied initially by Hoffmann [15], Erné [11], Erné and Wilke [13]; and later by Erné [12], Lawson [16], and others. Since, in general, the sets of $\mathcal{S}^c$ do not belong to the texture $\mathcal{S}$, this link with C-spaces has not been exploited in the literature on textures so far[1]. The referee has pointed out, however, that the *interior relation* for the topology $(S, \mathcal{S}^c)$, used by the aforementioned authors to characterize and study the C-spaces, can be used to give an algebraic characterization of textural concepts and results that may be more economic than the internal expression using p-sets and q-sets, and which at the same time could make the work on ditopological texture spaces of more immediate interest to workers in the field of C-spaces. With this in mind, therefore, we devote this section to a sketch of the relation between the internal and algebraic characterizations of the principle concepts used in the study of textures, particularly those involved in the main part of this paper on quotients.

We recall that for a topological space $(X, \mathcal{T})$ the *interior relation* ω is defined by $x \,\omega\, y \iff y \in (\bigcap\{G \in \mathcal{T} \mid x \in G\})^o$. For the topology $(S, \mathcal{S}^c)$ of the texture $(S, \mathcal{S})$ we therefore have

$$\begin{aligned} s \,\omega\, t &\iff t \in (\bigcap\{A^c \mid A \in \mathcal{S},\ s \notin A\})^o \\ &\iff t \in (S \setminus \bigcup\{A \in \mathcal{S} \mid s \notin A\})^o \\ &\iff t \notin \overline{\bigcup\{A \in \mathcal{S} \mid s \notin A\}} \\ &\iff P_t \not\subseteq Q_s, \end{aligned}$$

which immediately shows the relevance of the relation ω to the study of textures. In particular we note that the idempotency, $\omega = \omega^2$, of ω in this case can be read off directly from Lemma 1.2 (1) and (4). We will refer to ω as the *interior relation of* $(S, \mathcal{S})$, and denote it by ω_S if necessary, to avoid confusion.

[1] With the exception of a subsection in [9] relating simple textures with sober topologies

For a relation φ from X to Y, $A \subseteq X$ and $B \subseteq Y$, we set

$$A\varphi = \{y \in Y \mid \exists a \in A,\ a\varphi y\} \text{ and } \varphi B = \{x \in X \mid \exists b \in B,\ x\varphi b\}$$

as usual, abbreviating $\{a\}\varphi$ to $a\,\varphi$ and $\varphi\{b\}$ to $\varphi\,b$, respectively. As above we will often write $s\,\varphi\,t$ in preference to $(s,t) \in \varphi$. We note that:

Lemma 2.1. *Let $(S, \mathcal{S})$ be a texture with interior relation ω_S.*

(1) $A \in \mathcal{S} \iff A^c\omega_S = A^c$.
(2) *For* $A \in \mathcal{S}$, $A \not\subseteq Q_s \iff s \in \omega_s A \iff s \in A\omega_S^{-1}$.
(3) *For* $A \in \mathcal{S}$ *we have* $A^\flat = \omega_S A = A\omega_S^{-1}$.
(4) *For* $s \in S$, $P_s\omega_S^{-1} = s\omega_S^{-1}$ *and* $Q_s = (s\omega_S)^c$.

In the present paper membership of a given set in a given texture will generally be known, so Lemma 2.1 (3) will be particularly useful in establishing $A \subseteq B$ from $A\omega_S^{-1} \subseteq B\omega_S^{-1}$.

Direlations: Direlations give a suitable notion of (binary) relation for textures. For example, along with the notion of dicover, they play an important role in defining uniformities on textures [17].

Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. Generally, $\mathcal{S} \otimes \mathcal{T}$ is too small to support a useful notion of relation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. We consider instead $\mathcal{P}(S) \otimes \mathcal{T}$. We use $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ to denote the p–sets and q–sets for $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$, and likewise $\overline{P}_{(t,s)}$, $\overline{Q}_{(t,s)}$ those for $\mathcal{P}(T) \otimes \mathcal{S}$.

An element of $\mathcal{P}(S) \otimes \mathcal{T}$ is related to $\mathcal{T}$ but not to $\mathcal{S}$. We add conditions relating it to $\mathcal{S}$ in two different ways, producing dual concepts of relation and corelation [6].

$r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.

R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.

$R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.

CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

A pair (r, R) consisting of a relation r and corelation R is now called a *direlation*.

Where necessary we will refer to an arbitrary relation between the base sets as a *point relation* to avoid possible confusion with the relations and corelations in the above sense.

Example 2.2. For any texture $(S, \mathcal{S})$ the *identity direlation* (i, I) on $(S, \mathcal{S})$ is given by

$$i = \bigvee\{\overline{P}_{(s,s)} \mid s \in S\} \text{ and } I = \bigcap\{\overline{Q}_{(s,s)} \mid s \in S\}.$$

Direlations are ordered by $(r_1, R_1) \sqsubseteq (r_2, R_2) \iff r_1 \subseteq r_2$ and $R_2 \subseteq R_1$, and the direlation (r, R) on $(S, \mathcal{S})$ is called *reflexive* if $(i, I) \sqsubseteq (r, R)$.

Now we characterize direlations in terms of the interior relations ω_S, ω_T of $(S, \mathcal{S})$, $(T, \mathcal{T})$, respectively. First we note the following:

Lemma 2.3. *Let φ be a point relation from S to T.*

(1) $\varphi \in \mathcal{P}(S) \otimes \mathcal{T} \iff \varphi^c = \varphi^c \omega_T$.
(2) *If* $\varphi \in \mathcal{P}(S) \otimes \mathcal{T}$ *then* $\varphi \not\subseteq \overline{Q}_{(s,t)} \iff s\, \varphi\omega_T^{-1}\, t$.
(3) *If* $\varphi \in \mathcal{P}(S) \otimes \mathcal{T}$ *then* $\varphi^\flat = \varphi\omega_T^{-1}$.

Proof. (1) Left to the interested reader.

(2) If $\varphi \not\subseteq \overline{Q}_{(s,t)}$ then by Lemma 1.2 (4) applied to the texture $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$ we have $t' \in T$ with $\varphi \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t')} \not\subseteq \overline{Q}_{(s,t)}$. By Lemma 1.2 (6) we have $s\, \varphi\, t'$, while on the other hand $P_{t'} \not\subseteq Q_t$, so $t'\, \omega_T^{-1}\, t$. Hence $s\, \varphi\omega_T^{-1}\, t$. The converse is straightforward, and we omit the details.

(3) Immediate from (2) and Lemma 1.2 (2). □

If $\varphi \in \mathcal{P}(S) \otimes \mathcal{T}$ then by (3), $\varphi\omega_T^{-1}$ determines φ as the smallest set in the texture containing it, that is as its closure in the corresponding topology. See the comments following Lemma 2.1.

Proposition 2.4. *Let r, R be point relations from S to T.*

(1) *r is a relation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ if and only if*

$$R_a : r^c = r^c \omega_T \text{ and } R_b : r\omega_T^{-1} = \omega_S^{-1} r \omega_T^{-1}.$$

(2) *R is a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ if and only if*

$$CR_a : R^c = R^c \omega_T \text{ and } CR_b : R^c = \omega_S R^c.$$

Proof. (1) R_a is just Lemma 2.3 (1). On the other hand it is easy to verify that r satisfies *R1* if and only if $\omega_S^{-1} r \omega_T^{-1} \subseteq r\omega_T^{-1}$, and that it satisfies *R2* if and only if $r\omega_T^{-1} \subseteq \omega_S^{-1} r \omega_T^{-1}$.

(2) Left to the interested reader. □

We note in passing that the identity direlation (i_S, I_S) on $(S, \mathcal{S})$ is characterized by $i_S \omega_S^{-1} = \omega_S^{-1}$ and $I_S = \omega_S^c$, as the reader may easily verify.

Inverse of direlations: The *inverse* of (r, R) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is the direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ from $(T, \mathcal{T})$ to $(S, \mathcal{S})$ given by

$$r^{\leftarrow} = \bigcap \{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\}, \quad R^{\leftarrow} = \bigvee \{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\}.$$

The direlation (r, R) on $(S, \mathcal{S})$ is called *symmetric* if $(r, R)^{\leftarrow} = (r, R)$.

Proposition 2.5. *Let (r, R) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow}) : (T, \mathcal{T}) \to (S, \mathcal{S})$ is characterized by:*

(1) $(R^{\leftarrow})\omega_S^{-1} = (R^c)^{-1}$, *and*
(2) $r^{\leftarrow} = (\omega_T r^{-1})^c$.

Proof. (1) In view of CR_b it will be sufficient to prove $(R^{\leftarrow})\omega_S^{-1} = (\omega_S R^c)^{-1}$. If $t\,(R^{\leftarrow})\omega_S^{-1}\,s$ then $R^{\leftarrow} \not\subseteq \overline{Q}_{(t,s)}$ by Lemma 2.3 (2), so we have $s' \in S$ with $\overline{P}_{(t,s')} \not\subseteq \overline{Q}_{(t,s)}$ and $\overline{P}_{(s',t)} \not\subseteq R$. This gives $s\,\omega_S\,s'$, $s'\,R^c\,t$, so $s\,\omega_S R^c\,t$ and we have shown

$$(R^{\leftarrow})\omega_S^{-1} \subseteq (\omega_T R^c)^{-1}. \tag{2.1}$$

Conversely, if $t(\omega_S R^c)^{-1}s$ then $s(\omega_S R^c)t$ and so for some $s' \in S$ we have $s\,\omega_S\,s'$ and $s'\,R^c\,t$. This gives $\overline{P}_{(s',t)} \not\subseteq R$, whence $\overline{P}_{(t,s')} \subseteq R^{\leftarrow}$ and so $t\,R^{\leftarrow}s'$, while $s'\,\omega_S^{-1}\,s$ and so $t\,(R^{\leftarrow})\omega_S^{-1}\,s$. Thus

$$(\omega_S R^c) \subseteq (R^{\leftarrow})\omega_S^{-1}. \tag{2.2}$$

The result now follows from (2.1) and (2.2).

(2) Left to the interested reader. □

Composition of direlations: If $(r_1, R_1) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ and $(r_2, R_2) : (S_2, \mathcal{S}_2) \to (S_3, \mathcal{S}_3)$ are direlations, their *composition* $(r_2, R_2) \circ (r_1, R_1) = (r_2 \circ r_1, R_2 \circ R_1)$ from $(S_1, \mathcal{S}_1)$ to $(S_3, \mathcal{S}_3)$ is given by

$$r_2 \circ r_1 = \bigvee \{\overline{P}_{(s,u)} \mid \exists\, t \in S_2 \text{ with } r_1 \not\subseteq \overline{Q}_{(s,t)} \text{ and } r_2 \not\subseteq \overline{Q}_{(t,u)}\},$$

$$R_2 \circ R_1 = \bigcap \{\overline{Q}_{(s,u)} \mid \exists\, t \in S_2 \text{ with } \overline{P}_{(s,t)} \not\subseteq R_1 \text{ and } \overline{P}_{(t,u)} \not\subseteq R_2\}.$$

It is known [6] that composition is associative and that (i, I) is the identity under composition.

The direlation (r, R) on $(S, \mathcal{S})$ is *transitive* if $(r, R) \circ (r, R) \sqsubseteq (r, R)$.

Proposition 2.6. *With the notation above, and denoting by ω_k the interior point relation for $(S_k, \mathcal{S}_k)$, $1 \le k \le 3$, we have*

(1) $(r_2 \circ r_1)\omega_3^{-1} = r_1 r_2 \omega_3^{-1}$,
(2) $R_2 \circ R_1 = (R_1^c R_2^c)^c$.

Proof. Straightforward. □

Sections: Let $(r, R) : (S, \mathcal{S}) \to (T, \mathcal{T})$ be a direlation, $A \in \mathcal{S}$. The *A-sections* of r, R are given by

$$r^{\to} A = \bigcap \{Q_t \mid \forall\, s,\ r \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\} \in \mathcal{T},$$

$$R^{\to} A = \bigvee \{P_t \mid \forall\, s,\ \overline{P}_{(s,t)} \not\subseteq R \implies P_s \subseteq A\} \in \mathcal{T},$$

respectively.

Proposition 2.7. *With the notation above:*

(1) $(r^{\to} A)\omega_T^{-1} = A r \omega_T^{-1}$,
(2) $(R^{\to} A) = (A^c R^c)^c$.

Proof. (1) $t \in (r^{\to} A)\omega_T^{-1}$ gives $r^{\to} A \not\subseteq Q_t$ by Lemma 2.1 (2), so we have $s \in S$ with $r \not\subseteq \overline{Q}_{(s,t)}$ and $A \not\subseteq Q_s$. Now $s\,(r\omega_S^{-1})\,t$ by Lemma 2.3 (2), and $s \in A\omega_S^{-1}$ by Lemma 2.1 (2), so $t \in A\omega_S^{-1} r \omega_T^{-1}$. Hence

$$(r^{\to} A)\omega_T^{-1} \subseteq A\omega_S^{-1} r \omega_T^{-1}. \tag{2.3}$$

Conversely take $t \in A\omega_S^{-1} r \omega_T^{-1} = A\omega_S^{-1} r \omega_T^{-1} \omega_T^{-1}$ since ω_T is idempotent. Now we have $s \in A$ with $s\,(A\omega_S^{-1} r \omega_T^{-1} \omega_T^{-1})\,t$, and hence there exist $s' \in S$, $t' \in T$ with $s\,\omega_S^{-1}\,s'$, $s'\,(r\omega_T^{-1})\,t'$ and $t'\,\omega_T^{-1}\,t$. From Lemma 2.3 (2) we have $r \not\subseteq \overline{Q}_{(s',t')}$. We claim that $P_{t'} \subseteq r^{\to} A$.

To see this assume the contrary. Then there exists $t'' \in T$ with $P_{t'} \not\subseteq Q_{t''}$ for which

$$r \not\subseteq \overline{Q}_{(z,t'')} \implies A \subseteq Q_z \tag{2.4}$$

for all $z \in S$. However we easily deduce $r \not\subseteq \overline{Q}_{(s',t'')}$, so implication (2.4) with $z = s'$ gives $A \subseteq Q_{s'}$, which contradicts $s\,\omega_S^{-1}\,s'$ since $s \in A$. This establishes $P_{t'} \subseteq r^{\to}A$, and $P_{t'} \not\subseteq Q_t$ so $r^{\to}A \not\subseteq \overline{Q}_{(s,t)}$ which gives $t \in (r^{\to}A)\omega_T^{-1}$ by Lemma 2.3 (2). Thus,

$$A\omega_S^{-1}r\omega_T^{-1} \subseteq (r^{\to}A)\omega_T^{-1}. \tag{2.5}$$

From (2.3), (2.5) and R_b we have $(r^{\to}A)\omega_T^{-1} = A\omega_S^{-1}r\omega_T^{-1} = Ar\omega_T^{-1}$, as required.

(2) Essentially dual to (1), and is omitted. □

Presections: For $B \in \mathcal{T}$, the *B-presections* of r, R, are the B-sections $(r^{\leftarrow})^{\to}B$, $(R^{\leftarrow})^{\to}B$ of the inverses $r^{\leftarrow}$, $R^{\leftarrow}$, respectively. Normally these are written as $r^{\leftarrow}B$, $R^{\leftarrow}B$, respectively.

Proposition 2.8. *With the notation as above we have:*

(1) $r^{\leftarrow}B = (r\omega_T B^c)^c$,

(2) $(R^{\leftarrow}B)\omega_S^{-1} = R^cB$.

Proof. (1) Noting that $r^{\leftarrow}$ is a corelation from $(T,\mathcal{T})$ to $(S,\mathcal{S})$, we have $r^{\leftarrow}B = (r^{\leftarrow})^{\to}B = (B^c(r^{\leftarrow})^c)^c$ by Proposition 2.7 (2). Now using Proposition 2.5 (2), $r^{\leftarrow}B = (B^c\omega_T r^{-1})^c = (B^c(r\omega_T^{-1})^{-1})^c = (r\omega_T B^c)^c$, as required.

(2) In a similar way, using Propositions 2.7 (1) and 2.5 (1),

$$(R^{\leftarrow}B)\omega_S^{-1} = ((R^{\leftarrow})^{\to}B)\omega_S^{-1} = B(R^{\leftarrow})\omega_S^{-1} = B(R^c)^{-1} = R^cB\ ,$$

as required. □

Difunctions: A *difunction* from $(S,\mathcal{S})$ to $(T,\mathcal{T})$ is a direlation (f,F) from $(S,\mathcal{S})$ to $(T,\mathcal{T})$ satisfying the conditions

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

A difunction (f,F) is *surjective* if the inverse direlation $(f,F)^{\leftarrow}$ satisfies *DF1*, and *injective* if $(f,F)^{\leftarrow}$ satisfies *DF2*.

A characteristic property of a difunction (f,F) is that $f^{\leftarrow}B = F^{\leftarrow}B$ for all $B \in \mathcal{T}$ [6].

Proposition 2.9. *Given textures* $(S,\mathcal{S})$, $(T,\mathcal{T})$*:*

(1) *Let* $(r,R):(S,\mathcal{S}) \to (T,\mathcal{T})$ *be a direlation.*
   (a) (r,R) *satisfies DF1 if and only if* $\omega_S \subseteq R^c\omega_T r^{-1}$.
   (b) (r,R) *satisfies DF2 if and only if* $\omega_T r^{-1}R^c \subseteq \omega_T$
(2) *Let* $(f,F):(S,\mathcal{S}) \to (T,\mathcal{T})$ *be a difunction.*
   (a) (f,F) *is surjective if and only if* $\omega_T \subseteq \omega_T f^{-1}F^c$.
   (b) (f,F) *is injective if and only if* $F^c\omega_T r^{-1} \subseteq \omega_S$.

Proof. (1) According to [6, Lemma 2.23 (1 ii)], (r,R) satisfies *DF1* if and only if $r^{\leftarrow} \circ R \subseteq I_S$. Now using Propositions 2.6 (2), 2.5 (2) and $I_S = \omega_S^c$ gives

$$r^{\leftarrow} \circ R \subseteq I_S \iff (R^c(r^{\leftarrow})^c)^c \subseteq \omega_S^c \iff (R^c\omega_T r^{-1})^c \subseteq \omega_S^c,$$

which is equivalent to $\omega_S \subseteq R^c\omega_T r^{-1}$ on negating each side. This gives (a), and (b) follows similarly using [6, Lemma 2.23 (2 ii)].

(2) (f,F) is surjective if and only if $(F^{\leftarrow}, f^{\leftarrow}) : (T,\mathcal{T}) \to (S,\mathcal{S})$ satisfies *DF1*, so using (1 i) we obtain $\omega_T \subseteq (r^{\leftarrow})^c\omega_S(R^{\leftarrow})^{-1}$, which by Proposition 2.5 (2) is equivalent to $\omega_T \subseteq \omega_T r^{-1}\omega_S(R^{\leftarrow})^{-1} = \omega_T r^{-1}((R^{\leftarrow})\omega_S^{-1})^{-1}$. Applying Proposition 2.5 (1) now gives (a), and the proof of (b) is similar and is omitted. □

In case $(f,F) : (S_1,\mathcal{S}_1,\tau_1,\kappa_1) \to (S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is a difunction we make the following definitions:

(1) (f,F) is *continuous* if $G \in \tau_2 \implies F^{\leftarrow}G \in \tau_1$.
(2) (f,F) is *cocontinuous* if $K \in \kappa_2 \implies f^{\leftarrow}K \in \kappa_1$.
(3) (f,F) is *bicontinuous* if it is continuous and cocontinuous.

Also:

(1) (f,F) is *open* (*co-open*) if $G \in \tau_1 \implies f^{\rightarrow}G \in \tau_2$ ($F^{\rightarrow}G \in \tau_2$).
(2) (f,F) is *closed* (*coclosed*) if $K \in \kappa_1 \implies f^{\rightarrow}K \in \kappa_2$ ($F^{\rightarrow}K \in \kappa_2$).

It is shown in [7] that the category **dfDitop** of ditopological texture spaces and bicontinuous difunctions is topological over the category **dfTex** of textures and difunctions, thereby justifying our use of the term topology in naming the pair (τ, κ).

Difunctions and Point Functions: Only in certain special cases do difunctions correspond to point functions between the base sets. We recall in particular [6, Lemma 3.4] that if $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures and the point function $\varphi : S \to T$ satisfies

(a) $s, s' \in S,\ P_s \not\subseteq Q_{s'} \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$,

then the equalities

$$f = f_\varphi = \bigvee\{\overline{P}_{(s,t)} \mid \exists\, v \in S \text{ with } P_s \not\subseteq Q_v \text{ and } P_{\varphi(v)} \not\subseteq Q_t\},$$
$$F = F_\varphi = \bigcap\{\overline{Q}_{(s,t)} \mid \exists\, v \in S \text{ with } P_v \not\subseteq Q_s \text{ and } P_t \not\subseteq Q_{\varphi(v)}\},$$

define a point function $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$. Moreover [6, Lemma 3.9], if φ also satisfies

(b) $s \in S,\ P_{\varphi(s)} \not\subseteq B,\ B \in \mathcal{T} \implies \exists\, s' \in S \text{ with } P_{\varphi(s')} \not\subseteq B$,

then $f^{\leftarrow}B = \varphi^{-1}[B] = F^{\leftarrow}B$ for all $B \in \mathcal{T}$.

Proposition 2.10. *With the notation as above:*

(1) *φ satisfies (a) if and only if $\omega_S^{-1}\varphi \subseteq \varphi\omega_T^{-1}$.*
(2) *$(f, F) = (f_\varphi, F_\varphi)$ is characterized by*

$$f\omega_T^{-1} = \omega_S^{-1}\varphi\omega_T^{-1} \text{ and } F = (\omega_S\varphi\omega_T)^c.$$

(3) *φ satisfies (b) if and only if $\omega_T\varphi^{-1} \subseteq \omega_T\varphi^{-1}\omega_S$.*

Proof. (1) Suppose (a) holds. Then if $s\,\omega_S^{-1}\varphi\, t$ then we have $s' \in S$ with $s\,\omega_S^{-1}\, s'$ and $s'\,\varphi\, t$, that is $t = \varphi(s')$. Now $P_s \not\subseteq Q_{s'}$ so $P_{\varphi(s)} \not\subseteq Q_t$ by (a). This gives $s\,\varphi\omega_T^{-1}\, t$, whence $\omega_S^{-1} \subseteq \varphi\omega_T^{-1}$. The proof of the converse is similar, and is omitted.

(2) $s\, f\omega_T^{-1}\, t \iff f \not\subseteq \overline{Q}_{(s,t)} \iff \exists t' \in T, v \in S$ with $\overline{P}_{(s,t')} \not\subseteq \overline{Q}_{(s,t)}$, $P_s \not\subseteq Q_v$ and $P_{\varphi(v)} \not\subseteq Q_{t'} \iff s\,\omega_S^{-1}\varphi\omega_T^{-1}\omega_T^{-1}\, t$. Hence $f\omega_T^{-1} = \omega_S^{-1}\varphi\omega_T^{-1}$ since ω_T is idempotent. The proof of the formula for F is left to the reader.

(3) Straightforward on noting that in the definition of (b) it is sufficient to take B to be a q-set. $\square$

Finally we present a result which will be needed in the next section.

Proposition 2.11. *Let* φ *be onto and satisfy* (*a*) *and* (*b*). *Then* (f_φ, F_φ) *is surjective.*

Proof. For $(f, F) = (f_\varphi, F_\varphi)$ we have

$$\omega_T f^{-1} F^c = (f\omega_T^{-1})^{-1} F^c = (\omega_S^{-1}\varphi^{-1}\omega_T^{-1})^{-1}\omega_S\varphi\omega_T = \omega_T\varphi^{-1}\omega_S\varphi\omega_T$$

by Proposition 2.10 (2). On the other hand, since φ is onto we have $\Delta_T \subseteq \varphi^{-1}\varphi$, where Δ_T is the diagonal of T, so using the above characterization of (b),

$$\omega_T = \omega_T \Delta_T \omega_T \subseteq \omega_T\varphi^{-1}\varphi\omega_T \subseteq \omega_T\varphi^{-1}\omega_S\varphi\omega_T.$$

This shows that $\omega_T \subseteq \omega_T f^{-1} F^c$, whence (f, F) is surjective by Proposition 2.9 (2 a). □

The converse of this result is false, even if (f, F) is bijective [7, Example 2.14].

3. Equivalence Direlations and the Quotient Texture

Definition 3.1. Let $(S, \mathcal{S})$ be a texture. A direlation (r, R) on $(S, \mathcal{S})$ is called an *equivalence direlation* if it is reflexive, symmetric and transitive.

We have at once:

Lemma 3.2. (r, R) *is an equivalence direlation if and only if* $\omega^{-1} \subseteq r\omega^{-1}$, $(r\omega^{-1})^2 \subseteq r\omega^{-1}$ *and* $R = r^{\to} = (\omega r^{-1})^c$.

Corollary 3.3. *Let* (r, R) *is a equivalence direlation. Then* $r\omega^{-1}$ *is idempotent.*

We also note the following:

Lemma 3.4. *Let* (r, R) *be an equivalence direlation on* $(S, \mathcal{S})$. *Then for* $s, t \in S$ *the following are equivalent.*

(i) $v \in S,\ P_s \not\subseteq Q_v \implies r \not\subseteq \overline{Q}_{(t,v)}$.
(ii) $s\,\omega^{-1} \subseteq t\,r\omega^{-1}$.
(iii) $v \in S,\ P_s \not\subseteq Q_v \implies \overline{P}_{(v,t)} \not\subseteq R$.
(iv) $P_s \subseteq r^{\to} P_t$.

Proof. (1) $\Longleftrightarrow$ (2) Immediate from Lemma 2.3 (2).

(3) $\Longleftrightarrow$ (2) Clearly (3) is equivalent to $s\,\omega^{-1} \subseteq t\,(R^c)^{-1}$, while $(R^c)^{-1} = (R^{\leftarrow})\omega^{-1} = r\omega^{-1}$ by Proposition 2.5 (1) and the fact that (r, R) is symmetric.

(4) $\Longleftrightarrow$ (2) $P_s \subseteq r^{\rightarrow}P_t$ is equivalent to $P_s\omega^{-1} \subseteq (r^{\rightarrow}P_t)\omega^{-1}$, and by Proposition 2.7 (1) and R_b we have $(r^{\rightarrow}P_t)\omega^{-1} = P_t r\omega^{-1} = P_t\omega^{-1}r\omega^{-1}$. The proof is now completed by applying Lemma 2.1 (4), and using R_b again. □

The sense in which an equivalence direlation gives rise to a quotient texture is described in the following theorem.

Theorem 3.5. *Let (r, R) be an equivalence direlation on $(S, \mathcal{S})$. Then there exists a point equivalence relation ρ on S, a texturing $\mathcal{U}$ of the quotient set $U = S/\rho$ and a surjective difunction (f, F) on $(S, \mathcal{S})$ to $(U, \mathcal{U})$ satisfying $r = F^{\leftarrow} \circ f$ and $R = f^{\leftarrow} \circ F$.*

Proof. We will present the proof as a series of lemmas. Let us begin by associating an equivalence point relation ρ with (r, R).

Lemma 3.6. *If (r, R) is an equivalence direlation on $(S, \mathcal{S})$ then the point relation ρ defined by*

$$s\,\rho\,t \iff s\,\omega^{-1} \subseteq t\,r\omega^{-1} \text{ and } t\,\omega^{-1} \subseteq s\,r\omega^{-1}$$

is an equivalence point relation on S.

Proof. By Lemma 3.2 we have $\omega^{-1} \subseteq r\omega^{-1}$, whence $s\,\omega^{-1} \subseteq s\,r\omega^{-1}$, which gives $s\,\rho\,s$ for all $s \in S$. It is immediate that ρ is symmetric, so it remains to show transitivity. For $s, t, v \in S$ let $s\,\rho\,t$ and $t\,\rho\,v$. Then $s\,\omega^{-1} \subseteq t\,r\omega^{-1} = t\,\omega^{-1}r\omega^{-1} \subseteq v\,r\omega^{-1}r\omega^{-1} \subseteq v\,r\omega^{-1}$ by R_b and Lemma 3.2. Likewise, $v\,\omega^{-1} \subseteq s\,r\omega^{-1}$, so $s\,\rho\,v$ as required. □

Lemma 3.7. *Let (r, R) be an equivalence direlation on $(S, \mathcal{S})$ and ρ the equivalence point relation defined above. Then*

1. *For all $s, t \in S$, $s\,\rho\,t \iff P_s \subseteq r^{\rightarrow}P_t$ and $P_t \subseteq r^{\rightarrow}P_s$.*
2. *For $A \in \mathcal{S}$, $r^{\rightarrow}A = A \iff R^{\rightarrow}A = A \iff r^{\rightarrow}A = R^{\rightarrow}A$. We denote by $\mathcal{R}$ the family of all $A \in \mathcal{S}$ satisfying these equivalent conditions.*
3. *$A \in \mathcal{R} \iff Ar\omega^{-1} = A\omega^{-1}$. In particular the elements of $\mathcal{R}$ are saturated with respect to ρ.*

(4) *$r^{\to}A \in \mathcal{R}$ and $R^{\to}A \in \mathcal{R}$ for all $A \in \mathcal{S}$.*
(5) *$\mathcal{R}$ is closed under arbitrary joins and intersections.*
(6) *For $A \in \mathcal{S}$, $r^{\to}A = \bigcap\{B \in \mathcal{R} \mid A \subseteq B\}$ and $R^{\to}A = \bigvee\{B \in \mathcal{R} \mid B \subseteq A\}$.*

Proof. (1) This is just the equivalence of (ii) and (iv) in Lemma 3.4.

(2) If $r^{\to}A = A$ then $A \subseteq r^{\leftarrow}(r^{\to}A) = r^{\leftarrow}(A) = R^{\to}A$ by [6, Proposition 2.10 (2)] and the fact that (r, R) is symmetric. On the other hand $R^{\to}A \subseteq A$ since R is reflexive, so $R^{\to}A = A$. The converse is proved likewise, so $r^{\to}A = A \iff R^{\to}A = A$. The second equivalence is clear since $R^{\to}A \subseteq A \subseteq r^{\to}A$ for all $A \in \mathcal{S}$ by reflexivity.

(3) $A \in \mathcal{R} \iff f^{\to}A = A \iff (f^{\to}A)\omega^{-1} = A\omega^{-1} \iff Ar\omega^{-1} = A\omega^{-1}$ by Proposition 2.7 (1). For $s \in A \in \mathcal{R}$, $s\,\rho\,t$, we have $t\omega^{-1} \subseteq sr\omega^{-1} \subseteq Ar\omega^{-1} = A\omega^{-1}$, whence $t \in A$ and so A is saturated.

(4) For $A \in \mathcal{S}$ let $B = r^{\to}A$. Then $Br\omega^{-1} = (r^{\to}A)\omega^{-1}r\omega^{-1} = Ar\omega^{-1}r\omega^{-1} = Ar\omega^{-1} = B\omega^{-1}$ by R_b, Proposition 2.7 (1) and Corollary 3.3, so $B \in \mathcal{R}$ by (3). The proof of $R^{\to}A \in \mathcal{R}$ is left to the interested reader.

(5) Take $A_i \in \mathcal{R}$, $i \in I$. For $B = \bigvee_{i\in I} A_i$, $r^{\to}B = r^{\to}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} r^{\to}A_i = \bigvee_{i\in I} A_i = B$ by [6, Corollary 2.12 (2)] and (2), so again by (2), $\bigvee_{i\in I} A_i \in \mathcal{R}$. In the same way $B = \bigcap_{i\in I} A_i$ satisfies $R^{\to}B = B$, so we also have $\bigcap_{i\in I} A_i \in \mathcal{R}$.

(6) Take $A \in \mathcal{S}$. If $A \subseteq B$ for $B \in \mathcal{R}$ then $r^{\to}A \subseteq r^{\to}B = B$, so $r^{\to}A \subseteq \bigcap\{B \in \mathcal{R} \mid A \subseteq B\}$. The reverse inclusion is clear on noting that $B = r^{\to}A$ satisfies $B \in \mathcal{R}$ and $A \subseteq B$. The dual proof of the second equality is omitted. □

We will denote by U the quotient set S/ρ and by φ the canonical quotient mapping $\varphi : S \to U$, $\varphi : s \mapsto \bar{s}$, where $\bar{s}$ denotes the equivalence class of s. Let us define $\mathcal{U} = \{A \subseteq U \mid \varphi^{-1}[A] \in \mathcal{R}\}$. Note that, since the sets in $\mathcal{R}$ are saturated with respect to ρ, we also have $\mathcal{U} = \{\varphi[A] \mid A \in \mathcal{R}\}$. Now let us verify that $\mathcal{U}$ is a texturing of U, and identify the p-sets and q-sets in $(U, \mathcal{U})$.

Lemma 3.8. *With the notation above* $\mathcal{U} = \{A \mid \varphi^{-1}[A] \in \mathcal{R}\}$ *is a texturing of* U*. Moreover, for all* $s \in S$*,*

(i) $P_{\bar{s}} = \varphi[r^{\to} P_s]$, *and*

(ii) $Q_{\bar{s}} = \bigvee\{\varphi[R^{\to} Q_{s'}] \mid P_s \not\subseteq Q_{s'}\} \subseteq \varphi[R^{\to} Q_s]$.

Proof. Note first that $S = r^{\to} S \in \mathcal{R}$ implies $U = \varphi[S] \in \mathcal{U}$. Likewise, $\emptyset = R^{\to}\emptyset \in \mathcal{R}$ gives $\emptyset = \varphi[\emptyset] \in \mathcal{U}$.

Now let us take $A_j \in \mathcal{U}$, $j \in J$, and put $A = \bigcap_{j\in J} A_j$. Then $\varphi^{-1}[A] = \bigcap_{j\in J} \varphi^{-1}[A_j] \in \mathcal{R}$ by Lemma 3.7 (5), so $\bigcap_{j\in J} A_j \in \mathcal{U}$. Thus, $\mathcal{U}$ is a complete lattice of subsets of U. Now let $A = \bigvee_{j\in J} A_j$. We will verify that $\varphi^{-1}[A] = \bigvee_{j\in J} \varphi^{-1}[A_j]$. It is clear that $\bigvee_{j\in J} \varphi^{-1}[A_j] \subseteq \varphi^{-1}[A]$. On the other hand by Lemma 3.7 (5) we have $\bigvee_{j\in J} \varphi^{-1}[A_j] \in \mathcal{R}$, so

$$\bigcup_{j\in J} A_j \subseteq \varphi\Big[\bigcup_{i\in I} \varphi^{-1}[A_j]\Big] \subseteq \varphi\Big[\bigvee_{j\in J} \varphi^{-1}[A_j]\Big] \in \mathcal{U}.$$

Thus $A \subseteq \varphi[\bigvee_{j\in J} \varphi^{-1}[A_j]] \in \mathcal{U}$, which gives

$$\varphi^{-1}[A] \subseteq \varphi^{-1}[\varphi\Big[\bigvee_{j\in J} \varphi^{-1}[A_i]\Big]] = \bigvee_{j\in J} \varphi^{-1}[A_j]$$

since the sets of $\mathcal{R}$ are saturated. Thus $\varphi^{-1}[A] = \bigvee_{j\in J} \varphi^{-1}[A_j]$.

We see from the above that both joins and intersections are preserved under the mapping $\varphi^{-1} : \mathcal{U} \to \mathcal{S}$. Moreover, this mapping is 1-1 since φ is onto. Hence, since $\mathcal{S}$ is completely distributive, $\mathcal{U}$ is completely distributive too. We have already seen that meet in $\mathcal{U}$ coincides with intersection, but we need to verify that finite joins coincide with unions. Let I be finite. Then $\varphi^{-1}[\bigvee_{i\in I} A_i] = \bigvee_{i\in I} \varphi^{-1}[A_i] = \bigcup_{i\in I} \varphi^{-1}[A_i] = \varphi^{-1}\big[\bigcup_{i\in I} A_i\big]$ since I is finite, whence $\bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$ for $A_i \in \mathcal{U}$, $i \in I$ and I finite, as required.

In order to show $\mathcal{U}$ is a texturing it remains to show that it separates the points of U. Take $s, t \in S$ with $\varphi(s) \neq \varphi(t)$. Then $s\,\rho^c\,t$, so by Lemma 3.7 (1) we have $P_s \not\subseteq r^{\to} P_t$ or $P_t \not\subseteq r^{\to} P_s$. In the first case we have $\bar{t} \in \varphi[r^{\to} P_t] \in \mathcal{U}$ since $t \in P_t \subseteq r^{\to} P_t$ and $\bar{s} \notin \varphi[r^{\to} P_t]$ since $r^{\to} P_t \in \mathcal{R}$ is saturated. Likewise, the second case gives $\bar{s} \in \varphi[r^{\to} P_s]$ and $\bar{t} \notin \varphi[r^{\to} P_s]$, which completes the proof that $(U, \mathcal{U})$ is a texture.

Finally, let us verify (i) and (ii). For (i), first note that $\bar{s} \in \varphi[r^{\rightarrow}P_s] \in \mathcal{U}$, so $P_{\bar{s}} \subseteq \varphi[r^{\rightarrow}P_s]$. On the other hand, every element of $\mathcal{U}$ has the form $\varphi[A]$ for $A \in \mathcal{R}$ and, since the elements of $\mathcal{R}$ are saturated, $\bar{s} \in \varphi[A] \implies P_s \subseteq A \implies r^{\rightarrow}P_s \subseteq r^{\rightarrow}A = A \implies \varphi[r^{\rightarrow}P_s] \subseteq \varphi[A]$, whence $P_{\bar{s}} = \varphi[r^{\rightarrow}P_s]$.

For (ii), first note that $Q_{\bar{s}} = \bigvee\{\varphi[A] \mid A \in \mathcal{R},\ P_{\bar{s}} \not\subseteq \varphi[A]\}$ and that $P_{\bar{s}} \not\subseteq \varphi[A]$ is equivalent to $P_s \not\subseteq A$ since $A \in \mathcal{R}$ is saturated. Hence, for $P_s \not\subseteq Q_{s'}$ we have

$$\begin{aligned}
\varphi^{-1}(Q_{\bar{s}}) &= \varphi^{-1}\Big(\bigvee\{\varphi[A] \mid A \in \mathcal{R},\ P_s \not\subseteq A\}\Big)\\
&= \bigvee\{\varphi^{-1}[\varphi[A]] \mid A \in \mathcal{R},\ P_s \not\subseteq A\}\\
&= \bigvee\{A \in \mathcal{R} \mid P_s \not\subseteq A\} \text{ since } A \in \mathcal{R} \text{ is saturated}\\
&\supseteq \bigvee\{A \in \mathcal{R} \mid A \subseteq Q_{s'}\} = R^{\rightarrow}Q_{s'} = \varphi^{-1}[\varphi[R^{\rightarrow}Q_{s'}]],
\end{aligned}$$

where we have used Lemma 3.7 (6). Hence $\varphi[R^{\rightarrow}Q_{s'}] \subseteq Q_{\bar{s}}$, so $\bigvee\{\varphi[R^{\rightarrow}Q_{s'}] \mid P_s \not\subseteq Q_{s'}\} \subseteq Q_{\bar{s}}$. For the reverse inclusion take $t \in S$ with $P_{\bar{s}} \not\subseteq P_{\bar{t}}$, whence $P_s \not\subseteq r^{\rightarrow}P_t$. By the equivalence of (iii) and (iv) in Lemma 3.4 we have $s' \in S$ with $P_s \not\subseteq Q_{s'}$ and $\overline{P}_{(s',t)} \subseteq R$. Now $P_t \subseteq R^{\rightarrow}Q_{s'}$ by [6, Lemma 2.6 (2)], whence $r^{\rightarrow}P_t \subseteq R^{\rightarrow}Q_{s'}$ since $R^{\rightarrow}Q_{s'} \in \mathcal{R}$, and we deduce $P_{\bar{t}} = \varphi[r^{\rightarrow}P_t] \subseteq \varphi[R^{\rightarrow}Q_{s'}]$. Hence we have shown $P_{\bar{s}} \not\subseteq P_{\bar{t}} \implies P_{\bar{t}} \subseteq \varphi[R^{\rightarrow}Q_{s'}]$ for some $P_s \not\subseteq Q_{s'}$, which gives us $Q_{\bar{s}} \subseteq \bigvee\{\varphi[R^{\rightarrow}Q_{s'}] \mid P_s \not\subseteq Q_{s'}\}$, as required.

To complete the proof of (ii) we need only note that if $P_s \not\subseteq Q_{s'}$ then $Q_{s'} \subseteq Q_s$ and so $\varphi[R^{\rightarrow}Q_{s'}] \subseteq \varphi[R^{\rightarrow}Q_s]$, from which the required inclusion is clear. □

Corollary 3.9.

(1) *For $(U, \mathcal{U})$ the interior point relation ω_U is given by*

$$\varphi(s)\,\omega_U^{-1}\,\varphi(t) \iff s\,r\omega_S^{-1}\Big(\bigcap\{r\omega_S^{-1}\,t' \mid t\,\omega_S^{-1}\,t'\}\Big).$$

(2) *For $s, t \in S$, $r \not\subseteq \overline{Q}_{(s,t)} \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(t)}$.*

(3) *φ satisfies the conditions (a) and (b) mentioned earlier.*

Proof. (1) Using the facts proved above we have:

$$\begin{aligned}
P_{\varphi(s)} \not\subseteq Q_{\varphi(t)} &\iff \varphi^{-1}[P_{\varphi(s)}] \not\subseteq \varphi^{-1}[Q_{\varphi(t)}]\\
&\iff r^{\rightarrow}P_s \not\subseteq \bigvee\{R^{\rightarrow}Q_{t'} \mid P_t \not\subseteq Q_{t'}\}\\
&\iff \exists\, u \in (r^{\rightarrow}P_s)\omega_S^{-1} \text{ with } u \notin R^{\rightarrow}Q_{t'}\ \forall\, t\,\omega_S^{-1}\,t'.
\end{aligned}$$

By Proposition 2.7 (1), $u \in (r^{\to}P_s)\omega_S^{-1}$ is equivalent to $u \in sr\omega_S^{-1}$, while for $t\,\omega_S^{-1}\,t'$ we have $u \in (R^{\to}Q_{t'})^c = (r^{\leftarrow}Q_{t'})^c = r\omega_S^{-1}Q_{t'}^c$ by Proposition 2.8 (1). By Lemma 2.1 (4) we deduce that $u \in r\omega_S^{-1}t'$, whence $u \in \bigcap\{r\omega_S^{-1}\,t' \mid t\,\omega_S^{-1}\,t'\}$, as required.

(2) Suppose $r \not\subseteq \overline{Q}_{(s,t)}$. Then $s\,r\omega_S^{-1}\,t$, and since $\omega_S^{-1} \subseteq r\omega_S^{-1}$ we have $t \in r\omega_S^{-1}\,t'$ for all $t\,\omega_S^{-1}\,t'$. Hence $\varphi(s)\,\omega_U^{-1}\,\varphi(t)$ by (1), that is $P_{\varphi(s)} \not\subseteq Q_{\varphi(t)}$.

(3) (a) If $s\,\omega_S^{-1}\varphi\,u$ then we have $s\,\omega_S^{-1}\,s'$ and $u = \varphi(s')$. Now $s\,r\omega_S^{-1}\,s'$ and by (2) we deduce $s\,\varphi\omega_U^{-1}\,u$. Hence $\omega_S^{-1}\varphi \subseteq \varphi\omega_U^{-1}$, so φ satisfies (a) by Proposition 2.10 (1).

(b) If we take $s\,\varphi\omega_U^{-1}\,\varphi(t)$ then $\varphi(s)\,\omega_T^{-1}\,\varphi(t)$, so by (1) $s\,r\omega_S^{-1}\,u$ with $u \in r\omega_S^{-1}\,u' \,\forall\, t\,\omega^{-1}\,u'$. But by R_b we have $s\,\omega_S^{-1}r\omega_S^{-1}\,u$, whence $s\,\omega_S^{-1}\varphi\omega_U^{-1}\,\varphi(t)$. Thus $\varphi\omega_U^{-1} \subseteq \omega_S^{-1}\varphi\omega_U^{-1}$ and (b) is satisfied by Proposition 2.10 (3). □

It follows from Corollary 3.9 (3) that φ gives rise to a difunction $(f, F) = (f_\varphi, F_\varphi) : (S, \mathcal{S}) \to (U, \mathcal{U})$, which we will call the *canonical quotient difunction*. Moreover

$$f^{\leftarrow}B = \varphi^{-1}[B] = F^{\leftarrow}B \text{ for all } B \in \mathcal{U},$$

while from Proposition 2.11 we see that the canonical quotient difunction (f, F) is surjective since φ is onto. Moreover:

Lemma 3.10. *$f^{\to}A = \varphi[A] = F^{\to}A$ for all $A \in \mathcal{R}$.*

Proof. Take $A \in \mathcal{R}$. Since (f, F) is surjective we have $F^{\to}A \subseteq f^{\to}A$ by [6, Corollary 2.33 (1 i)], so it will be sufficient to prove $f^{\to}A \subseteq \varphi[A] \subseteq F^{\to}A$.

Since $\varphi[A] \in \mathcal{U}$, the first inclusion will follow from $(f^{\to}A)\omega_U^{-1} \subseteq A\varphi\omega_U^{-1}$, which by Proposition 2.7 (1) is equivalent to $A\omega_S^{-1}\varphi\omega_U^{-1} \subseteq A\varphi\omega_U^{-1}$. By Proposition 2.10 (1) this follows from (a).

By Propositions 2.7 (2) and 2.10 (2) we have $F^{\to}A = (A^c\omega_S\varphi\omega_U)^c$, so $\varphi[A] \not\subseteq F^{\to}A$ gives the existence of $u \in A\varphi\omega_U^{-1} \cap A^c\omega_S\varphi\omega_U$. Hence we have $a \in A, b \notin A$ with $a\,\varphi\omega_U^{-1}\varphi^{-1}\omega_S^{-1}\,b$ since ω_U is idempotent. Now for some $b' \in S$ with $b'\,\omega_S^{-1}\,b$ we have $\varphi(a)\,\omega_U^{-1}\,\varphi(b')$, so by Corollary 3.9 (1) $a\,r\omega_S^{-1}r\omega^{-1}\,b$, that is $a\,r\omega_S^{-1}\,b$. Thus $b \in Ar\omega_S^{-1} \subseteq A$ by Lemma 3.7 (3), which is a contradiction. □

Finally, we must show that $r = F^{\leftarrow} \circ f$, $R = f^{\leftarrow} \circ F$ following by taking the inverse of both sides. Further, since $r, F^{\leftarrow} \circ f \in \mathcal{P}(S) \times \mathcal{S}$, it will be sufficient to show $r\omega_S^{-1} = (F^{\leftarrow} \circ f)\omega_S^{-1}$. Now by Proposition 2.6 (1) and R_b for $F^{\leftarrow}$ we have $(F^{\leftarrow} \circ f)\omega_S^{-1} = f(F^{\leftarrow})\omega_S^{-1} = f\omega_U^{-1}(F^{\leftarrow})\omega_S^{-1}$, whence using Propositions 2.5 (1), 2.10 (2) and the idempotency of ω_U^{-1} we obtain

$$(F^{\leftarrow} \circ f)\omega_S^{-1} = \omega_S^{-1}\varphi\omega_U^{-1}\varphi^{-1}\omega_S^{-1}.$$

If $s\,\omega_S^{-1}\varphi\omega_U^{-1}\varphi^{-1}\omega_S^{-1}\,t$ we have $s', t' \in S$ satisfying $s\,\omega_S^{-1}\,s'$, $t'\,\omega_S^{-1}\,t$ and $\varphi(s')\,\omega_U^{-1}\,\varphi(t')$. We deduce $s\,r\omega_S^{-1}r\omega_S^{-1}\,t$ from Corollary 3.9 (1), whence $s\,r\omega_S^{-1}\,t$.

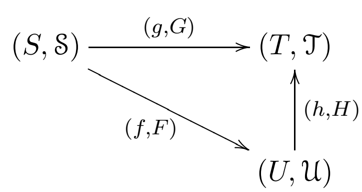
Conversely let $s\,r\omega_S^{-1}\,t$. Then $s\,\omega_S^{-1}r\omega_S^{-1}\omega_S^{-1}\,t$, so we have $s', t' \in S$ with $s\,\omega_S^{-1}\,s'$, $s'\,r\omega_S^{-1}\,t'$ and $t'\,\omega_S^{-1}\,t$. By Corollary 3.9 (2) we have $\varphi(s')\,\omega_U^{-1}\,\varphi(t')$, so $s\,\omega_S^{-1}\varphi\omega_U^{-1}\varphi^{-1}\omega_S^{-1}\,t$.

This shows $r = F^{\leftarrow} \circ f$, and the proof is complete. □

Example 3.11. For a discrete texture $(X, \mathcal{P}(X))$ the interior relation is just the identity relation on X, so (r, R) is an equivalence direlation if and only if the point relation r is reflexive and transitive, and $R = r^{\leftarrow} = (r^c)^{-1}$. Clearly $\rho = r \cap r^{-1}$ and $(f, F) = (\varphi, \varphi^c)$. Finally $\varphi(x)\,\omega_U^{-1}\,\varphi(y) \iff x\,r\,y$, so $(U, \mathcal{U})$ will be a discrete texture if and only if $r = \rho$, that is if and only if r is also symmetric as a point relation.

In the reverse direction we have:

Theorem 3.12. *Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures and (g, G) a difunction on $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (r, R) defined by $r = G^{\leftarrow} \circ g$ and $R = g^{\leftarrow} \circ G$ is an equivalence direlation on $(S, \mathcal{S})$. Moreover, if $(U, \mathcal{U})$ is the quotient texture associated with (r, R) as in Theorem 3.5, and (f, F) the canonical quotient difunction from $(S, \mathcal{S})$ to $(U, \mathcal{U})$, then $(h, H) = (g, G) \circ (f, F)^{\leftarrow}$ is an injective difunction on $(U, \mathcal{U})$ to $(T, \mathcal{T})$ satisfying $(h, H) \circ (f, F) = (g, G)$. Finally, if (g, G) is surjective then (h, H) is bijective.*



Proof. Since $r^{\leftarrow} = (G^{\leftarrow} \circ g)^{\leftarrow} = g^{\leftarrow} \circ (G^{\leftarrow})^{\leftarrow} = g^{\leftarrow} \circ G = R$ by [6, Proposition 2.17 (2) and Lemma 2.4 (2)], we see (r, R) is symmetric. Since (g, G) is a difunction we have $R = g^{\leftarrow} \circ G \subseteq I_S$ by [6, Lemma 2.23 (1 ii)], whence $i_S = I_S^{\leftarrow} \subseteq R^{\leftarrow} = r$ and the direlation (r, R), is reflexive. To show that (r, R) is transitive note first that $r \circ r = (G^{\leftarrow} \circ g) \circ (G^{\leftarrow} \circ g) = G^{\leftarrow} \circ [(g \circ G^{\leftarrow}) \circ g]$ by [6, Proposition 2.17 (3)]. Since (g, G) is a difunction, $g \circ G^{\leftarrow} \subseteq i_T$ by [6, Lemma 2.23 (2 ii)]. Hence by [6, Proposition 2.17 (4)] we have $G^{\leftarrow} \circ [(g \circ G^{\leftarrow}) \circ g] \subseteq G^{\leftarrow} \circ (i_T \circ g)$, and $i_T \circ g = g$ by [6, Proposition 2.17 (1)], so $r \circ r \subseteq G^{\leftarrow} \circ g = r$. $R \subseteq R \circ R$ follows by symmetry, whence (r, R) is transitive and hence an equivalence direlation.

Let us now show that $(g, G) \circ (f, F)^{\leftarrow} = (g \circ F^{\leftarrow}, G \circ f^{\leftarrow})$ is a difunction. For $B \in \mathcal{T}$ we have $g^{\leftarrow}B = G^{\leftarrow}B \in \mathcal{R}$ by [6, Theorem 2.24] and the fact that $r^{\rightarrow}(g^{\leftarrow}B) = (G^{\leftarrow} \circ g)^{\rightarrow}(g^{\leftarrow}B) = G^{\leftarrow}(g^{\rightarrow}(g^{\leftarrow}B)) \subseteq G^{\leftarrow}B = g^{\leftarrow}B$ by [6, Lemma 2.9 (1), Lemma 2.7], which proves $r^{\rightarrow}(g^{\leftarrow}B) = g^{\leftarrow}B$ since r is reflexive. Now, using Lemma 2.8 (2) we have

$$(g \circ F^{\leftarrow})^{\leftarrow}B = F^{\rightarrow}(g^{\leftarrow}B) = f^{\rightarrow}(g^{\leftarrow}B) = f^{\rightarrow}(G^{\leftarrow}B) = (G \circ f^{\leftarrow})^{\leftarrow}B$$

which gives the required result by [6, Theorem 2.24].

To show $(g, G) \circ (f, F)^{\leftarrow}$ is injective we must show that $((g, G) \circ (f, F)^{\leftarrow})^{\leftarrow} = (f \circ G^{\leftarrow}, F \circ g^{\leftarrow})$ satisfies *DF2*. Note that for $B \in \mathcal{T}$ we have

$$(f \circ G^{\leftarrow})^{\rightarrow}((F \circ g^{\leftarrow})^{\leftarrow}B) = f^{\rightarrow}(G^{\leftarrow} \circ g)^{\leftarrow}(F^{\leftarrow}B)) = f^{\rightarrow}(r^{\rightarrow}(F^{\leftarrow}B)).$$

Since $F^{\leftarrow}B = \varphi^{-1}[B] \in \mathcal{R}$ we have $r^{\rightarrow}(F^{\leftarrow}B) = F^{\leftarrow}B$, while $f^{\rightarrow}(F^{\leftarrow}B) \subseteq B$ by [6, Theorem 2.24 (2 b)]. Hence

$$(f \circ G^{\leftarrow})^{\rightarrow}((F \circ g^{\leftarrow})^{\leftarrow}B) \subseteq B$$

and so $(f \circ G^{\leftarrow}, F \circ g^{\leftarrow})$ satisfies *DF2* by [6, Lemma 2.23 (2)].

It remains to show that if (g, G) is surjective, then so is $(g, G) \circ (f, F)^{\leftarrow}$. But in this case the direlation $(g, G)^{\leftarrow}$ satisfies *DF1* by [6, Theorem 2.31 (1)], whence $[(g, G) \circ (f, F)^{\leftarrow}]^{\leftarrow} = (f, F) \circ (g, G)^{\leftarrow}$ also satisfies *DF1* since (f, F) does. But now $(g, G) \circ (f, F)^{\leftarrow}$ is surjective, again by [6, Theorem 2.31 (1)], and the proof of the theorem is complete. □

Example 3.13. If $(g, G) : (X, \mathcal{P}(X)) \to (Y, \mathcal{P}(Y))$ is a difunction then $g : X \to Y$ is a point function, $r = gg^{-1}$ is symmetric as a point relation, and $x \,\rho\, x' \iff g(x) = g(x')$. Hence in this case Theorem 3.12 gives the discrete texturing on the usual quotient associated with g. Naturally, difunctions (g, G) from $(X, \mathcal{P}(X))$ to non-discrete textures will generally produce quotients of $(X, \mathcal{P}(X))$ which are not discrete.

The following result will also prove useful:

Theorem 3.14. *For $k = 1, 2$ let $(S_k, \mathcal{S}_k)$ be a texture, (r_k, R_k) an equivalence direlation on $(S_k, \mathcal{S}_k)$, $(U_k, \mathcal{U}_k)$ the corresponding quotient texture and $(g, G) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ a difunction which is compatible in the sense that $A \in \mathcal{R}_2 \implies g^{\leftarrow} A \in \mathcal{R}_1$. Then there exists a difunction $(\bar{g}, \bar{G}) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ making the following diagram commutative.*

$$\begin{array}{ccc}
(S_1, \mathcal{S}_1) & \xrightarrow{(f_1, F_1)} & (U_1, \mathcal{U}_1) \\
\Big\downarrow{\scriptstyle (g,G)} & & \Big\downarrow{\scriptstyle (\bar{g},\bar{G})} \\
(S_2, \mathcal{S}_2) & \xrightarrow{(f_2, F_2)} & (U_2, \mathcal{U}_2)
\end{array}$$

Proof. Consider the mapping $\theta : \mathcal{U}_2 \to \mathcal{U}_1$ defined by $\theta(A) = \varphi_1[g^{\leftarrow} \varphi_2^{-1}[A]]$ for all $A \in \mathcal{U}_2$. We have noted above that the mapping $A \mapsto \varphi_2^{-1}[A]$ from $\mathcal{U}_2$ to $\mathcal{R}_2$, being an isomorphism, preserves arbitrary intersections and joins. By hypothesis, for $B \in \mathcal{R}_2$ we have $g^{\leftarrow} B \in \mathcal{R}_1$, so $B \mapsto g^{\leftarrow} B$ is a mapping from $\mathcal{R}_2$ to $\mathcal{R}_1$, and this too preserves arbitrary intersections and joins by [6, Corollary 2.26]. Finally the mapping $C \mapsto \varphi_1[C]$ from $\mathcal{R}_1$ to $\mathcal{U}_1$ is an isomorphism and therefore preserves arbitrary intersections and joins. Thus, the same is true of the mapping θ, and by [7, Proposition 4.1] we deduce the existence of a difunction $(\bar{g}, \bar{G}) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ satisfying $\bar{g}^{\leftarrow} A = \bar{G}^{\leftarrow} A = \theta(A)$ for all $A \in \mathcal{U}_2$.

It remains to show the commutativity of the above diagram. We will establish $\bar{g} \circ f_1 = f_2 \circ g$, leaving the dual proof of $\bar{G} \circ F_1 = F_2 \circ G$ to the interested reader. It will be sufficient to show that $(\bar{g} \circ f_1)^{\leftarrow} = (f_2 \circ g)^{\leftarrow}$, whence by [6, Lemma 2.7] we need only show that $(\bar{g} \circ f_1)^{\leftarrow} A = (f_2 \circ g)^{\leftarrow} A$ for all $A \in \mathcal{U}_2$. However

$$\begin{aligned}(\bar{g} \circ f_1)^{\leftarrow} A &= f_1^{\leftarrow}(\bar{g}^{\leftarrow} A) = \varphi_1^{-1}[\theta(A)] \\ &= \varphi_1^{-1}[\varphi_1[g^{\leftarrow} \varphi_2^{-1}[A]] = g^{\leftarrow} \varphi_2^{-1}[A] \\ &= g^{\leftarrow}(f_2^{\leftarrow} A) = (f_2 \circ g)^{\leftarrow} A,\end{aligned}$$

which gives the required result. □

We end this section by presenting a method for generating equivalence direlations.

Proposition 3.15. *Let $(S, \mathcal{S})$ be a texture, $\mathcal{B} \subseteq \mathcal{S}$ and define the direlation (r, R) on $(S, \mathcal{S})$ by*

$$r = \bigvee \{\overline{P}_{(s,t)} \mid \exists\, u \text{ with } P_s \not\subseteq Q_u \text{ and } P_t \subseteq B \text{ or } B \subseteq Q_u \forall\, B \in \mathcal{B}\},$$

$$R = \bigcap \{\overline{Q}_{(s,t)} \mid \exists\, v \text{ with } P_v \not\subseteq Q_s \text{ and } P_v \subseteq B \text{ or } B \subseteq Q_t \forall\, B \in \mathcal{B}\}.$$

Then:

(1) (a) $s\, r\omega_S^{-1}\, t \iff \exists\, u \in s\omega_S^{-1},\ v \in \omega_S^{-1} t$ *satisfying* $u \in B\omega_S^{-1} \implies v \in B\omega_S^{-1}\ \forall\, B \in \mathcal{B}$,

(b) $(s, t) \in R^c \iff \exists\, v \in \omega_S^{-1} s,\ u \in t\omega_S^{-1}$ *satisfying* $u \in B\omega_S^{-1} \implies v \in B\omega_S^{-1}$.

(2) *(r, R) is an equivalence direlation on $(S, \mathcal{S})$.*

(3) *$\mathcal{B}$ generates the set $\mathcal{R}$ associated with (r, R) in the sense that $\mathcal{B} \subseteq \mathcal{R}$ and every element of $\mathcal{R}$ can be written as an intersection of joins of elements of $\mathcal{B}$.*

Proof. (1) If $s\, r\omega_S^{-1}\, t$ then $r \not\subseteq \overline{Q}_{(s,t)}$, which gives $P_{t'} \not\subseteq Q_t$, $P_s \not\subseteq Q_u$ satisfying $B \not\subseteq Q_u \implies P_{t'} \subseteq B$. Taking $P_{t'} \not\subseteq Q_v$, $P_v \not\subseteq Q_t$ now gives $u \in s\omega_S^{-1}$, $v \in \omega_S^{-1}$ with $u \in B\omega_S^{-1} \implies v \in B\omega_S^{-1}$, and the converse is easily established. This shows (a), and the proof of (b) is similar and is left to the reader.

(2) If $s\,\omega_S^{-1}\, t$ then $s\,\omega_S^{-1}\omega_S^{-1}\, t$ so we have $u \in S$ with $s\,\omega_S^{-1}\, u$, $u\,\omega_S^{-1}\, t$, and taking $v = u$ in (1 a) gives $s\, r\omega_S^{-1}\, t$. Hence $\omega_S^{-1} \subseteq r\omega_S^{-1}$. Suppose $s\, r\omega_S^{-1} r\omega_S^{-1}\, t$ and take $w \in S$ with $s\, r\omega_S^{-1}\, w$, $w\, r\omega_S^{-1}\, t$. Now by (1 a) we have $u \in s\omega_S^{-1}$, $v \in \omega_S^{-1} w$ and $u' \in w\omega_S^{-1}$, $v' \in \omega_S^{-1} t$ satisfying the appropriate implications. Hence

$$\begin{aligned}u \in B\omega_S^{-1} \implies v \in B\omega_S^{-1} &\implies u' \in B\omega_S^{-1}\omega_S^{-1}\omega_S^{-1} = B\omega_S^{-1} \\ &\implies v' \in B\omega_S^{-1},\end{aligned}$$

which shows that $s\, r\omega_S^{-1}\, t$ by (1 a), and so $r\omega_S^{-1} r\omega_S^{-1} \subseteq r\omega_S^{-1}$.

Finally from (1) we note that $s\,r\omega_S^{-1}\,t \iff t\,R^c\,s$ so $r\omega_S^{-1} = (R^c)^{-1} = (R^{\leftarrow})\omega_S^{-1}$ by Proposition 2.5 (1), whence $r = R^{\leftarrow}$ and so $R = r^{\leftarrow}$. By Corollary 3.2 this completes the proof that (r, R) is an equivalence direlation.

(3) Take $B_0 \in \mathcal{B}$. By Lemma 3.7 (3), $B_0 \in \mathcal{R}$ if and only if $B_0 r\omega_S^{-1} = B_0\omega_S^{-1}$, and since $\omega_S^{-1} \subseteq r\omega_S^{-1}$ it will clearly suffice to show $B_0 r\omega_S^{-1} \subseteq B_0\omega_S^{-1}$. Take $t \in B_0 r\omega_S^{-1}$. Then for some $b \in B$ we have $b\,r\omega_S^{-1}\,t$, so there exists $u \in b\omega_S^{-1}$, $v \in \omega_S^{-1}t$ with $u \in B\omega_S^{-1} \implies v \in B\omega_S^{-1}$ for all $B \in \mathcal{B}$ by (1 a). But $u \in B_0\omega_S^{-1}$ so $v \in B_0\omega_S^{-1}$, whence $t \in B_0\omega_S^{-1}\omega_S^{-1} = B_0\omega_S^{-1}$, as required. This establishes $\mathcal{B} \subseteq \mathcal{R}$.

Finally, we take $A \in \mathcal{R}$. Suppose that $P_t \not\subseteq A$. Then $P_t \not\subseteq r^{\rightarrow}A$ since $r^{\rightarrow}A = A$ for $A \in \mathcal{R}$, so we have $t' \in S$ with $P_t \not\subseteq Q_{t'}$ satisfying $r \not\subseteq \overline{Q}_{(s,t')} \implies A \subseteq Q_s$. Take any $s \in S$ with $A \not\subseteq Q_s$ and take $s' \in S$ with $A \not\subseteq Q_{s'}$, $P_{s'} \not\subseteq Q_s$. By the above implication $r \subseteq \overline{Q}_{(s',t')}$, whence $\overline{P}_{(s',t)} \not\subseteq r$ and we may choose $B_s^t \in \mathcal{B}$ satisfying $P_t \not\subseteq B_s^t \not\subseteq Q_s$.

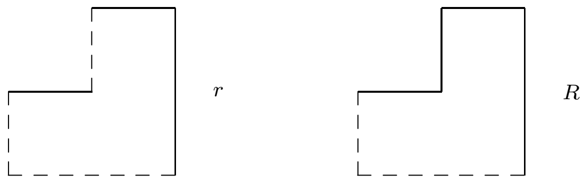
Let $B_t = \bigvee\{B_s^t \mid A \not\subseteq Q_s\}$. Then $A \subseteq B_t$. Indeed, if not we shall have $s \in S$ satisfying $A \not\subseteq Q_s$, $P_s \not\subseteq B_t$ and so the contradiction $P_s \not\subseteq B_s^t \subseteq B_s^t$. Hence $A \subseteq \bigcap\{B_t \mid P_t \not\subseteq A\}$. If we do not have equality we may take $w \in S$ satisfying $\bigcap\{B_t \mid P_t \not\subseteq A\} \not\subseteq Q_w$, $P_w \not\subseteq A$. In particular this gives $B_w \not\subseteq Q_w$, so there exists $s \in S$ satisfying $B_s^w \not\subseteq Q_w$, which contradicts $P_w \not\subseteq B_s^w$. This shows that $A = \bigcap\{B_t \mid P_t \not\subseteq A\}$, so A has the required representation. □

Corollary 3.16. *If* (r, R) *is the equivalence direlation generated by* $\mathcal{B}$ *as in Proposition 3.15, the elements of* $\mathcal{B}$ *are saturated for* (r, R).

Proof. Immediate from Lemma 3.7 (3). □

Example 3.17. Consider the texture $(L, \mathcal{L})$ of Examples 1.1 (2), and set $\mathcal{B} = \{(0, \frac{1}{2}]\}$. Then the corresponding equivalence direlation is easily seen to be:

$$r = \{(s,t) \mid 0 < t \le \tfrac{1}{2} \text{ or } \tfrac{1}{2} < s \le 1\}, \text{ and}$$
$$R = \{(s,t) \mid 0 < t \le \tfrac{1}{2} \text{ or } \tfrac{1}{2} \le s \le 1\}.$$



Clearly, $\mathcal{R} = \{\emptyset,\ (0, \frac{1}{2}],\ L\}$, while

$$s\rho t \iff 0 < s \leq \tfrac{1}{2},\ 0 < t \leq \tfrac{1}{2} \text{ or } \tfrac{1}{2} < s \leq 1,\ \tfrac{1}{2} < t \leq 1.$$

Hence there are two equivalence classes which we denote by $\boldsymbol{\frac{1}{2}}$ and $\mathbf{1}$. This gives us $\varphi(s) = \begin{cases} \boldsymbol{\frac{1}{2}} & 0 < s \leq \frac{1}{2} \\ \mathbf{1} & \frac{1}{2} < s \leq 1 \end{cases}$, $U = \{\boldsymbol{\frac{1}{2}}, \mathbf{1}\}$ and $\mathcal{U} = \{\emptyset,\ \{\boldsymbol{\frac{1}{2}}\},\ U\}$.

The following result will be useful when applying Theorem 3.14 to equivalence direlations obtained as in Proposition 3.15.

Lemma 3.18. *Let (r_k, R_k) be equivalence direlations on $(S_k, \mathcal{S}_k)$ generated by $\mathcal{B}_k \subseteq \mathcal{S}_k$, $k = 1, 2$ as in Proposition 3.15. Then if the difunction $(g, G) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ satisfies $B \in \mathcal{B}_2 \implies g^{\leftarrow}B \in \mathcal{R}_1$ it satisfies $A \in \mathcal{R}_2 \implies g^{\leftarrow}A \in \mathcal{R}_1$.*

Proof. By Proposition 3.15, $A \in \mathcal{R}_2$ may be written as an intersection of joins of sets of $\mathcal{B}_2$. On the other hand the mapping $(g^{\leftarrow})^{\rightarrow}$ preserves arbitrary intersections and joins, so by hypothesis $g^{\leftarrow}A = (g^{\leftarrow})^{\rightarrow}A$ can be written as an intersection of joins of elements of $\mathcal{R}_1$. However, $\mathcal{R}_1$ is closed under arbitrary intersections and joins by Lemma 3.7 (5), whence $g^{\leftarrow}A \in \mathcal{R}_1$. □

4. Quotient Ditopologies and Quotient Difunctions

Now let (τ, κ) be a ditopology on $(S, \mathcal{S})$. If, as above, (r, R) is an equivalence direlation on $(S, \mathcal{S})$, $(U, \mathcal{U})$ the corresponding quotient texture and $(f, F) : (S, \mathcal{S}) \to (U, \mathcal{U})$ the canonical quotient difunction then clearly $\tau_U = \{H \in \mathcal{U} \mid F^{\leftarrow}H \in \tau\}$ is a topology and $\kappa_U = \{K \in \mathcal{U} \mid f^{\leftarrow}K \in \kappa\}$ a cotopology on $(U, \mathcal{U})$.

Definition 4.1. With the notation above, (τ_U, κ_U) is called the *quotient ditopology of* (τ, κ) *on* $(U, \mathcal{U})$, and $(U, \mathcal{U}, \tau_U, \kappa_U)$ the *quotient ditopological texture space of* $(S, \mathcal{S}, \tau, \kappa)$ *modulo* (r, R).

Clearly the quotient ditopology is the finest ditopology on the quotient $(U, \mathcal{U})$ making the canonical quotient difunction bicontinuous. Just as for quotients of topological spaces we have:

Proposition 4.2. *Let $(U, \mathcal{U}, \tau_U, \kappa_U)$ be a quotient of $(S, \mathcal{S}, \tau, \kappa)$ and (f, F) the canonical quotient difunction. If $(g, G) : (U, \mathcal{U}, \tau_U, \kappa_U) \to (S', \mathcal{S}', \tau', \kappa')$ is a difunction, then (g, G) is continuous (cocontinuous, bicontinuous) if and only if $(g, G) \circ (f, F)$ is continuous (respectively, cocontinuous, bicontinuous).*

Proof. This follows from the equalities $(g \circ f)^{\leftarrow} A = f^{\leftarrow}(g^{\leftarrow} A)$ and $(G \circ F)^{\leftarrow} A = F^{\leftarrow}(G^{\leftarrow} A)$ for all $A \in \mathcal{S}'$, established in [6, Lemma 2.16 (2)]. □

Before proceeding it will be useful to have a notion of homeomorphism for difunctions.

Definition 4.3. An isomorphism in the category **dfTex** of textures and difunctions which, together with its inverse, is bicontinuous, will be called a *difunctional homeomorphism*, or *dihomeomorphism* for short.

Dihomeomorphisms, therefore, coincide with the isomorphisms of the category **dfDitop** of ditopological texture spaces and bicontinuous difunctions.

It is proved in [6, Proposition 3.14 (5)] that a difunction (h, H) is an isomorphism if and only if it is bijective, and it is clear from the proof of that proposition that the inverse is then the inverse direlation $(h, H)^{\leftarrow} = (H^{\leftarrow}, h^{\leftarrow})$. The following gives several useful characterizations of a dihomeomorphism, and is the textural analogue of [10, Proposition 1.4.18].

Proposition 4.4. *Let $(h, H) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a bijective difunction. Then the following are equivalent.*

(i) *(h, H) is a dihomeomorphism.*
(ii) *(h, H) is bicontinuous, open and coclosed.*
(iii) *(h, H) is bicontinuous, closed and co-open.*
(iv) $h^{\to} A = H^{\to} A \in \tau_2 \iff A \in \tau_1$, *and*
$h^{\to} A = H^{\to} A \in \kappa_2 \iff A \in \kappa_1$.
(v) $h^{\leftarrow} B = H^{\leftarrow} B \in \tau_1 \iff B \in \tau_2$, *and*
$h^{\leftarrow} B = H^{\leftarrow} B \in \kappa_1 \iff B \in \kappa_2$.

Proof. (i) $\Longrightarrow$ (ii) The inverse $(H^{\leftarrow}, h^{\leftarrow})$ of (h, H) is continuous, so

$$U_1 \in \tau_1 \implies h^{\rightarrow}U_1 = ((h^{\leftarrow})^{\leftarrow})^{\rightarrow}U_1 = (h^{\leftarrow})^{\leftarrow}U_1 \in \tau_2,$$

which shows that (h, H) is open. Likewise, the cocontinuity of $(H^{\leftarrow}, h^{\leftarrow})$ implies the coclosedness of (h, H).

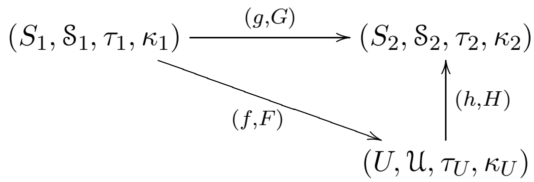
(ii) $\Longrightarrow$ (iii) By [6, Corollary 2.33 (1, 2)] we see that since (h, H) is bijective, $h^{\rightarrow}A = H^{\rightarrow}A$ for all $A \in \mathcal{S}_1$. Hence (h, H) is closed if and only if it is coclosed, and open if and only if it is co-open.

(iii) $\Longrightarrow$ (iv) $A \in \tau_1 \implies H^{\rightarrow}A \in \tau_2$ since (h, H) is co-open, and we have already noted that $h^{\rightarrow}A = H^{\rightarrow}A$ since (h, H) is bijective. On the other hand if $h^{\rightarrow}A = H^{\rightarrow}A \in \tau_2$ then $A = H^{\leftarrow}(h^{\rightarrow}A) \in \tau_1$ by the continuity of (h, H), the injectivity of (h, H) and [6, Corollary 2.33 (2)]. The proof of the second result is dual to this and is omitted.

(iv) $\Longrightarrow$ (v) Take $B \in \mathcal{S}_2$ and set $A = H^{\leftarrow}B$. Then, since (h, H) is surjective, $B = h^{\rightarrow}(H^{\leftarrow}B) = h^{\rightarrow}A$ by [6, Corollary 2.33 (1)]. If $B \in \tau_2$ then $h^{\rightarrow}A \in \tau_2$ and we deduce $H^{\leftarrow}B = A \in \tau_1$ by (iv), and we know that $H^{\leftarrow}B = h^{\leftarrow}B$ since (h, H) is a difunction. On the other hand if we assume that $h^{\leftarrow}B = H^{\leftarrow}B \in \tau_1$ then $A \in \tau_1$ and by (iv) we have $B = h^{\rightarrow}A \in \tau_2$. The proof of the remaining results is dual to this, and is omitted.

(v) $\Longrightarrow$ (i) It is clear from (v) that (h, H) is bicontinuous. To show $(H^{\leftarrow}, h^{\leftarrow})$ is continuous take $U_1 \in \tau_1$. Then $U_1 = H^{\leftarrow}(h^{\rightarrow}U_1) \in \tau_1$ by [6, Corollary 2.33 (2)] since (h, H) is injective, whence $h^{\rightarrow}U_1 \in \tau_2$ by (v) applied to $B = h^{\rightarrow}U_1$. But, as noted above, $h^{\rightarrow}U_1 = (h^{\leftarrow})^{\leftarrow}U_1$, so $(h^{\leftarrow}, H^{\leftarrow})$ is continuous. Likewise $(H^{\leftarrow}, h^{\leftarrow})$ is cocontinuous and we have shown that (h, H) is a dihomeomorphism. □

Now let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$ be ditopological texture spaces and $(g, G) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ a bicontinuous difunction. As in Theorem 3.12, (g, G) gives rise to a quotient texture $(U, \mathcal{U})$, a canonical quotient difunction (f, F) and an injective difunction (h, H) from $(U, \mathcal{U})$ to $(S_2, \mathcal{S}_2)$ satisfying $(h, H) \circ (f, F) = (g, G)$. We take the quotient ditopology (τ_U, κ_U) on $(U, \mathcal{U})$.



By Proposition 4.2 we see that the difunction (h, H) is bicontinuous. In case (g, G) is surjective (h, H) is a bijection, but in general it need not be a dihomeomorphism because the inverse may not be bicontinuous.

Definition 4.5. $(g, G) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is called a *quotient difunction* if it can be expressed as the composition of a canonical quotient difunction on $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and a dihomeomorphism onto $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$.

Proposition 4.6. *The following are equivalent for a surjective difunction* (g, G) *from* $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ *to* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *:*

(1) *The difunction* (g, G) *is quotient.*
(2) $G^{\leftarrow}B \in \tau_1 \iff B \in \tau_2$ *and* $g^{\leftarrow}B \in \kappa_1 \iff B \in \kappa_2$.
(3) *With the notation as in Theorem 3.12 and the commutative diagram above, the difunction* (h, H) *is a dihomeomorphism.*

Proof. (1) $\implies$ (2) Let $(g, G) = (k, K) \circ (f, F)$, where $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (U, \mathcal{U}, \tau_U, \kappa_U)$ is a canonical quotient difunction and $(k, K) : (U, \mathcal{S}, \tau_U, \kappa_U) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ a dihomeomorphism. By the definition of (τ_U, κ_U) we have $G^{\leftarrow}B = F^{\leftarrow}(K^{\leftarrow}B) \in \tau_1 \iff K^{\leftarrow}B \in \tau_U$, while by Proposition 4.4 (v) we have $K^{\leftarrow}B \in \tau_U \iff B \in \tau_2$. The proof of the second equivalence is dual to this and is omitted.

(2) $\implies$ (3) Since (h, H) is bijective because (g, G) is surjective we need only verify Proposition 4.4 (iv). Take $A \in \mathcal{U}$. Then $G^{\leftarrow}(h^{\rightarrow}A) = (H \circ F)^{\leftarrow}(h^{\rightarrow}A) = F^{\leftarrow}(H^{\leftarrow}(h^{\rightarrow}A)) = F^{\leftarrow}A$ by [6, Corollary 2.33 (2)] since (h, H) is injective. Hence

$$A \in \tau_U \iff F^{\leftarrow}A \in \tau_1 \iff G^{\leftarrow}(h^{\rightarrow}A) \in \tau_1 \iff h^{\rightarrow}A \in \tau_2$$

by (2). In just the same way $A \in \kappa_U \iff H^{\rightarrow}A \in \kappa_2$.

(3) $\implies$ (1) Immediate. □

Corollary 4.7. *A composition of quotient difunctions is a quotient difunction. On the other hand if* (g_1, G_1), (g_2, G_2) *are bicontinuous difunctions whose composition* $(g_2, G_2) \circ (g_1, G_1)$ *is a quotient difunction then* (g_2, G_2) *is a quotient difunction.*

Proof. The first statement is clear from Proposition 4.6 (2). If $(g_2 \circ g_1, G_2 \circ G_1)$ is a quotient difunction it is surjective so by [6, Theorem 3.32 (1)] we have

$$i \subseteq (g_2 \circ g_1) \circ (G_2 \circ G_1)^{\leftarrow} = g_2 \circ (g_1 \circ G_1^{\leftarrow}) \circ G_2^{\leftarrow} \subseteq g_2 \circ G_2^{\leftarrow}$$

since $g_1 \circ G_1^{\leftarrow} \subseteq i$ by [6, Lemma 2.23 (2)] applied to the difunction (g_1, G_1). Hence (g_2, G_2) is surjective. Since (g_1, G_1), (g_2, G_2) are bicontinous the result now follows easily from Proposition 4.6. □

Corollary 4.8. *An injective quotient difunction is a dihomeomorphism.*

Proof. In view of Proposition 4.6 (3) it will be sufficient to show that the canonical quotient difunction (f, F) of the quotient generated by a bijective difunction (g, G) as in Theorem 3.12 is a dihomeomorphism. But since, as we have noted before, $(G^{\leftarrow}, g^{\leftarrow})$ is now the inverse of (g, G) we have $r = G^{\leftarrow} \circ g = i$ and $R = g^{\leftarrow} \circ G = I$. Lemma 3.7 (1) now gives

$$s \rho t \iff P_s \subseteq P_t \text{ and } P_t \subseteq P_s \iff P_s = P_t \iff s = t.$$

It follows that the quotient $(U, \mathcal{U}, \tau_U, \kappa_U)$ may be identified with $(S, \mathcal{S}, \tau, \kappa)$, φ becomes the identity and so $(f, F) = (i, I)$, which is a dihomeomorphism. □

Corollary 4.9. *Surjective bicontinuous difunctions which are open or co-open, and closed or coclosed, are quotient difunctions.*

Proof. Clear from Proposition 4.6 (2) and [6, Corollary 2.33 (1) and Theorem 2.24 (3)]. □

Proposition 4.10. *Let* (r, R) *be an equivalence direlation on* $(S, \mathcal{S}, \tau, \kappa)$ *and* (f, F) *the canonical quotient difunction. Then* (f, F) *is:*

(1) *open if and only if* $A \in \tau \implies r^{\rightarrow} A \in \tau$.
(2) *co-open if and only if* $A \in \tau \implies R^{\rightarrow} A \in \tau$.
(3) *closed if and only if* $A \in \kappa \implies r^{\rightarrow} A \in \kappa$.
(4) *coclosed if and only if* $A \in \kappa \implies R^{\rightarrow} A \in \tau$.

Proof. We establish (1), leaving the other cases to the interested reader. By Theorem 3.5 we have $r = F^{\leftarrow} \circ f$, so for $A \in \mathcal{S}$ we have $r^{\rightarrow}A = (F^{\leftarrow} \circ f)^{\rightarrow}A = F^{\leftarrow}(f^{\rightarrow}A)$. By the definition of the quotient topology we deduce that $r^{\rightarrow}A \in \tau \iff f^{\rightarrow}A \in \tau_U$, from which (1) follows at once. □

Corollary 4.11. *Let $(g, G) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a quotient difunction. Then (g, G) is:*

(1) *open if and only if $A \in \tau_1 \implies G^{\leftarrow}(g^{\rightarrow}A) \in \tau_1$.*
(2) *co-open if and only if $A \in \tau_1 \implies G^{\leftarrow}(G^{\rightarrow}A) \in \tau_1$.*
(3) *closed if and only if $A \in \kappa_1 \implies g^{\leftarrow}(g^{\rightarrow}A) \in \kappa_1$.*
(4) *coclosed if and only if $A \in \kappa_1 \implies g^{\leftarrow}(G^{\rightarrow}A) \in \kappa_1$.*

Proof. With the notation as in Theorem 3.12 we have $(g, G) = (h, H) \circ (f, F)$, where (f, F) is the canonical quotient difunction for the equivalence direlation (r, R) given by $r = G^{\leftarrow} \circ g$, $R = g^{\leftarrow} \circ G$, and (h, H) is a dihomeomorphism by Proposition 4.6. Hence (g, G) and (f, F) share the properties in question, so the results follow at once from Proposition 4.10. □

Definition 4.12. An equivalence direlation (r, R) on a ditopological texture space is called *open* (*co-open, closed, coclosed*) if the canonical quotient difunction is open (co-open, closed, coclosed).

We recall from [8] that $(S, \mathcal{S}, \tau, \kappa)$ is T_1 if each set $A \in \mathcal{S}$ can be written in the form $A = \bigvee_{j \in J} K_j$ with $K_j \in \kappa$, $j \in J$; and that it is co-T_1 if each $A \in \mathcal{S}$ can be written in the form $A = \bigcap_{j \in J} G_j$ with $G_j \in \tau$, $j \in J$.

Proposition 4.13. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and (r, R) an equivalence direlation on $(S, \mathcal{S})$.*

(1) *If $(S, \mathcal{S}, \tau, \kappa)$ is T_1 and (r, R) is closed then every set in $\mathcal{R}$ can be written as a join of closed sets in $\mathcal{R}$.*
(2) *If $(S, \mathcal{S}, \tau, \kappa)$ is co-T_1 and (r, R) is co-open then every set in $\mathcal{R}$ can be written as an intersection of open sets in $\mathcal{R}$.*

Proof. (1) For $K_j \in \kappa$, $j \in J$, we have $r^{\rightarrow}(\bigvee_{j \in J} K_j) = \bigvee_{j \in J} r^{\rightarrow}K_j$ by [6, Corollary 2.12 (2)], and $r^{\rightarrow}K_j \in \kappa$ by Proposition 4.10 (3). Since $\mathcal{R} = \{r^{\rightarrow}A \mid A \in \mathcal{S}\}$ by Lemma 3.7 (2,4), the required result follows at once.

(2). Dual to (1). □

5. The T_0 reflector

We begin by showing that every ditopological texture space has a quotient that is a T_0 ditopological texture space. Let us recall the following characteristic property of T_0 ditopological spaces [8].

Definition 5.1. The ditopology (τ, κ) on $(S, \mathcal{S})$ is T_0 if given $Q_s \not\subseteq Q_t$ there exists $B \in \tau \cup \kappa$ satisfying $P_s \not\subseteq B \not\subseteq Q_t$.

Now we may give:

Theorem 5.2. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space, (r, R) the equivalence direlation generated by $\mathcal{B} = \tau \cup \kappa$ as in Proposition 3.15, $(U, \mathcal{U})$ the quotient texture of $(S, \mathcal{S})$ with respect to (r, R) and (τ_U, κ_U) the quotient ditopology on $(U, \mathcal{U})$. Then the quotient ditopological space $(U, \mathcal{U}, \tau_U, \kappa_U)$ is T_0.*

Proof. Take $s, t \in S$ with $Q_{\bar{s}} \not\subseteq Q_{\bar{t}}$. By Lemma 3.8 (ii) we have $Q_{\bar{s}} = \bigvee\{\varphi[R^{\to}Q_{s'}] \mid P_s \not\subseteq Q_{s'}\}$ and so there exists $s' \in S$ with $P_s \not\subseteq Q_{s'}$ satisfying $\varphi[R^{\to}Q_{s'}] \not\subseteq Q_{\bar{t}}$. On the other hand $Q_{\bar{t}} \in \mathcal{U}$ so we have $A \in \mathcal{R}$ with $\varphi[A] = Q_{\bar{t}}$, whence $\varphi[R^{\to}Q_{s'}] \not\subseteq \varphi[A]$ and so $R^{\to}Q_{s'} \not\subseteq A$. Now we have $w \in S$ with $P_w \not\subseteq A$ for which $\overline{P}_{(z,w)} \not\subseteq R \implies P_z \subseteq Q_{s'}$. If we choose $s'' \in S$ with $P_s \not\subseteq Q_{s''}$, $P_{s''} \not\subseteq Q_{s'}$, and set $z = s''$ in the above implication we obtain $\overline{P}_{(s'',w)} \subseteq R$. If now take $w' \in S$ with $P_w \not\subseteq Q_{w'}$ and $P_{w'} \not\subseteq A$ we obtain $R \not\subseteq \overline{Q}_{(s'',w')}$. From the definition of R, and bearing in mind that $P_s \not\subseteq Q_{s''}$, we see that there exists $B \in \mathcal{B} = \tau \cup \kappa$ satisfying $P_s \not\subseteq B \not\subseteq Q_{w'}$.

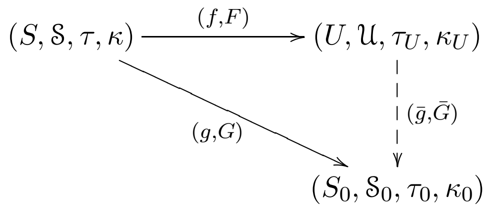
From $P_s \notin B$, and the fact that B is saturated with respect to ρ gives $\bar{s} \notin \varphi[B]$, that is $P_{\bar{s}} \not\subseteq \varphi[B]$. Also, $B \not\subseteq A$ and since A and B are both saturated with respect to ρ we obtain $\varphi[B] \not\subseteq \varphi[A] = Q_{\bar{t}}$. Finally, $\varphi[B] \in \varphi[\mathcal{B}] = \varphi[\tau \cup \kappa] = \tau_U \cup \kappa_U$, and we have established that $(U, \mathcal{U}, \tau_U, \kappa_U)$ is T_0. □

We obtain the subcategory **dfDitop**$_0$ of **dfDitop** by restricting to T_0 ditopological texture spaces.

Theorem 5.3. *The category* **dfDitop**$_0$ *is a full reflexive subcategory of* **dfDitop**.

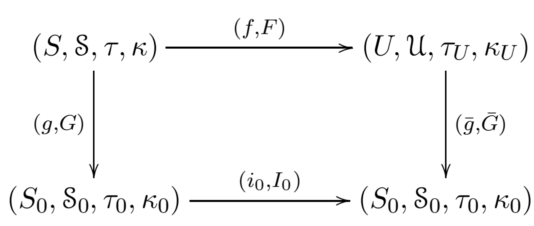
Proof. It is clear that **dfDitop**$_0$ is a full subcategory of **dfDitop**. Take $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}$**dfDitop** and let $(f, F) : (S, \mathcal{S}, \tau, \kappa) \to (U, \mathcal{U}, \tau_U, \kappa_U)$ be the canonical difunction onto the T_0 quotient.

We claim (f, F) is a reflection [1]. To show this take $(S_0, \mathcal{S}_0, \tau_0, \kappa_0) \in \mathrm{Ob}\,\mathbf{dfDitop}_0$ and $(g, G) \in \mathbf{dfDitop}((S, \mathcal{S}, \tau, \kappa), (S_0, \mathcal{S}_0, \tau_0, \kappa_0))$. We must show the existence of a $\mathbf{dfDitop}_0$ morphism $(\bar{g}, \bar{G})$ making the following diagram commutative.



Let us now show that since $(S_0, \mathcal{S}_0, \tau_0, \kappa_0)$ is T_0 the corresponding equivalence direlation (r_0, R_0) is the identity (i_0, I_0) on $(S_0, \mathcal{S}_0)$. By symmetry it will be sufficient to show that $R_0 = I_0$, and by reflexivity that $I_0 \subseteq R_0$. Suppose that $I_0 \not\subseteq R_0$ and take $s, t \in S$ with $I_0 \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq I_0$. The first result leads to $Q_s \not\subseteq Q_t$, whence we have $B_0 \in \tau \cup \kappa$ satisfying $P_s \not\subseteq B_0 \not\subseteq Q_s$.

From the second result we have $t' \in S$ with $\overline{P}_{(s,t)} \not\subseteq \overline{Q}_{(s,t')}$, and then $v \in S$ with $P_v \not\subseteq Q_s$ for which $P_v \subseteq B$ or $B \subseteq Q_{t'}$ for all $B \in \tau \cup \kappa$. But $P_t \not\subseteq Q_{t'}$ gives $Q_{t'} \subseteq Q_t$, and $P_v \not\subseteq Q_s$ gives $P_s \subseteq P_v$, so $P_s \subseteq B$ or $B \subseteq Q_t \ \forall\, B \in \tau \cup \kappa$. Taking $B = B_0$ now gives a contradiction, so $(r_0, R_0) = (i_0, I_0)$ as required. Arguing as in the proof of Corollary 4.8 we may identify the quotient with $(S_0, \mathcal{S}_0, \tau_0, \kappa_0)$, and the canonical quotient difunction becomes (i_0, I_0). Also, the bicontinuity of (g, G) implies that $g^{\leftarrow}(\tau_0 \cup \kappa_0) \subseteq \tau \cup \kappa = \mathcal{B} \subseteq \mathcal{R}$, so by Lemma 3.18, Theorem 3.14 gives a difunction $(\bar{g}, \bar{G})$ making the diagram



commutative. Finally, let us show that $(\bar{g}, \bar{G})$ is bicontinuous. For $A \in \mathcal{S}_0$ we have $\bar{g}^{\leftarrow}A = \bar{G}^{\leftarrow}A = \varphi[g^{\leftarrow}\varphi_0^{-1}[A]] = \varphi[g^{\leftarrow}A]$ by the proof of Theorem 3.14. and the fact that the quotient mapping on S_0 is the identity. Now if $A \in \tau_0$ or $A \in \kappa_0$ then $g^{\leftarrow}A \in \tau \cup \kappa = \mathcal{B} \subseteq \mathcal{R}$ so the set $g^{\leftarrow}A$ is saturated with respect to ρ and so,

$$F^{\leftarrow}(\bar{G}^{\leftarrow}A) = f^{\leftarrow}(\bar{g}^{\leftarrow}A) = \varphi^{-1}[\varphi[g^{\leftarrow}A]] = g^{\leftarrow}A = G^{\leftarrow}A.$$

Hence $A \in \tau_0 \implies G^{\leftarrow}A \in \tau \implies F^{\leftarrow}(\bar{G}^{\leftarrow}A) \in \tau \implies \bar{G}^{\leftarrow}A \in \tau_U$, which proves that $(\bar{g}, \bar{G})$ is continuous. Likewise it is cocontinuous and hence is a $\mathbf{dfDitop}_0$-morphism. □

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