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A NATURAL FUNCTOR FOR HYPERSPACES

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Abstract. Let $(X,\mathcal{T})$, $(Y,\mathcal{T}')$ be Hausdorff topological spaces and $CL(X)$, $CL(Y)$ respectively the families of all non empty closed subsets of X and Y and be assigned some hypertopologies. For each function $f\colon X \to Y$ there is a natural function: $F\colon CL(X) \to CL(Y)$, where for each $A \in CL(X)$, $F(A) = clf(A) \in CL(Y)$. In this paper, we study the relationship between f and F. We are primarily interested in finding necessary and sufficient conditions on f for the continuity of F. **To avoid trivial situations, we will assume that Y contains at least two points and an arc and f is a surjection.** Since the base spaces X, Y are embedded in their hyperspaces, we always have f continuous. We use our recent study of Bombay topologies to get the general solution and derive results in the case of various known hypertopologies. Sample results are:

(1) Let X and Y be metric spaces and $CL(X)$, $CL(Y)$ be assigned the corresponding Hausdorff metric topologies. Then f is uniformly continuous if, and only if, F is (uniformly) continuous.

(2) Let X and Y be topological spaces and $CL(X)$, $CL(Y)$ be assigned the corresponding Vietoris topologies. Then f is continuous if, and only if, F is continuous.

(3) Let X and Y be Hausdorff topological spaces and $CL(X)$, $CL(Y)$ be assigned the corresponding Fell topologies. Then f is continuous and strongly compact if, and only if, F is continuous.

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1. Introduction

Let $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ be Hausdorff topological spaces with compatible LO-proximities γ, γ_1, γ_2 ($\gamma_1 \leq \gamma_2$) on X and α, α_1, α_2 ($\alpha_1 \leq \alpha_2$) on Y. We also study situations wherein X and Y have compatible uniformities $\mathcal{U}$ and $\mathcal{V}$ respectively which naturally induce EF-proximities. Let $CL(X)$, $CL(Y)$ denote respectively the families of all non empty closed subsets of X and Y. Corresponding to each function $f\colon X \to Y$ there is a natural function

$F\colon CL(X) \to CL(Y)$, where for each $A \in CL(X), F(A) = cl f(A) \in CL(Y)$.

We assign hyperspace topologies τ, τ' related respectively to topologies, (proximities, metrics or uniformities) on $CL(X), CL(Y)$. Our aim in this paper is to find necessary and sufficent conditions on f for the continuity of F (see [12] where a few cases are discussed). In order to avoid trivial cases, we assume that

(a) Y **contains at least two points and an arc**;
(b) f **is not constant and is a surjection**.

In all hyperspace topologies that we discuss here, X (respectively Y) is embedded in $CL(X)$ (respectively $CL(Y)$) by the map $x \to \{x\}$ (respectively $y \to \{y\}$). Hence the continuity of F implies the continuity of f. So also we assume

(c) f **is continuous**.

There is a vast literature on the *comparison* of various hypertopologies with one another or similar hypertopologies corresponding to different parameters (e.g. different metrics or uniformities) on the same space. These are special cases of our results when f is the identity function and F is the associated function from $CL(X) \to CL(X)$ with different hypertopologies on the domain and the range.

We give the standard notations for concepts in X; similar ones apply to Y. For each subset E of X, $cl_X E$, $int E$ and E^c stand for the closure, interior and complement of E in X. The symbol γ_0 denotes the fine or Wallman LO-proximity on X given by

$A \gamma_0 B$ if, and only if, $cl A \cap cl B \neq \emptyset$.

The symbol $\gamma^\sharp$ denotes the fine EF-proximity on a Tychonoff space X given by

$A \not\gamma^\sharp B$ if, and only if, there is a continuous function f on X to $[0,1]$ with $f(A) = 0$ and $f(B) = 1$.

For $E \subset X$ and a LO-proximity γ, $\gamma(E) = \{F \subset X : E\gamma F\}$.

$K(X) =$ the family of all non-empty compact subsets of X.

Λ denotes a non-empty subfamily of $CL(X)$ which is closed under finite unions, and contains all singletons. We call Λ a **cobase**. Important examples of cobases are $CL(X)$, $K(X)$, $Z(X)$ (the family of zero sets) and $F(X)$ (the family of finite sets). Given any family of non-empty closed sets, we can get a cobase by addition of singletons and finite unions of its members to the family.

For any set $E \subset X$, $\mathcal{E} \subset \mathcal{T}$ and a LO-proximity γ on X we use the following notation:

$E^- = \{A \in CL(X) : A \cap E \neq \emptyset\}$;
$\mathcal{E}^- = \{A \in CL(X) : A \cap E \neq \emptyset$ for each $E \in \mathcal{E}\}$;
$E^{++}_\gamma = \{A \in CL(X) : A \ll_\gamma E$, i.e. $A \not\gamma E^c\}$;
$E^+ = \{A \in CL(X) : A \subset E$, i.e. $A \ll_{\gamma_0} E\}$.

The *upper proximal Delta-topology* (w.r.t. γ) $\sigma(\gamma,\Lambda)^+$ is generated by the basis $\{E^{++}_\gamma : E^c \in \Lambda\}$.

The *upper Delta-topology* $\tau(\Lambda)^+ = \sigma(\gamma_0,\Lambda)^+$.

The *upper* **U**-*topology* $\tau(U)^+$ has as a basis $\{E^+$, where either $E^c \in K(X)$ or $clE \in K(X)\}$ (cf. [2]).

The *upper far-miss ball topology* $\tau(FM)^+$ has a local basis at $A \in CL(X)$, $\{E^+ : E \in \mathcal{B}_d$ and $A \in E^{++}\}$ and it coincides with the *upper Wijman topology* $\tau(W_d)^+$ (cf. [8]).

The *lower Vietoris (or finite) topology* $\tau(V^-)$ has a basis $\{\mathcal{E}^- : \mathcal{E} \subset \mathcal{T}$ is finite$\}$.

The *lower locally finite topology* $\tau(LF^-)$ has a basis $\{\mathcal{E}^- : \mathcal{E} \subset \mathcal{T}$ is locally finite$\}$.

The $\mathbb{L}$-*lower locally finite topology* $\tau(\mathbb{L}^-)$ has a basis $\{\mathcal{E}^- : \mathcal{E} \in \mathbb{L}\}$, where $\mathbb{L} \subset \{\mathcal{E}^- : \mathcal{E} \subset \mathcal{T}$ is locally finite$\}$ satisfies a simple *filter* condition, i.e. given $\mathcal{E},\mathcal{F} \in \mathbb{L}$ there is a $\mathcal{G} \in \mathbb{L}$ such that $\mathcal{G}^- \subset \mathcal{E}^- \cap \mathcal{F}^-$.

Let $(X,\mathcal{T})$ a Tychonoff space, $\mathcal{U}$ a compatible uniformity which induces the EF-proximity γ. Then $\mathbb{L}\mathcal{U}$ is the collection of families of open sets of the form $\{U(x) : x \in Q \subset A\}$, where $A \in CL(X)$, $U \in \mathcal{U}$ and Q is U-discrete (note that each member of $\mathbb{L}\mathcal{U}$ is discrete and hence locally finite).

The $\mathbb{LU}$*-lower locally finite topology* $\tau(\mathbb{LU})^-)$ has a basis $\{\mathcal{E}^- : \mathcal{E} \subset \mathbb{LU}\}$ (cf. [8]).

The *proximal (finite) Delta-topology* (w.r.t. γ) $\sigma(\gamma, \Lambda) = \sigma(\gamma, \Lambda)^+ \vee \tau(V^-)$.

We omit γ if it is obvious from the context and write $\sigma(\Lambda)$ for $\sigma(\gamma, \Lambda)$.

The *Delta-topology* $\tau(\Lambda) = \tau(\Lambda)^+ \vee \tau(V^-)$.

The **U**-*topology* $\tau(U) = \tau(U)^+ \vee \tau(V^-)$.

The *proximal locally finite Delta-topology* $\sigma(LF, \gamma, \Lambda) = \sigma(\gamma, \Lambda)^+ \vee \tau(LF^-)$.

The *proximal* $\mathbb{L}$*-locally finite Delta-topology* $\sigma(\mathbb{L}, \gamma, \Lambda) = \sigma(\gamma, \Lambda)^+ \vee \tau(\mathbb{L}^-)$.

We omit the prefix *"proximal"* and replace σ by τ if $\gamma = \gamma_0$.

Well known special cases are:

(a) when $\Lambda = CL(X)$,

$\tau(\Lambda) = \tau(V)$ the *Vietoris or finite topology*;

$\sigma(\gamma, \Lambda) = \sigma(\Lambda)$ the *proximal topology*;

$\tau(LF, \Lambda) = \tau(LF)$ the *locally finite topology*;

$\sigma(LF, \gamma, \Lambda) = \sigma(LF, \gamma)$ the *proximal locally finite topology.*

$\sigma(\mathbb{LU})$ the *proximal* $\mathbb{LU}$*-locally finite topology* and coincides with the *Hausdorff uniformity* $\tau(H_{\mathcal{U}})$ (cf. [8]). If X is equipped with a metric d and $\mathcal{U}$ is the metric uniformity associated with d, then $\tau(H_{\mathcal{U}})$ is the *Hausdorff metric topology* $\tau(H_d)$ (cf. [8]).

(b) When $\Lambda = K(X)$ and γ is EF,

$\tau(\Lambda) = \tau(F) = \sigma(F)$ the *Fell topology.*

(c) If (X, d) is a metric space, γ is the metric proximity induced by d and Λ denotes the family $\mathcal{B}_d$ of finite unions of closed d-balls of non-negative radii, then

$\tau(\Lambda) = \tau(\mathcal{B}_d)$ the *Ball topology*;

$\sigma(\Lambda) = \sigma(\mathcal{B}_d)$ the *proximal Ball topology.*

$\tau(FM)$ the *hit-far-miss ball topology* which coincides with the *Wijsman topology* $\tau(W_d)$ (cf. [8]).

The *upper Bombay topology* $\sigma(\gamma_1, \gamma_2, \Lambda)^+$ has two proximal parameters γ_1, γ_2, with $\gamma_1 \le \gamma_2$, and cobase Λ. The neighbourhoods of $A_0 \in CL(X)$ are

$$\{E^{++}_{\gamma_2} : E^c \in \Lambda \quad \text{and} \quad A_0 \not\gamma_1 E^c\}.$$

The $\mathbb{L}$-*locally finite Bombay topology* (with parameters γ_1, γ_2, Λ) is $\sigma(\mathbb{L}, \gamma_1, \gamma_2, \Lambda) = \sigma(\gamma_1, \gamma_2, \Lambda)^+ \vee \tau(\mathbb{L}^-)$.

It was shown in [3] that all known hypertopologies are special cases of the above.

When $\mathbb{L}$ consists of finite families, we have the *finite Bombay topology*.

(i) $\tau(F) \subset \tau(U) \subset \sigma(\gamma) \subset \tau(V) \subset \tau(LF)$.
(ii) $\tau(F) \subset \tau(U) \subset \sigma(\gamma) \subset \tau(H_d) \subset \tau(LF)$.
(iii) $\tau(F) \subset \tau(W_d) \subset \sigma(\mathcal{B}_d) \subset \sigma(\gamma) \subset \tau(V) \subset \tau(LF)$.
(iv) $\tau(F) \subset \tau(W_d) \subset \tau(\mathcal{B}_d) \subset \sigma(\gamma) \subset \tau(H_d) \subset \tau(LF)$.

A *proper subset* is a non-empty subset which is not equal to the whole space.

Definition 1.1. Let X, Y be Hausdorff spaces with LO-proximities γ, α. A function $f\colon (X, \gamma) \to (Y, \alpha)$ is *p-continuous* if for each A, $B \subset Y$, $A \not\alpha B$ implies $f^{-1}(A) \not\gamma f^{-1}(B)$ (cf. [5], [10]).

Definition 1.2. Let X, Y Hausdorff spaces, γ, α LO-proximities and Λ, Δ cobases of X and Y respectively.

A function $f\colon (X, \gamma) \to (Y, \alpha)$ is Δ-Λ-*p-continuous* if for each $B \in \Delta$ with $B \ll_\alpha W \neq Y$, W open, there exists a $B' \in \Lambda$ such that $f^{-1}(B) \subset B' \ll_\gamma f^{-1}(W)$ (note that if $\Delta = CL(Y)$, $\Lambda = CL(X)$, then Δ-Λ-p continuity coincides with p-continuity).

Special cases are:

(a) f is *closed p-continuous* if for each pair of disjoint closed sets A, B of Y, $f^{-1}(A) \not\gamma f^{-1}(B)$.
(b) f is *compactly p-continuous* if for each compact set K in Y disjoint from a closed set S in Y, $f^{-1}(K) \not\gamma f^{-1}(S)$.
(c) f is *p-ball-p-continuous* if for each e-ball B in Y and $S \in CL(Y)$, $B \not\alpha S$ implies $f^{-1}(B) \not\gamma f^{-1}(S)$.
(d) f is *ball-p-continuous* if for each e-ball B in Y and $S \in CL(Y)$, $B \cap S = \emptyset$ implies $f^{-1}(B) \not\gamma f^{-1}(S)$.
(e) f is *strongly compact* if for each proper compact set $K \subset Y$, $f^{-1}(K)$ is compact.

Let (X, d) be a metric space and γ the metric proximity induced by d. We recall that a subset E of X is said to be *weakly totally bounded* or *w-TB* in an open set W if there exists a $B \in \mathcal{B}_d$ such that $E \ll_\gamma B \ll_\gamma W$ (cf. [4], [3] and [1] where the term *strictly d-included* is adopted).

Definition 1.3. Let X, Y be metrizable spaces with metrics d, e respectively, γ the metric proximity induced by d, Δ a cobase on $CL(Y)$ and f a function from X into Y.

(1) f is *Δ-weakly totally bounded* if for each $B \in \Delta$ and each proper open set W in Y, with $B \subset W$, $f^{-1}(B)$ is weakly totally bounded in $f^{-1}(W)$.

Special cases are:
(a) when $\Delta = CL(Y)$, f is *closed-weakly totally bounded*.
(b) When $\Delta = K(Y)$, f is *compactly-weakly totally bounded*.
(c) When $\Delta = \mathcal{B}_e$, f is *ball-weakly totally bounded*.

(2) f is *ε-Δ-weakly totally bounded* if for each proper $B \in \Delta$ and each $\varepsilon > 0$, $f^{-1}(B)$ is weakly totally bounded in $f^{-1}(S_e(B, \varepsilon))$.

Special cases are:
(a) when $\Delta = CL(Y)$, f is *ε-closed-weakly totally bounded*.
(b) When $\Delta = K(Y)$, f is *ε-compactly-weakly totally bounded*.
(c) When $\Delta = \mathcal{B}_e$, f is *ε-ball-weakly totally bounded*.

Definition 1.4. Let X, Y be metrizable spaces with metrics d, e respectively, $\mathcal{B}_d$ the family of all finite unions of closed proper d-balls on X, Δ a cobase on $CL(Y)$ and f a function from X into Y.

(1) f is *bounded on* Δ if for each $B \in \Delta$ and each proper open set W in Y, with $B \subset W$, there is a $B' \in \mathcal{B}_d$ such that $f^{-1}(B) \subset B' \subset f^{-1}(W)$.

Special cases are:
(a) when $\Delta = CL(Y)$, f is *bounded on closed sets*.
(b) When $\Delta = K(Y)$, f is *bounded on compact sets*.
(c) When $\Delta = \mathcal{B}_e$, f is *bounded on closed balls*.

(2) f is *ε-bounded on* Δ if for each proper $B \in \Delta$ and each $\varepsilon > 0$, there is a $B' \in \mathcal{B}_d$ such that $f^{-1}(B) \subset B' \subset f^{-1}(S_e(B, \varepsilon))$.

Special cases are:

(a) when $\Delta = CL(Y)$, f is *ε-bounded on closed sets*.
(b) When $\Delta = K(Y)$, f is *ε-bounded on compact sets*.
(c) When $\Delta = \mathcal{B}_e$ f is *ε-bounded on closed balls*.

Standard references are [1] for hyperspaces, [5], [10], [11], [13] for proximities. For recent publications see [3].

2. Finite Bombay topologies

We recall that $f\colon X \to Y$ is a continuous surjection. The following Theorem plays a key role.

Theorem 2.1. (cf. [3, Theorem 3.5]) *Let $CL(X)$, $CL(Y)$ be assigned the upper Bombay topologies $\sigma(\gamma_1, \gamma_2, \Lambda)^+$, $\sigma'(\alpha_1, \alpha_2, \Delta)^+$ respectively. The following are equivalent:*

(a) *$F\colon CL(X) \to CL(Y)$ is continuous;*
(b) *for each $B \in \Delta$ with $B \ll_{\alpha_1} W \neq Y$, W open, there exists a $B' \in \Lambda$ such that*
  (i) $f^{-1}(B) \subset B' \ll_{\gamma_1} f^{-1}(W)$
  i.e. $f\colon (X, \gamma_1) \to (Y, \alpha_1)$ is Δ-Λ-p-continuous (cf. Definition 1.2)*, and*
  (ii) $f^{-1}[\alpha_2(B)] \subset \gamma_2(B')$
  i.e. $f\colon (X, \gamma_2) \to (Y, \alpha_2)$ is a **regular map** (see [6])*.*

Proof. Suppose $A_0 \in CL(X)$ and $F(A_0) \in U^{++}_{\alpha_2}$, where $B = U^c \in \Delta$ and $F(A_0) \not\delta_1 U^c$. We may assume that $W = [F(A_0)]^c \neq \emptyset$. We note that that F is continuous at A_0 if, and only if, F is continuous at $A_1 = f^{-1}(cl f(A_0)) \in CL(X)$ if, and only if, there is a $B' = V^c \in \Lambda$ such that $A_1 \in V^{++}_{\gamma_2} \subset f^{-1}(U^{++}_{\alpha_2})$ and $A_1 \not\eta_1 V^c$. These conditions are equivalent to (i) and (ii). □

The results given below follow easily from Theorem 2.1 and so we omit the proofs.

Theorem 2.2. (cf. [3, Remarks 4.1(f)]) *Let X, Y be metrizable spaces with metrics d, e respectively. Let $CL(X)$, $CL(Y)$ be assigned the Wijsman topologies $\tau(W_d)$, $\tau(W_e)$ respectively. The following are equivalent:*

(a) *$F\colon CL(X) \to CL(Y)$ is continuous;*

(b) *f is ε-ball-weakly totally bounded* (cf. 2.(c) in (Definition 1.3).

Theorem 2.3. *Let X, Y be Hausdorff spaces with LO-proximities γ, α and cobases Λ, Δ respectively. Let $CL(X)$, $CL(Y)$ be assigned the proximal Delta-topologies $\sigma(\gamma, \Lambda)$, $\sigma'(\alpha, \Delta)$ respectively. The following are equivalent:*

(a) *$F\colon CL(X) \to CL(Y)$ is continuous;*
(b) *for each $B \in \Delta$ with $B \ll_\alpha W \neq Y$, W open, there exists a $B' \in \Lambda$ such that*
 (i) *$f^{-1}(B) \subset B' \ll_\gamma f^{-1}(W)$*
 i.e. $f\colon (X, \gamma) \to (Y, \alpha)$ is Δ-Λ-p-continuous, and
 (ii) *$f^{-1}[\alpha(B)] \subset \gamma(B')$*
 i.e. $f\colon (X, \gamma) \to (Y, \alpha)$ is regular.

Corollary 2.4. *Let X, Y be Hausdorff spaces with LO-proximities γ, α and $CL(X)$, $CL(Y)$ be assigned the proximal topologies $\sigma(\gamma)$, $\sigma'(\alpha)$. The following are equivalent:*

(a) *$F\colon CL(X) \to CL(Y)$ is continuous;*
(b) *f is p-continuous and regular.*

Theorem 2.5. *Let X, Y be Hausdorff spaces with cobases Λ, Δ respectively. Let $CL(X)$, $CL(Y)$ be assigned the Delta-topologies $\tau(\Lambda)$, $\tau'(\Delta)$ respectively. The following are equivalent:*

(a) *$F\colon CL(X) \to CL(Y)$ is continuous;*
(b) *$B \in \Delta$ with $B \subset W \neq Y$, W open, there exists a $B' \in \Lambda$ such that $f^{-1}(B) \subset B' \subset f^{-1}(W)$.*

From Theorem 2.5 we obtain

Corollary 2.6. *Let X, Y be Hausdorff spaces and let $CL(X)$, $CL(Y)$ be assigned the Fell topologies. The following are equivalent:*

(a) *$F\colon CL(X) \to CL(Y)$ is continuous;*
(b) *f is strongly compact* (cf. Definition 1.2 (e)).

Corollary 2.7. *Let X, Y be Hausdorff spaces and let $CL(X)$, $CL(Y)$ be assigned the Vietoris topologies. Then $F\colon CL(X) \to CL(Y)$ is always continuous* (see (c) in Introduction).

3. Locally finite Bombay topologies

Definition 3.1 (Filter condition $(\star)$)**.** Let $(X,\mathcal{T})$ be a Hausdorff topological space and $\mathbb{L} \subset \{\mathcal{E} \subset \mathcal{T} : \mathcal{E} \text{ is locally finite}\}$. The family $\mathbb{L}$ satisfy the filter condition $(\star)$ if whenever $\mathcal{P}$, $\mathcal{Q} \in \mathbb{L}$, there is $\mathcal{R} \in \mathbb{L}$ such that $\mathcal{R}^- \subset \mathcal{P}^- \cap \mathcal{Q}^-$.

We start with the following useful Lemma and Corollary.

Lemma 3.2. *Let $(X,\mathcal{T})$, $(Y,\mathcal{T}')$ be Hausdorff topological spaces, $f\colon X \to Y$ a continuous surjection, $\mathbb{L} \subset \{\mathcal{E} \subset \mathcal{T} : \mathcal{E}$ is locally finite$\}$, $\mathbb{M} \subset \{\mathcal{G} \subset \mathcal{T}' : \mathcal{G}$ is locally finite$\}$, satisfying the filter condition $(\star)$. The following are equivalent:*

(a) *$F\colon (CL(X), \tau(\mathbb{L}^-)) \to (CL(Y), \tau'(\mathbb{M}^-))$ is continuous;*
(b) *for each $\mathcal{G} \in \mathbb{M}$, there is an $\mathcal{E} \in \mathbb{L}$ such that $\mathcal{E}^- \subset f^{-1}(\mathcal{G}^-)$;*
(c) *for each $A \in CL(X)$ and for each $\mathcal{G} \in \mathbb{M}$ with $cl f(A) \in \mathcal{G}^-$, there is $\mathcal{E} \in \mathbb{L}$ such that $A \in \mathcal{E}^- \subset f^{-1}(\mathcal{G}^-)$.*

Proof. It is trivial. □

Remark 3.3. Note that condition (b) in Lemma 3.2 can be viewed, in hyperspace theory, as a kind of semicontinuity of $f\colon X \to Y$ with respect to the lower parts of locally finite families on X and Y.

Corollary 3.4. *Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be separated uniform spaces, let $\mathcal{U}$, $\mathcal{V}$ induce the topologies $\mathcal{T}$, $\mathcal{T}'$ on X, Y respectively, $f\colon X \to Y$ a continuous surjection, $\mathbb{M} \subset \{\mathcal{G} \subset \mathcal{T}' : \mathcal{G}$ is locally finite$\}$, satisfying the filter condition $(\star)$. The following are equivalent:*

(a) *$F\colon (CL(X), \tau(V^-)) \to (CL(Y), \tau'(\mathbb{M}^-))$ is continuous;*
(b) *Y is totally bounded.*

Proof. It suffices to show $(a) \Rightarrow (b)$. Assume not. Then, there exists a sequence $\{y_n\}$ of distinct points of Y wich is W-discrete for some $W \in \mathcal{V}$. Let $V \in \mathcal{V}$ be such that $V^3 \subset W$ and $K = \bigcup_{n\in\mathbb{N}} clV[y_n]$. It follows that $A^- = f^{-1}(K^{-1})$ is a discrete family of X which does not fulfill (b) in Lemma 3.2. □

Let X, Y be Hausdorff spaces and let $CL(X)$, $CL(Y)$ be assigned the locally finite Bombay topologies $\sigma(\mathbb{L},\gamma_1,\gamma_2,\Lambda) = \sigma(\gamma_1,\gamma_2,\Lambda)^+ \vee \tau(\mathbb{L}^-)$ and $\sigma'(\mathbb{M},\alpha_1,\alpha_2,\Delta) = \sigma'(\alpha_1,\alpha_2,\Delta)^+ \vee \tau'(\mathbb{M}^-)$, respectively.

It is well known that the upper and lower hypertopologies act separately. So, $F\colon CL(X)\to CL(Y)$ is continuous if, and only if, it is continuous w.r.t. the upper and lower parts separately.

Hence we have the following:

Theorem 3.5 (Main Theorem)**.** *Let $(X,\mathcal{T})$, $(Y,\mathcal{T}')$ be Hausdorff topological spaces, γ_1, γ_2, α_1, α_2 compatible LO-proximities on X and Y respectively with $\gamma_1\le\gamma_2$, $\alpha_1\le\alpha_2$, $f\colon X\to Y$ a continuous surjection, $\mathbb{L}\subset\{\mathcal{E}\subset\mathcal{T}:\mathcal{E}$ is locally finite$\}$, $\mathbb{M}\subset\{\mathcal{G}\subset\mathcal{T}':\mathcal{G}$ is locally finite$\}$, satisfying the filter condition $(\star)$ and Λ, Δ cobases. Let $CL(X)$, $CL(Y)$ be assigned the locally finite Bombay topologies $\sigma(\mathbb{L},\gamma_1,\gamma_2,\Lambda)=\sigma(\gamma_1,\gamma_2,\Lambda)^+\vee\tau(\mathbb{L}^-)$ and $\sigma'(\mathbb{M},\alpha_1,\alpha_2,\Delta)=\sigma'(\alpha_1,\alpha_2,\Delta)^+\vee\tau'(\mathbb{M}^-)$, respectively. The following are equivalent:*

(a) *$F\colon CL(X)\to CL(Y)$ is continuous;*

(b) (1) *for each $B\in\Delta$ with $B\ll_{\alpha_1}W\neq Y$, W open, there exists a $B'\in\Lambda$ such that*
  - (i) *$f^{-1}(B)\subset B'\ll_{\gamma_1}f^{-1}(W)$ i.e. $f\colon(X,\gamma_1)\to(Y,\alpha_1)$ is Δ-Λ-p-continuous, and*
  - (ii) *$f^{-1}[\alpha_2(B)]\subset\gamma_2(B')$, i.e. $f\colon(X,\gamma_2)\to(Y,\alpha_2)$ is regular.*

  (2) *for each $\mathcal{E}\in\mathbb{M}$, there is an $\mathcal{F}\in\mathbb{L}$ such that $\mathcal{F}^-\subset f^{-1}(\mathcal{E}^-)$, i.e. f is lower semicontinuous w.r.t. $\mathbb{L}$ and $\mathbb{M}$.*

Corollary 3.6. *If on $CL(X)$ and $CL(Y)$ are assigned the finite Bombay topology $\sigma(\gamma_1,\gamma_2,\Lambda)=\sigma(\gamma_1,\gamma_2,\Lambda)^+\vee\tau(V^-1)$ and the locally finite Bombay topology $\sigma'(\mathbb{M},\alpha_1,\alpha_2,\Delta)=\sigma'(\gamma_1,\gamma_2,\Delta)^+\vee\tau'(\mathbb{M}^-)$ respectively. The following are equivalent:*

(a) *$F\colon CL(X)\to CL(Y)$ is continuous;*

(b) (1) *for each $B\in\Delta$ with $B\ll_{\alpha_1}W\neq Y$, W open, there exists a $B'\in\Lambda$ such that*
  - (i) *$f^{-1}(B)\subset B'\ll_{\gamma_1}f^{-1}(W)$ i.e. $f\colon(X,\gamma_1)\to(Y,\alpha_1)$ is Δ-Λ-p-continuous, and*
  - (ii) *$f^{-1}[\alpha_2(B)]\subset\gamma_2(B')$, i.e. $f\colon(X,\gamma_2)\to(Y,\alpha_2)$ is regular.*

  (2) *Y is totally bounded.*

Uniform case:

Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be separated uniform spaces, let $\mathcal{U}$, $\mathcal{V}$ induce EF-proximities γ, α which, in turn, induce the topologies $\mathcal{T}$, $\mathcal{T}'$ on X, Y respectively. Let Λ, Δ be cobases which are closed hereditary. Let $\mathbb{L}\Lambda$ be the collection of all families of open sets of the form

$\{U(x) : x \in Q \subset A\}$, where $A \in \Lambda$, $U \in \mathcal{U}$ and Q is U-discrete. We define $\mathbb{L}\Delta$ similarly using $\mathcal{V}$.

The *Hausdorff-Bourbaki or H-B topology* on $CL(X)$ (*associated with* Λ) is defined by $\sigma(\mathbb{L}\Lambda,\gamma) = \sigma(\gamma)^+ \vee \tau(\mathbb{L}\Lambda^-)$ (see also section 1).

Theorem 3.7. *Let $CL(X)$, $CL(Y)$ be assigned the topologies $\sigma(\mathbb{L}\Lambda,\gamma)$, $\sigma'(\mathbb{L}\Delta,\alpha)$ respectively. The following are equivalent:*

(a) *$F\colon CL(X) \to CL(Y)$ is continuous;*

(b) (1) *for each $B \in \Delta$ with $B \ll_\alpha W \neq Y$, W open, there exists a $B' \in \Lambda$ such that*

(i) *$f^{-1}(B) \subset B' \ll_\gamma f^{-1}(W)$*
i.e. $f\colon (X,\gamma) \to (Y,\alpha)$ is Δ-Λ-p-continuous, and

(ii) *$f^{-1}[\alpha(B)] \subset \gamma(B')$,*
i.e. $f\colon (X,\gamma) \to (Y,\alpha)$ is regular.

(2) *for each $\mathcal{E} \in \mathbb{L}\Delta$, there is an $\mathcal{F} \in \mathbb{L}\Lambda$ such that $\mathcal{F}^- \subset f^{-1}(\mathcal{E}^-)$, i.e. f is lower semicontinuous w.r.t. $\mathbb{L}\Delta$ and $\mathbb{L}\Lambda$.*

Corollary 3.8. ([12], [9]) *Let X and Y be normal spaces. Then the locally finite topologies on $CL(X)$, $CL(Y)$ are induced by the fine uniformities on X and Y respectively. Thus the map $F\colon (CL(X),\tau(LF)) \to (CL(Y),\tau'(LF))$ is always continuous* (see (c) in Introduction).

Metric case:

Let X and Y be metrizable spaces with compatible metrics d, e, $\mathcal{U}$, $\mathcal{V}$ the metric uniformities generated by d, e, and γ, α be the metric proximities. Let Λ, Δ be the families of all non-empty closed metrically bounded subsets of (X,d), (Y,e). Then $\sigma(\mathbb{L}\Lambda,\gamma)$, $\sigma(\mathbb{L}\Delta,\alpha)$ are precisely the *Attouch-Wets or AW-topologies* $\tau(AW_d)$ on $CL(X)$, $\tau'(AW_e)$ on $CL(Y)$ respectively. We note that the metric uniformity is the finest member of the metric proximity class [10, Page 77] and hence uniform continuity and proximal continuity coincide. Thus we have the following:

Theorem 3.9. (cf. [1, Theorem (3.3.3) Page 93]) *Let* (X,d), (Y,e) *be metric spaces and* $CL(X)$, $CL(Y)$ *be assigned the AW-topologies* $\tau(AH_d)$, $\tau'(AH_e)$ *respectively. The following are equivalent:*

(a) $F\colon (CL(X),\tau(AW_d)) \to (CL(Y),\tau'(AW_e))$ *is continuous;*
(b) (1) f^{-1} *preserves bounded sets, and*
  (2) f *is uniformly continuous on bounded subsets of* X.

Corollary 3.10. *Let* (X,d), (Y,e) *be metric spaces and* $CL(X)$, $CL(Y)$ *be assigned the Hausdorff metric topologies* $\tau(H_d)$, $\tau'(H_e)$ *respectively. Then* $F\colon CL(X) \to CL(Y)$ *is (uniformly) continuous if, and only if,* f *is uniformly continuous.*

4. Different hypertopologies; Fell domain

In the previous sections we studied the same type of hypertopology on both $CL(X)$ and $CL(Y)$. Now, beginning with this section, we study the problem when the hyperpaces are assigned different hypertopologies. Most of the results follow easily from the main results 2.1, 3.5.

In this section, we study the continuity of $F\colon CL(X) \to CL(Y)$ when $CL(X)$ is assigned the Fell topology $\tau(F)$ and $CL(Y)$ is assigned a different hypertopology. From Corollary 2.6 it follows that $f\colon X \to Y$ must be continuous and strongly compact.

We note that in this case, $\Lambda = K(X)$, $\Delta = K(Y)$ and 2.1 $(b)(i)$ is equivalent to $f^{-1}(\Delta) \subset \Lambda$.

Theorem 4.1. (Fell $\to$ **U**)
Let X, Y *be Hausdorff spaces. The following are equivalent:*

(a) $F\colon (CL(X),\tau(F)) \to (CL(Y),\tau'(U))$ *is continuous;*
(b) (1) f *strongly compact, and*
  (2) *either* (i) X, Y *are both compact or* (ii) Y *has no open set with compact closure.*

Proof. The first case is the existence of an open set U of Y with clU compact. Then if $A \in [f^{-1}(U^+)]$, then there is an open set V in X such that V^c is compact and $A \subset V \subset f^{-1}(U)$. This shows that $X = V^c \cup f^{-1}(clU)$ is compact.

The second case is that Y has no open set with compact closure and so $\tau'(F) = \tau'(U)$ and the result follows from 2.6. □

We recall that a metric space (Y,e) has *nice closed balls* if each its proper closed balls is compact ([1]).

Theorem 4.2. (Fell → Wijsman, proximal Ball, Ball)
Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(F)) \to (CL(Y),\tau'(W_e))$ *is continuous;*
(b) $F\colon (CL(X),\tau(F)) \to (CL(Y),\sigma'(\alpha,\mathcal{B}_e))$ *is continuous;*
(c) $F\colon (CL(X),\tau(F)) \to (CL(Y),\tau'(\mathcal{B}_e))$ *is continuous;*
(d) (1) *f is strongly compact, and*
(2) *Y has nice closed balls.*

Proof. The result follows from the fact that $f^{-1}(B)$ is compact for each proper closed ball in Y and so B itself is compact. □

Using Theorem 2.1 and Theorem 3.5 we have

Theorem 4.3. (Fell → Proximal, Vietoris, Hausdorff metric, Locally finite, proximal Locally finite)
Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(F)) \to (CL(Y),\sigma'(\alpha))$ *is continuous;*
(b) $F\colon (CL(X),\tau(F)) \to (CL(Y),\tau'(V))$ *is continuous;*
(c) $F\colon (CL(X),\tau(F)) \to (CL(Y),\tau'(H_e))$ *is continuous;*
(d) $F\colon (CL(X),\tau(F)) \to (CL(Y),\tau'(LF))$ *is continuous;*
(e) $F\colon (CL(X),\tau(F)) \to (CL(Y),\sigma'(LF,\alpha))$ *is continuous;*
(f) (1) *f is strongly compact, and*
(2) *X, Y are both compact.*

5. U domain

Definition 5.1. A family $\Lambda \subset CL(X)$ is a **U**-*family* if for each $B \in \Lambda$ either B or clB^c is compact.

Theorem 5.2. (**U** → Fell)
Let X, Y be Hausdorff spaces. The following are equivalent:

(a) $F\colon (CL(X),\tau(U)) \to (CL(Y),\tau'(F))$ *is continuous;*
(b) *for each proper compact set $K \subset Y$, either $f^{-1}(K)$ is compact or $clf^{-1}(K^c)$ is compact, i.e. the family of inverse images of all the compact subsets K in Y is a* **U**-*family.*

Theorem 5.3. (**U** $\to$ Wijsman, proximal Ball, Ball)
Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(U)) \to (CL(Y),\tau'(W_e))$ *is continuous;*
(b) $F\colon (CL(X),\tau(U)) \to (CL(Y),\sigma'(\alpha,\mathcal{B}_e))$ *is continuous;*
(c) $F\colon (CL(X),\tau(U)) \to (CL(Y),\tau'(\mathcal{B}_e))$ *is continuous;*
(d) (1) *f is strongly compact, and*
(2) *each proper $B \in \mathcal{B}_e$ is either compact or clB^c is compact, i.e. the family of all proper closed e-balls B in Y is a **U**-family.*

Theorem 5.4. (**U** $\to$ Proximal, Vietoris)
Let (X,γ), (Y,α) be EF-proximity spaces. The following are equivalent:

(a) $F\colon (CL(X),\tau(U)) \to (CL(Y),\sigma'(\alpha))$ *is continuous;*
(b) $F\colon (CL(X),\tau(U)) \to (CL(Y),\tau'(V))$ *is continuous;*
(c) (1) *f is continuous and strongly compact, and*
(2) *each proper open set W in Y satisfies W^c is compact or clW is compact, i.e. the family of all proper closed subsets S in Y is a **U**-family.*

Theorem 5.5. (**U** $\to$ Hausdorff metric)
Let (X,d), (Y,e) be metric spaces. The following are equivalent:

(a) $F\colon (CL(X),\tau(U)) \to (CL(Y),\tau'(H_e))$ *is continuous;*
(b) (1) *f is strongly compact and regular,*
(2) *the family of all proper closed subsets S in Y is a **U**-family, and*
(3) *Y is totally bounded.*

Theorem 5.6. (**U** $\to$ Locally finite, Proximal locally finite)
Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(U)) \to (CL(Y),\tau'(LF))$ *is continuous;*
(b) $F\colon (CL(X),\tau(U)) \to (CL(Y),\sigma'(LF,\alpha))$ *is continuous;*
(c) (1) *f is strongly compact,*
(2) *the family of all proper closed subsets S in Y is a **U**-family, and*
(3) *Y is totally bounded.*

6. Wijsman domain

We recall, again, that if (X, d) is a metric space and γ is the metric proximity induced by d, then a subset E of X is *weakly totally bounded* in an open set W if there exists a $B \in \mathcal{B}_d$ such that $E \ll_\gamma B \ll_\gamma W$ (cf. Introduction).

Theorem 6.1. (Wijsman → Fell)
Let (X, d), (Y, e) be metric spaces. The following are equivalent:
(a) $F\colon (CL(X), \tau(W_d)) \to (CL(Y), \tau'(F))$ *is continuous;*
(b) f *is compactly-weakly totally bounded* (cf. 1.(b) in Definition 1.3).

Theorem 6.2. (Wijsman → **U**)
Let (X, d), (Y, e) be metric spaces. The following are equivalent:
(a) $F\colon (CL(X), \tau(W_d)) \to (CL(Y), \tau'(U))$ *is continuous;*
(b) (1) f *is compactly-weakly totally bounded, and*
(2) *for each open subsets U, V in Y with $clU \subset V$ and clU compact, $f^{-1}(U)$ is weakly totally bounded in $f^{-1}(V)$.*

Theorem 6.3. (Wijsmann → proximal Ball)
Let (X, d), (Y, e) be metric spaces, α the metric proximity induced by e. The following are equivalent:
(a) $F\colon (CL(X), \tau(W_d)) \to (CL(Y), \sigma'(\alpha, \mathcal{B}_e))$ *is continuous;*
(b) f *is ε-ball-weakly totally bounded* (cf. 2.(c) in Definition 1.3).

Theorem 6.4. (Wijsman → Ball)
Let (X, d), (Y, e) be metric spaces. The following are equivalent:
(a) $F\colon (CL(X), \tau(W_e)) \to (CL(Y), \tau'(\mathcal{B}_e))$ *is continuous;*
(b) f *is ball-weakly totally bounded* (cf. 1.(c) in Definition 1.3).

Theorem 6.5. (Wijsman → Proximal)
Let (X, d), (Y, e) be metric spaces, α the metric proximity induced by e. The following are equivalent:
(a) $F\colon (CL(X), \tau(W_d)) \to (CL(Y), \sigma'(\alpha))$ *is continuous;*
(b) f *is ε-closed-weakly totally bounded* (cf. 2.(a) in Definition 1.3).

Theorem 6.6. (Wijsman → Vietoris)
Let (X, d), (Y, e) be metric spaces. The following are equivalent:
(a) $F\colon (CL(X), \tau(W_d)) \to (CL(Y), \tau'(V))$ *is continuous;*

(b) *f is closed-weakly totally bounded* (cf. 1.(a) in Definition 1.3).

Theorem 6.7. (Wijsman $\to$ Hausdorff metric, Proximally locally finite)

Let (X, d), (Y, e) be metric spaces, α the metric induced by e. The following are equivalent:

(a) $F\colon (CL(X), \tau(W(d))) \to (CL(Y), \tau'(H_e))$ *is continuous;*

(b) $F\colon (CL(X), \tau(W(d))) \to (CL(Y), \sigma'(LF, \alpha))$ *is continuous;*

(c) (1) *f is ε-closed-weakly totally bounded, and*

(2) *Y is totally bounded.*

Theorem 6.8. (Wijsman $\to$ Locally finite)

Let (X, d), (Y, e) be metric spaces. The following are equivalent:

(a) $F\colon (CL(X), \tau(W_d)) \to (CL(Y), \tau'(LF))$ *is continuous;*

(b) (1) *f is closed-weakly totally bounded, and*

(2) *Y is totally bounded.*

7. Proximal Ball domain

Theorem 7.1. (Proximal Ball $\to$ Fell)

Let (X, d), (Y, e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \tau'(F))$ *is continuous;*

(b) *f is compactly p-continuous* (cf. (b) in Definition 1.2).

Theorem 7.2. (Proximal Ball $\to$ **U**)

Let (X, d), (Y, e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \tau'(U))$ *is continuous;*

(b) (1) *f is compactly p-continuous, and*

(2) *for each open subsets U, V in Y with $clU \subset V$ and clU compact, $f^{-1}(U)$ is weakly totally bounded in $f^{-1}(V)$.*

Theorem 7.3. (Proximal Ball $\to$ Wijsman)

Let (X, d), (Y, e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \tau'(W(e)))$ *is continuous;*

(b) *f is ε-ball-weakly totally bounded* (cf. 2.(c) in Definition 1.3).

Theorem 7.4. (Proximal Ball → Ball)

Let (X, d), (Y, e) *be metric spaces,* γ *the metric proximity induced by* d*. The following are equivalent:*

(a) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \tau'(\mathcal{B}_e))$ *is continuous;*

(b) f *is ball p-continuous* (cf. (d) in Definition 1.2).

Theorem 7.5. (Proximal Ball → Proximal)

Let (X, d), (Y, e) *be metric spaces,* γ, α *the metric proximities induced by* d *and* e *respectively. The following are equivalent:*

(a) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \sigma'(\alpha))$ *is continuous;*

(b) f *is* ε*-closed-weakly totally bounded* (cf. 2.(a) in Definition 1.3).

Theorem 7.6. (Proximal Ball → Vietoris)

Let (X, d), (Y, e) *be metric spaces,* γ *the metric proximity induced by* d*. The following are equivalent:*

(a) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \tau'(V))$ *is continuous;*

(b) f *is closed-weakly totally bounded* (cf. 1.(a) in Definition 1.3).

Theorem 7.7. (Proximal Ball → Hausdorff metric, Proximal locally finite)

Let (X, d), (Y, e) *be metric spaces,* γ, α *the metric proximity induced by* d *and* e *respectively. The following are equivalent:*

(a) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \tau'(H_e))$ *is continuous;*

(b) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \sigma'(LF, \alpha))$ *is continuous;*

(c) (1) f *is* ε*-closed-weakly totally bounded, and*
(2) Y *is totally bounded.*

Theorem 7.8. (Proximal Ball → Locally finite)

Let (X, d), (Y, e) *be metric spaces,* γ *the metric proximity induced by* d*. The following are equivalent:*

(a) $F\colon (CL(X), \sigma(\gamma, \mathcal{B}_d)) \to (CL(Y), \tau'(LF))$ *is continuous;*

(b) (1) f *is closed-weakly totally bounded, and*
(2) Y *is totally bounded.*

8. Ball domain

Theorem 8.1. (Ball → Fell)

Let (X, d), (Y, e) *be metric spaces. The following are equivalent:*

(a) $F\colon (CL(X), \tau(\mathcal{B}_d)) \to (CL(Y), \tau'(F))$ *is continuous;*

(b) f *is bounded on compact sets* (cf. 1.(b) in Definition 1.4).

Theorem 8.2. (Ball $\to$ **U**)
Let (X,d), (Y,e) be metric spaces. The following are equivalent:

(a) $F\colon (CL(X),\tau(\mathcal{B}_d)) \to (CL(Y),\tau'(U))$ *is continuous;*
(b) (1) f *is bounded on compacts, and*
(2) *for each open subsets U, V in Y with $clU \subset V$ and clU compact, $f^{-1}(U)$ is weakly totally bounded in $f^{-1}(V)$.*

Theorem 8.3. (Ball $\to$ Wijsman, Proximal Ball)
Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(\mathcal{B}_d)) \to (CL(Y),\tau'(W_e))$ *is continuous;*
(b) $F\colon (CL(X),\tau(\mathcal{B}_d)) \to (CL(Y),\sigma'(\alpha,\mathcal{B}_e))$ *is continuous;*
(c) f *is ε-bounded on closed balls* (cf. 2.(c) in Definition 1.4).

Theorem 8.4. (Ball $\to$ Proximal)
Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(\mathcal{B}_d)) \to (CL(Y),\sigma'(\alpha))$ *is continuous;*
(b) f *is ε-bounded on closed sets* (cf. 2.(a) in Definition 1.4).

Theorem 8.5. (Ball $\to$ Vietoris)
Let (X,d), (Y,e) be metric spaces. The following are equivalent:

(a) $F\colon (CL(X),\tau(\mathcal{B}_d)) \to (CL(Y),\tau'(V))$ *is continuous;*
(b) f *is bounded on closed sets* (cf. 1.(a) in Definition 1.4).

Theorem 8.6. (Ball $\to$ Hausdorff metric, Proximal locally finite)
Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(\mathcal{B}_d)) \to (CL(Y),\tau'(H_e))$ *is continuous;*
(b) $F\colon (CL(X),\tau(\mathcal{B}_d)) \to (CL(Y),\sigma'(LF,\alpha))$ *is continuous;*
(c) (1) f *is ε-bounded on closed sets, and*
(2) Y *is totally bounded.*

Theorem 8.7. (Ball $\to$ Locally finite)
Let (X,d), (Y,e) be metric spaces. The following are equivalent:

(a) $F\colon (CL(X),\tau(\mathcal{B}_d)) \to (CL(Y),\tau'(LF))$ *is continuous;*
(b) (1) f *is bounded on closed sets, and*
(2) Y *is totally bounded.*

9. Proximal domain

Theorem 9.1. (Proximal $\to$ Fell, **U**)

Let (X,d), (Y,e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X),\sigma(\gamma)) \to (CL(Y),\tau'(F))$ *is continuous;*
(b) $F\colon (CL(X),\sigma(\gamma)) \to (CL(Y),\tau'(U))$ *is continuous;*
(c) f *is compactly p-continuous* (cf. (b) in Definition 1.2).

Theorem 9.2. (Proximal $\to$ Wijsman)

Let (X,d), (Y,e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X),\sigma(\gamma)) \to (CL(Y),\tau'(W_e))$ *is continuous;*
(b) (1) *for each proper e-ball B in Y and $\varepsilon > 0$, there is a $T \in CL(X)$ such that $f^{-1}(B) \subset T \ll_\gamma f^{-1}(S_e(B,\varepsilon))$, and*
(2) f *is p-ball-p-continuous* (cf. (c) in Definition 1.2).

Theorem 9.3. (Proximal $\to$ Proximal Ball)

Let (X,d), (Y,e) be metric spaces, γ, α the metric proximities induced by d and e respectively. The following are equivalent:

(a) $F\colon (CL(X),\sigma(\gamma)) \to (CL(Y),\sigma'(\mathcal{B}_e,\alpha))$ *is continuous;*
(b) f *is p-ball-p-continuous.*

Theorem 9.4. (Proximal $\to$ Ball)

Let (X,d), (Y,e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X),\sigma(\gamma)) \to (CL(Y),\tau'(\mathcal{B}_e))$ *is continuous;*
(b) f *is ball p-continuous* (cf. (d) in Definition 1.2).

Theorem 9.5. (Proximal $\to$ Vietoris)

Let (X,d), (Y,e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X),\sigma(\gamma)) \to (CL(Y),\tau'(V))$ *is continuous;*
(b) f *is closed p-continuous* (cf. Definition 1.2 (a)).

Theorem 9.6. (Proximal $\to$ Hausdorff metric, Proximal locally finite)

Let (X,d), (Y,e) be metric spaces, γ, α the metric proximities induced by d and e respectively. The following are equivalent:

(a) $F\colon (CL(X),\sigma(\gamma)) \to (CL(Y),\tau'(H_e))$ *is continuous;*

(b) *$F\colon (CL(X), \sigma(\gamma)) \to (CL(Y), \sigma'(LF, \alpha))$ is continuous;*
(c) (1) *f is uniformly continuous, and*
(2) *Y is totally bounded.*

Theorem 9.7. (Proximal → Locally finite)
Let (X, d), (Y, e) be metric spaces, and γ the metric proximity induced by d. The following are equivalent:
(a) *$F\colon (CL(X), \sigma(\gamma)) \to (CL(Y), \tau'(LF))$ is continuous;*
(b) (1) *f is closed p-continuous, and*
(2) *Y is totally bounded.*

10. Vietoris domain

Theorem 10.1. (Vietoris → Fell, Ball)
Let (X, d), (Y, e) be metric spaces. Then the maps $F\colon (CL(X), \tau(V)) \to (CL(Y), \tau'(F))$ and $F\colon (CL(X), \tau(V)) \to (CL(Y), \tau'(\mathcal{B}_e))$ are always continuous.

Theorem 10.2. (Vietoris → **U**)
Let (X, d), (Y, e) be metric spaces. The following are equivalent:
(a) *$F\colon (CL(X), \tau(V)) \to (CL(Y), \tau'(U))$ is continuous;*
(b) *f is compactly p-continuous* (cf. (b) in Definition 1.2).

Theorem 10.3. (Vietoris → Wijsman)
Let (X, d), (Y, e) be metric spaces. The following are equivalent:
(a) *$F\colon (CL(X), \tau(V)) \to (CL(Y), \tau'(W_e))$ is continuous;*
(b) *for each proper e-ball B in Y and $\varepsilon > 0$, there is a $T \in CL(X)$ such that $f^{-1}(B) \subset T \subset f^{-1}(S_e(B, \varepsilon))$.*

Theorem 10.4. (Vietoris → Proximal Ball)
Let (X, d), (Y, e) be metric spaces, α the metric proximity induced by e. The following are equivalent:
(a) *$F\colon (CL(X), \tau(V)) \to (CL(Y), \sigma'(\alpha, \mathcal{B}_e))$ is continuous;*
(b) *for each proper e-ball B in Y and $\varepsilon > 0$, there is a $T \in CL(X)$ such that*
(i) *$f^{-1}(B) \subset T \subset f^{-1}(S_e(B, \varepsilon))$, and*
(ii) *$f^{-1}(\alpha(B)) \subset \gamma_0(T)$.*

Theorem 10.5. (Vietoris → Proximal)
Let (X, d), (Y, e) be metric spaces and α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X), \tau(V)) \to (CL(Y), \sigma'(\alpha))$ *is continuous;*
(b) (1) $f\colon (X, \gamma_0) \to (Y, \alpha)$ *is p-continuous, and*
  (2) *for each pair of* α*-far closed sets* S, T *in* Y, *there is a closed set* E *in* X *such that*
   (i) $f^{-1}(S) \subset E \subset f^{-1}(T)$, *and*
   (ii) $f^{-1}(\alpha(S)) \subset \gamma_0(E)$.

Theorem 10.6. (Vietoris $\to$ Hausdorff metric, Proximal locally finite)

Let (X, d), (Y, e) *be metric spaces,* α *the metric proximity induced by* e. *The following are equivalent:*

(a) $F\colon (CL(X), \tau(V)) \to (CL(Y), \tau'(H_e))$ *is continuous;*
(b) $F\colon (CL(X), \tau(V)) \to (CL(Y), \sigma'(LF, \alpha))$ *is continuous;*
(c) (1) $f\colon (X, \gamma_0) \to (Y, \alpha)$ *is p-continuous, and*
  (2) *for each pair of* α*-far closed sets* S, T *in* Y, *there is a closed set* E *in* X *such that*
   (i) $f^{-1}(S) \subset E \subset f^{-1}(T)$,
   (ii) $f^{-1}(\alpha(S)) \subset \gamma_0(E)$, *and*
  (3) Y *is totally bounded.*

Theorem 10.7. (Vietoris $\to$ Locally finite)

Let (X, d), (Y, e) *be metric spaces. The following are equivalent:*

(a) $F\colon (CL(X), \tau(V)) \to (CL(Y), \tau'(LF))$ *is continuous;*
(b) Y *is totally bounded.*

11. Hausdorff metric domain

Theorem 11.1. (Hausdorff metric $\to$ Fell, **U**)

Let (X, d), (Y, e) *be metric spaces. The following are equivalent:*

(a) $F\colon (CL(X), \tau(H_d)) \to (CL(Y), \tau'(F)$ *is continuous;*
(b) $F\colon (CL(X), \tau(H_d)) \to (CL(Y), \tau'(U))$ *is continuous;*
(c) f *is compactly p-continuous* (cf. (b) in Definition 1.2).

Theorem 11.2. (Hausdorff metric $\to$ Wijsman)

Let (X, d), (Y, e) *be metric spaces,* γ *the metric proximity induced by* d. *The following are equivalent:*

(a) $F\colon (CL(X), \tau(H_d)) \to (CL(Y), \tau'(W_e))$ *is continuous;*
(b) (1) f *is ball p-continuous* (cf. Definition 1.2(d)), *and*
  (2) *for each proper* e*-ball* B *in* Y *and* $\varepsilon > 0$, *there is a* $T \in CL(X)$ *such that* $f^{-1}(B) \subset T \ll_\gamma f^{-1}(S_e(B, \varepsilon))$.

Theorem 11.3. (Hausdorff metric → Proximal Ball, Ball)
Let (X,d), (Y,e) be metric spaces, γ, α the metric proximities induced by d and e respectively. The following are equivalent:

(a) $F\colon (CL(X),\tau(H_d)) \to (CL(Y),\sigma'(\alpha,\mathcal{B}_e))$ *is continuous;*
(b) $F\colon (CL(X),\tau(H_d)) \to (CL(Y),\tau'(\mathcal{B}_e))$ *is continuous;*
(c) f *is p-ball-p-continuous* (cf. Definition 1.2(c)).

We recall that if (X,d), (Y,e) are metric spaces and γ, α are metric proximities induced by d and e respectively, then a function f from X into Y is p-continuous if, and only if, f is uniformly continuous (see for example [5, Problem 8.5.19(a)]). Thus, from the Main Theorem 3.5 we have

Theorem 11.4. (Hausdorff metric → Proximal)
Let (X,d), (Y,e) be metric spaces, γ, α the metric proximities induced by d and e respectively. The following are equivalent:

(a) $F\colon (CL(X),\tau(H_d)) \to (CL(Y),\sigma'(\alpha))$ *is continuous;*
(b) f *is uniformly continuous.*

Similarly, the below results follow easily from Theorem 3.5.

Theorem 11.5. (Hausdorff metric → Vietoris)
Let (X,d), (Y,e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X),\tau(H_d)) \to (CL(Y),\tau'(V))$ *is continuous;*
(b) f *is p-closed continuous* (cf. Definition 1.2(c)).

Theorem 11.6. (Hausdorff metric → Locally finite)
Let (X,d), (Y,e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X),\tau(H_d)) \to (CL(Y),\tau'(LF))$ *is continuous;*
(b) (1) f *is p-closed continuous, and*
(2) *for each locally finite family $\mathcal{G}$ in Y, there is a discrete family of open balls with equal radii $\mathcal{E}$ in X such that $\mathcal{E}^- \subset f^{-1}(\mathcal{G}^-)$.*

Remark 11.7. We point out that (b)2 in the above Theorem expresses the fact that the discrete families of open balls are enough to guarantee the lower semicontinuity of f with respect to locally finite families of open sets.

Theorem 11.8. (Hausdorff metric → Proximal locally finite)

Let (X,d), (Y,e) be metric spaces, γ, α the metric proximities induced by d and e respectively. The following are equivalent:

(a) $F\colon (CL(X),\tau(H_d)) \to (CL(Y),\sigma'(LF,\alpha))$ *is continuous;*
(b) (1) *f is uniformly continuous, and*
 (2) *for each locally finite open family $\mathcal{G}$ in Y, there is a discrete family of open balls with equal radii $\mathcal{E}$ in X such that $\mathcal{E}^- \subset f^{-1}(\mathcal{G}^-)$.*

12. Locally finite domain

Theorem 12.1. (Locally finite → Fell, Ball, Vietoris)

Let (X,d), (Y,e) be metric spaces. Then, $F\colon (CL(X),\tau(LF)) \to (CL(Y),\tau'(F))$, $F\colon (CL(X),\tau(LF)) \to (CL(Y),\tau'(\mathcal{B}_e))$ and $F\colon (CL(X),\tau(LF)) \to (CL(Y),\tau'(V))$ are always continuous.

Theorem 12.2. (Locally finite → **U**)

Let (X,d), (Y,e) be metric spaces. The following are equivalent:

(a) $F\colon (CL(X),\tau(LF)) \to (CL(Y),\tau'(U))$ *is continuous;*
(b) *f is compactly p-continuous* (cf. Definition 1.2(c)).

Theorem 12.3. (Locally finite → Wijsman)

Let (X,d), (Y,e) be metric spaces. The following are equivalent:

(a) $F\colon (CL(X),\tau(LF)) \to (CL(Y),\tau'(W_e))$ *is continuous;*
(b) *for each proper e-ball B in Y and each $\varepsilon > 0$, there is a $T \in CL(X)$ such that $f^{-1}(B) \subset T \subset f^{-1}(S_e(B,\varepsilon))$.*

Theorem 12.4. (Locally finite → Proximal Ball)

Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(LF)) \to (CL(Y),\sigma'(\mathcal{B}_e,\alpha))$ *is continuous;*
(b) *for each proper e-ball B in Y and each $\varepsilon > 0$, there is a closed set T in X such that*
 (1) $f^{-1}(B) \subset T \subset f^{-1}(S_e(B,\varepsilon))$, *and*
 (2) $f^{-1}(\alpha(B)) \subset \gamma_0(T)$.

Theorem 12.5. (Locally finite → Proximal)

Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X),\tau(LF)) \to (CL(Y),\sigma'(\alpha))$ *is continuous;*

(b) *for each pair of nonempty α-far closed sets S, T in Y, there is a closed set E in X such that*
 (1) $f^{-1}(S) \subset E \subset f^{-1}(T^c)$, *and*
 (2) $f^{-1}(\alpha(S)) \subset \gamma_0(E)$.

Theorem 12.6. (Locally finite → Hausdorff metric, Proximal locally finite)

Let (X,d), (Y,e) be metric spaces, α the metric proximity induced by e. The following are equivalent:

(a) $F\colon (CL(X), \tau(LF)) \to (CL(Y), \tau'(H_e))$ *is continuous;*
(b) $F\colon (CL(X), \tau(LF)) \to (CL(Y), \sigma'(LF, \alpha))$ *is continuous;*
(c) (1) *for each pair of nonempty α-far closed sets S, T in Y, there is a set E in X such that $f^{-1}(S) \subset E \subset f^{-1}(T^c)$ and $f^{-1}(\alpha(S)) \subset \gamma_0(E)$, and*
 (2) *for each discrete family $\mathcal{G}$ of open balls of equal radii in Y, there is a locally finite open family $\mathcal{E}$ in X such that $\mathcal{E}^- \subset f^{-1}(\mathcal{G}^-)$.*

Remark 12.7. Condition (c)2 in 12.6 points out that we need locally finite open families $\mathcal{E}$ in X to guarantee the lower semicontinuity of f with respect to discrete families of open balls in Y

13. Proximal locally finite domain

Theorem 13.1. (Proximal locally finite → Fell, **U**)

Let (X,d), (Y,e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \tau'(F))$ *is continuous;*
(b) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \tau'(U))$ *is continuous;*
(c) f *is compactly p-continuous* (cf. Definition 1.2 (b)).

Theorem 13.2. (Proximal locally finite → Wijsman, Ball)

Let (X,d), (Y,e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \tau'(W_e))$ *is continuous;*
(b) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \tau'(W_e))$ *is continuous;*
(c) f *is ball p-continuous* (cf. Definition 1.2 (d)).

Theorem 13.3. (Proximal locally finite → Proximal Ball)

Let (X,d), (Y,e) be metric spaces, γ, α the metric proximities induced by d and e respectively. The following are equivalent:

(a) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \sigma'(\mathcal{B}_e, \alpha))$ *is continuous;*
(b) f *is p-ball-p-continuous* (cf. Definition 1.2 (c)).

Theorem 13.4. (Proximal locally finite → Proximal)
Let (X, d), (Y, e) be metric spaces, γ, α the metric proximities induced by d and e respectively. The following are equivalent:

(a) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \sigma'(\alpha))$ *is continuous;*
(b) f *is uniformly continuous.*

Theorem 13.5. (Proximal locally finite → Vietoris)
Let (X, d), (Y, e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \tau'(V))$ *is continuous;*
(b) f *is closed p-continuous* (cf. Definition 1.2 (a)).

Theorem 13.6. (Proximal locally finite → Hausdorff metric)
Let (X, d), (Y, e) be metric spaces, γ, α the metric proximities induced by d and e respectively. The following are equivalent:

(a) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \tau'(H_e))$ *is continuous;*
(b) (1) f *is uniformly continuous, and*
(2) *for each discrete family $\mathcal{G}$ of open balls of equal radii in Y, there is a locally finite open family $\mathcal{E}$ in X such that $\mathcal{E}^- \subset f^{-1}(\mathcal{G}^-)$.*

Theorem 13.7. (Proximal locally finite → Locally finite)
Let (X, d), (Y, e) be metric spaces, γ the metric proximity induced by d. The following are equivalent:

(a) $F\colon (CL(X), \sigma(LF, \gamma)) \to (CL(Y), \tau'(LF))$ *is continuous;*
(b) (1) f *is closed p-continuous* (cf. Definition 1.2 (a)), *and*
(2) *for each locally finite open family $\mathcal{G}$ in Y, there is a locally finite open family $\mathcal{E}$ in X such that $\mathcal{E}^- \subset f^{-1}(\mathcal{G}^-)$.*

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