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SUBMAXIMALITY IN LOCALES

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Abstract. We define submaximal locales analogously to submaximal topological spaces and investigate the extent to which some of the characterizations of submaximality in spaces can be carried over to locales. We note in particular that a locale is submaximal iff its complemented nowhere dense sublocales are closed; a result which is false for spaces. Among regular locales, we give a characterization in terms of elements of associated frames.

1. Introduction

Submaximal spaces were introduced by Bourbaki [3] and have since been studied by numerous authors such as Arhangelsk'iĭ and Collins [1], Dontchev [4], Kennedy and McCartan [7] and Schröder [13], amongst others. Recall that a topological space is submaximal if each of its dense subspaces is open.

A natural translation to locales is that one should call a locale submaximal if each of its dense sublocales is open. Since in general a topological space X has fewer subspaces than the locale $\mathrm{Lc}(X)$ it determines has sublocales, one would not expect this notion to be conservative in the sense that a topological space X is submaximal if and only if $\mathrm{Lc}(X)$ is. Indeed that is not the case.

This phenomenon, which arises from the difference between subspaces and sublocales coupled with the fact that every locale has a smallest dense sublocale, is reminiscent of what happened when an attempt was made to define Hausdorff locales. We shall, as in the case of the Hausdorff property, refer to this property as strong submaximality.

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Now if we want a conservative notion of submaximality in locales we might want to limit the number of dense sublocales that we wish to consider so that when viewing a space as a locale, we still deal with exactly the same sublocales. We shall therefore call a locale submaximal if each of its dense *complemented* sublocales is open. In this way we indeed have a conservative notion.

The purpose of this paper is to study side by side these two variants of submaximality and investigate the extent to which some of the characterizations of submaximality in spaces can be carried over to locales.

In [4], a number of characterizations of submaximality in spaces are given. An analysis of the proofs reveals that a good few of them are point-based. They also exploit the fact that every subspace is complemented in the Boolean algebra of all subspaces; the complement being the set-theoretic complement. Now even if a sublocale is complemented in the lattice of sublocales, the complement is not the set-theoretic complement. Furthermore, complemented dense sublocales have the "peculiar-looking" (from a classical topologist's point of view) property that their interiors are dense.

2. Background

Recall that a *frame* is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of a frame L by 1_L and 0_L respectively, or simply 1 and 0 if there is no possibility of confusion. The *pseudocomplement* of an element a of L is the element $a^* = \bigvee\{x \in L \,|\, x \wedge a = 0\}$. A *frame homomorphism* is a map between frames that preserves finite meets, including the top element, and arbitrary joins, including the bottom element. The resulting category is denoted by **Frm**. The opposite category is called the category of *locales* and denoted by **Loc**. If X is a locale we denote by $\mathcal{O}X$ the corresponding frame; and if $f : X \to Y$ is a morphism in **Loc** we denote the corresponding frame homomorphism by f^*. If X is a topological space, we denote by $\mathfrak{O}X$ the frame of its open sets, and by $\mathrm{Lc}(X)$ the locale it determines; that is, the locale for which $\mathcal{O}\mathrm{Lc}(X) = \mathfrak{O}X$.

The *right adjoint* of a frame homomorphism $h : L \to M$ is the map $h_* : M \to L$ given by $h_*(a) = \bigvee\{x \in L \mid h(x) \leq a\}$. A frame homomorphism is *dense* if its right adjoint maps the bottom to the bottom. On the other hand, a frame homomorphism is *codense* if it maps only the top to the top. For a continuous map of locales f, f_* will denote the right adjoint of the frame homomorphism f^*.

A *nucleus* on a frame L is a map $j : L \to L$ that satisfies for all a, b in L:

(1) $a \leq j(a)$
(2) $j(j(a)) = j(a)$
(3) $j(a \wedge b) = j(a) \wedge j(b)$.

If j is a nucleus on a frame L, the set $L_j = \{a \in L \mid j(a) = a\}$ (also denoted by $Fix(j)$) is a frame with the same meet as in L and join given by $\bigvee_{L_j} S = j(\bigvee_L S)$ for each $S \subseteq L_j$.

If X is a locale, a *sublocale* of X is a locale X_j with $\mathcal{O}(X_j) = (\mathcal{O}X)_j$ for some nucleus j on $\mathcal{O}X$. The lattice $\mathcal{S}(X)$ of all sublocales of a locale X is a coframe the zero element of which we shall denote by $\bot$.

A sublocale A of X (say, $A = X_j$) is *dense* in case $0_{\mathcal{O}X} \in \mathcal{O}A$. This is true iff $j(0) = 0$. On the other hand, a sublocale B of X is *nowhere dense* if $B \wedge \mathrm{d}(X) = \bot$. A complemented (in $\mathcal{S}(X)$) sublocale is dense if and only if its complement is nowhere dense.

Every locale X has a smallest dense sublocale which we denote by $\mathrm{d}(X)$. In fact $\mathcal{O}\mathrm{d}(X) = (\mathcal{O}X)_j$ for the nucleus j given by $j(a) = a^{**}$ for each $a \in \mathcal{O}X$. On the hand an element a of a frame is called *dense* if $a^* = 0$.

Let $a \to b$ denote the Heyting implication

$$a \to b = \bigvee\{x \in \mathcal{O}X \mid a \wedge x \leq b\}$$

in the frame $\mathcal{O}X$ of the locale X. Then for any $a \in \mathcal{O}X$, the maps $u_a : \mathcal{O}X \to \mathcal{O}X$ and $c_a : \mathcal{O}X \to \mathcal{O}X$ given by $x \rightsquigarrow a \to x$ and $x \rightsquigarrow a \vee x$ respectively are nuclei. The induced sublocales are called *open* and *closed* respectively. We also have that X_{u_a} and X_{c_a} are complementary in $\mathcal{S}(X)$. Not all sublocales are complemented in $\mathcal{S}(X)$.

The *closure* of a sublocale A of X, denoted by $\mathrm{cl}(A)$ or $\overline{A}$, is the sublocale

$$\overline{A} = \bigwedge\{T \in \mathcal{S}(X) \mid T \text{ is closed and } A \leq T\}.$$

In fact, if $A = X_j$, then $\overline{A} = X_{c_{j(0)}}$. The *interior* of A, denoted by $\mathrm{int}(A)$, is defined dually; to wit,

$$\mathrm{int}(A) = \bigvee\{U \in \mathcal{S}(X) \mid U \text{ is open and } U \leq A\}.$$

For sublocales S and T of X we set

$$S \setminus T = \bigvee\{A \in \mathcal{S}(X) \mid A \leq S \text{ and } A \wedge T = \bot\},$$

and note that if T is a complemented sublocale then $X \setminus T$ is the complement of T. If T is complemented then for any sublocale S, $S \setminus T = S \wedge (X - T)$, where $X - T$ denotes the complement of T. The *boundary* of a sublocale A of X is the sublocale

$$\mathrm{bd}(A) = \overline{A} \setminus \mathrm{int}(A).$$

Note that if A and B are complemented sublocales then so are $A \vee B$ and $A \wedge B$. Furthermore, $X - (A \vee B) = (X - A) \wedge (X - B)$ and $X - (A \wedge B) = (X - A) \vee (X - B)$. Also note that if A and B are complemented sublocales, then $X - A \leq B \Leftrightarrow X - B \leq A$.

Observation. If A is a complemented sublocale of X, then $X - \mathrm{int}(A) = \overline{X - A}$. To verify this let us note that since $\mathrm{int}(A) \leq A$, $X - A \leq X - \mathrm{int}(A)$. Since $X - \mathrm{int}(A)$ is closed, it follows that $\overline{X - A} \leq X - \mathrm{int}(A)$. On the other hand let F be a closed sublocale with $X - A \leq F$. Then $X - F \leq A$ since F and A are complemented. But now $X - F$ is open, so we have that $X - F \leq \mathrm{int}(A)$ whence $X - \mathrm{int}(A) \leq F$. Taking the meet of all such F we deduce that $X - \mathrm{int}(A) \leq \overline{X - A}$. Thus the two sublocales are equal.

So if A is a complemented sublocale then

$$\overline{A} \wedge \overline{X - A} = \overline{A} \wedge (X - \mathrm{int}(A)) = \overline{A} \setminus \mathrm{int}(A) = \mathrm{bd}(A).$$

We shall use this fact in one of the characterizations.

A sublocale A is called

- *regular open* if $A = \mathrm{int}(\overline{A})$
- *preopen* if $A \leq \mathrm{int}(\overline{A})$
- *locally closed* if $A = U \wedge F$ for some open U and closed F
- *semi-closed* if $\mathrm{int}(A) = \mathrm{int}(\overline{A})$
- *locally semi-closed* if $A = U \wedge S$ for some open U and semi-closed S
- *g-closed* if for any open sublocale U, $A \leq U \Rightarrow \overline{A} \leq U$.

For further results in locales we refer to Johnstone [6] and Plewe ([10], [11]). For the most part, our discourse will take place in **Loc**. There will however be exceptions necessitated by the fact that in those cases computation in **Frm** is more transparent. Thus, in the language of Isbell [5], our locales will at times intrude their frames into innocent conversation.

3. Submaximality

We commence by formalizing the discussion given in the Introduction.

Definition 3.1. A locale is *submaximal* in case each of its dense complemented sublocales is open. It is *strongly submaximal* if each of its dense sublocales is open.

In view of the fact that a complemented sublocale is dense if and only if its complement is nowhere dense, we have that *a locale is submaximal if and only if its complemented nowhere dense sublocales are closed.*

We shall on occasion speak "algebraically" and say a frame is submaximal to mean that the locale it determines is submaximal.

Let us note that if X is a topological space then X is submaximal if and only if the locale $\mathrm{Lc}(X)$ is submaximal. This follows from the fact, stated in the Introduction, that dense complemented sublocales of $\mathrm{Lc}(X)$ are dense subspaces of X; and dense subspaces of X are dense complemented sublocales of $\mathrm{Lc}(X)$.

On the other hand, strongly submaximal locales are included in the class of submaximal locales. The inclusion is strict as witnessed by the following example.

Example. Let X be a submaximal Hausdorff space that has no isolated points. Such a space does exist (see for instance Example 3.10 in [1]). Then the locale $\mathrm{Lc}(X)$ is submaximal. It is however not strongly submaximal because the dense sublocale $\mathrm{d}(\mathrm{Lc}(X))$ is not open; for if it were, then it would have been a subspace, which it is not since it is pointless.

There are however instances where the two notions coincide.

Example. It has been shown by Niefield and Rosenthal [9] that all sublocales of a spatial locale are spatial if and only if the space is weakly scattered. Thus, for a weakly scattered space X, $\mathrm{Lc}(X)$ is strongly submaximal iff it is submaximal.

Examples of strongly submaximal locales are given next.

Example. Boolean locales are strongly submaximal because each sublocale of a Boolean locale is open. So localic submaximality generalizes and transcends topological submaximality.

Remark 3.2. If X is a strongly submaximal locale then $\mathcal{O}X$ has a smallest dense element. This is in fact a consequence of $\mathrm{d}(X)$ being open. This was observed by Banaschewski and Pultr in [2]. We offer an alternative verification. Let $\mathrm{d}(X)$ be the open sublocale X_{u_a}, for some $a \in \mathcal{O}X$. Since $\mathrm{d}(X)$ is dense, $0 \in \mathcal{O}\mathrm{d}(X)$ and so $a \to 0 = 0$; that is $a^* = 0$. Now let r be another dense element of $\mathcal{O}X$. From $\mathrm{d}(X) = X_{u_a}$ we deduce that $a \to r = r^{**} = 0^* = 1$, which implies that $a \leq r$.

In fact, for regular locales, strong submaximality is characterizable "algebraically" in terms of elements of the underlying frames as shown below.

Proposition 3.3. *A regular locale X is strongly submaximal iff for every dense onto frame homomorphism $h : \mathcal{O}X \to M$, there is a smallest element $a \in \mathcal{O}X$ such that $h(a) = 1$.*

Proof. $(\Rightarrow)$: Let $h : \mathcal{O}X \to M$ be dense onto. Then, by strong submaximality, h is open; that is, there exists $a \in \mathcal{O}X$ such that the map $\bar{h} : \downarrow a \to M$, mapping as h, is an isomorphism. Thus $h(a) = 1$. Now suppose $b \in \mathcal{O}X$ such that $h(b) = 1$. Then $b \wedge a$ is an element of $\downarrow a$ with $\bar{h}(b \wedge a) = h(b) \wedge h(a) = 1$. Therefore $b \wedge a = a$, implying that $a \leq b$.
$(\Leftarrow)$: Let X_j be a dense sublocale of X. Then j defines a dense onto frame homomorphism $\mathcal{O}X \to (\mathcal{O}X)_j$ by $x \rightsquigarrow j(x)$. Pick $a \in \mathcal{O}X$ with the stated property. We claim that the restriction $\bar{j}$ of j to $\downarrow a$ is an isomorphism. For any $b \leq a$ with $\bar{j}(b) = 1$ we have that $b = a = 1_{\downarrow a}$; so that $\bar{j}$ is codense and therefore one-to-one since the frames involved are regular. To show ontoness, let $x \in (\mathcal{O}X)_j$.

Then $x \wedge a$ is an element of $\downarrow a$ with $\bar{j}(x \wedge a) = j(x) \wedge j(a) = j(x) = x$. Therefore X_j is open; proving the result. $\square$

In [11], Plewe calls a locale *scattered* if every nonzero closed sublocale contains a nonzero Boolean sublocale. He then shows that a locale is scattered if and only if each of its sublocales is complemented.

One of the referees has observed that the structure of strongly submaximal locales is simple to describe, namely: *A locale is strongly submaximal iff it is the union of two disjoint Boolean sublocales, one open and the other closed.* Furthermore, such locales are scattered. Consequently, the results which we prove for submaximal locales in the next two propositions and the corollary following them also hold for strongly submaximal locales either by this observation or by the same proofs mutatis mutandis.

The following lemma will be useful.

Lemma 3.4. *For any sublocale A of a locale X, the sublocale $A \vee (X \setminus \overline{A})$ is dense.*

Proof. Say $A = X_j$. Then $\overline{A} = X_{c_{j(0)}}$, so that the nucleus determining $X \setminus \overline{A}$ is $u_{j(0)}$. Thus the nucleus determining $A \vee (X \setminus \overline{A})$ is $j \wedge u_{j(0)}$. Now

$$\begin{aligned}(j \wedge u_{j(0)})(0) &= j(0) \wedge u_{j(0)}(0) \\ &= j(0) \wedge (j(0) \to 0) \\ &= j(0) \wedge (j(0))^* \\ &= 0,\end{aligned}$$

which proves the result. $\square$

As an application we show that open sublocales of a submaximal locale are precisely those that are expressible as meets of regular open sublocales with dense complemented sublocales.

Let us first note that if U is an open sublocale of X, say $U = X_{u_a}$, then $\mathrm{int}(\overline{U}) = X_{u_b}$ where $b = a^{**}$. Consequently, for any open sublocale U, the sublocale $\mathrm{int}(\overline{U})$ is regular open since $(\ )^{****} = (\ )^{**}$ in any frame.

Proposition 3.5. *A sublocale of a submaximal locale is open iff it is a meet of a complemented dense sublocale and regular open one.*

Proof. Let X be a submaximal locale and $A = D \wedge R$ where D is a complemented dense sublocale and R is regular open. Then we have that D is open and hence A is open.

Conversely, let U be an open sublocale. Now

$$\begin{aligned}
\overline{U} \setminus (\overline{U} \setminus U) &= \overline{U} \setminus \big(\overline{U} \wedge (X - U)\big) \\
&= \overline{U} \wedge \Big(X - \big(\overline{U} \wedge (X - U)\big)\Big) \\
&= \overline{U} \wedge \big((X - \overline{U}) \vee U\big) \\
&= \big(\overline{U} \wedge (X - \overline{U})\big) \vee (\overline{U} \wedge U) \\
&= \perp \vee U \\
&= U \vee \perp \\
&= \big(U \wedge \operatorname{int}(\overline{U})\big) \vee \big(\operatorname{int}(\overline{U}) \wedge (X - \overline{U})\big) \\
&= \operatorname{int}(\overline{U}) \wedge \big(U \vee (X \setminus \overline{U})\big),
\end{aligned}$$

the last but one step holding because $U \leq \operatorname{int}(\overline{U})$ as U is open, and $X - \overline{U} \leq X - \operatorname{int}(\overline{U})$. So we have that $U = \operatorname{int}(\overline{U}) \wedge (U \vee (X \setminus \overline{U}))$; a meet of a regular open sublocale and a dense one. □

We shall refer to the following result in some characterizations of submaximality.

Proposition 3.6. *Every complemented sublocale of a submaximal locale is submaximal.*

Proof. Let X be a submaximal locale, A a complemented sublocale of X and B be a sublocale of A that is complemented in A. By 1.10 in Isbell [5] there is a complemented sublocale C of X such that $B = C \wedge A$. Therefore B is complemented in X; and so the dense sublocale $B \vee (X \setminus \overline{B})$ is complemented in X (being a join of complemented sublocales) and hence is open in X by the hypothesis. Therefore the sublocale $A \wedge (B \vee (X \setminus \overline{B}))$ is open in A. But $A \wedge (B \vee (X \setminus \overline{B})) = (A \wedge B) \vee (A \wedge (X \setminus \overline{B})) = (A \wedge B) \vee \perp = A \wedge B = B$. Thus B is open in A as required. □

The corresponding result for strongly submaximal locales is that every sublocale of a strongly submaximal locale is strongly submaximal. Since every sublocale is dense in its closure, and the closure of any sublocale is complemented, we deduce from the foregoing result that:

Corollary. *Every complemented sublocale of a submaximal locale is open in its closure.*

We now give several characterizations of submaximality which are verbatim translations of similar characterizations in spaces. As remarked earlier, similar characterizations hold in the case of strong submaximality with the adjective "complemented" on sublocales dropped.

Proposition 3.7. *The following conditions on a locale X are equivalent:*

(1) *X is submaximal.*
(2) *Every preopen complemented sublocale of X is open.*
(3) *Every complemented sublocale of X is locally closed.*
(4) *Every complemented sublocale of X is locally semi-closed.*
(5) *Every dense complemented sublocale of X is locally semi-closed.*

Proof. (1) $\Rightarrow$ (2) : Let A be a preopen complemented sublocale of X. Since A is open in $\overline{A}$ there is an open $U \in \mathcal{S}(X)$ such that $A = U \wedge \overline{A}$. Thus $U \wedge \overline{A} \leq \text{int}(\overline{A})$ since A is preopen. This implies that $A = U \wedge \overline{A} \leq U \wedge \text{int}(\overline{A}) \leq U \wedge \overline{A} = A$, whence $A = U \wedge \text{int}(\overline{A})$ and so A is an open sublocale of X.
(2) $\Rightarrow$ (1) : Let A be a dense complemented sublocale of X. Then $\text{int}(\overline{A}) = \text{int}(X) = X$; which shows that A is preopen, and is therefore open.
(1) $\Rightarrow$ (3) : Let A be a complemented sublocale of X. Since A is open in $\overline{A}$, there is an open sublocale U of X such that $A = U \wedge \overline{A}$. So A is locally closed since $\overline{A}$ is closed in X.
(3) $\Rightarrow$ (4) : This is trivial since closed sublocales are semi-closed.
(4) $\Rightarrow$ (5) : Trivial.
(5) $\Rightarrow$ (1) : Let A be a dense complemented sublocale of X and find an open U and a semi-closed S such that $A = U \wedge S$. Since A is dense and $A \leq S$, S is also dense. Thus $\text{int(S)} = \text{int}(\overline{S}) = \text{int}(X) = X$. Consequently $A = U \wedge S = U \wedge X = U$, which is open. □

Corollary. *Every g-closed complemented sublocale of a submaximal locale is closed.*

Proof. Let A be a g-closed complemented sublocale of a submaximal locale X. Then there is an open sublocale O and a closed sublocale C such that $A = O \wedge C$. Therefore $A \leq O$, whence $\overline{A} \leq O$ since A is g-closed. But we also have $\overline{A} \leq C$ since $A \leq C$ and C is closed. Thus $\overline{A} \leq O \wedge C = A$, and therefore $A = \overline{A}$; which establishes the result. ☐

For our next characterization we note that if a locale has the property that each of its complemented sublocales is open, then the locale is in fact Boolean.

Proposition 3.8. *A locale is submaximal iff the boundaries of its complemented sublocales are Boolean.*

Proof. ($\Rightarrow$): Let A be a complemented sublocale of a submaximal locale X, and let S be a sublocale of $\mathrm{bd}(A)$ which is complemented in $\mathrm{bd}(A)$. Then $S \leq \overline{A}$ and $S \leq \overline{X - A}$. This implies that $S \vee A \leq \overline{A}$ and $S \vee (X - A) \leq \overline{X - A}$. Now $\mathrm{cl}_{\overline{A}}(S \vee A) = \overline{A} \wedge \overline{S \vee A} = \overline{A}$. Thus $S \vee A$ is dense in $\overline{A}$. We show that $S \vee A$ is complemented in $\overline{A}$. Since S is complemented in $\mathrm{bd}(A) = \overline{A} \wedge \overline{X - A}$, there is a complemented sublocale C of X such that $S = C \wedge \overline{A} \wedge \overline{X - A}$. Then $S \vee A = A \vee \big(C \wedge \overline{X - A} \wedge \overline{A}\big) = D \wedge \overline{A}$ for the complemented sublocale $D = A \vee (C \wedge \overline{X - A})$ of X. Now $\overline{A}$ is submaximal since it is a complemented sublocale of a submaximal locale; so $S \vee A$ is open in $\overline{A}$. Therefore there is an open sublocale U of X such that $S \vee A = U \wedge \overline{A}$. Similarly there is an open sublocale V of X such that $S \vee (X - A) = V \wedge \overline{X - A}$. Now

$$\begin{aligned} S &= S \vee \bot \\ &= S \vee (A \wedge (X - A)) \\ &= (S \vee A) \wedge (S \vee (X - A)) \\ &= (U \wedge \overline{A}) \wedge (V \wedge \overline{X - A}) \\ &= (U \wedge V) \wedge (\overline{A} \wedge \overline{X - A}), \end{aligned}$$

which shows that S is open in $\mathrm{bd}(A)$. So every complemented sublocale of $\mathrm{bd}(A)$ is open in $\mathrm{bd}(A)$, and therefore $\mathrm{bd}(A)$ is Boolean.
($\Leftarrow$): Let A be a complemented dense sublocale of a locale X with the stated property. Then $\mathrm{bd}(A) = X - \mathrm{int}(A)$. Since A is complemented in X, $A \wedge (X - \mathrm{int}(A))$ is complemented in $X - \mathrm{int}(A)$;

and is therefore open in $X-\mathrm{int}(A)$ by the hypothesis. So there is an open sublocale U of X such that $A\wedge(X-\mathrm{int}(A)) = U\wedge(X-\mathrm{int}(A))$. But now

$$\begin{aligned} A &= A \wedge \big(\mathrm{int}(A) \vee (X - \mathrm{int}(A)\big) \\ &= \big(A \wedge \mathrm{int}(A)\big) \vee \big(A \wedge (X - \mathrm{int}(A))\big) \\ &= \mathrm{int}(A) \vee \big(U \wedge (X - \mathrm{int}(A))\big) \\ &= \big(\mathrm{int}(A) \vee U\big) \wedge \big(\mathrm{int}(A) \vee (X - \mathrm{int}(A)\big) \\ &= \big(\mathrm{int}(A) \vee U\big) \wedge X \\ &= \mathrm{int}(A) \vee U. \end{aligned}$$

So A is a join of open sublocales of X, and is therefore open in X. Thus X is submaximal. □

As in spaces let us call a filter in the lattice $\mathcal{S}(X)$ a filter *on* X. Also, say a filter on X is *open-based* (resp. *closed-based*) in case it has a base consisting of open (resp. closed) sublocales. Furthermore, call a filter on X complemented in case it is a filter in the sublattice of complemented sublocales. We then have the following corollary.

Corollary. *Let X be a submaximal locale. Then every complemented ultrafilter on X is either open-based or closed-based.*

Proof. Let $\mathcal{F}$ be a complemented ultrafilter on X. If $\mathrm{int}(F) \neq \bot$ for each F in $\mathcal{F}$, then, in view of $\mathcal{F}$ being an ultrafilter, $\{\mathrm{int}(F) | F \in \mathcal{F}\}$ is a base for $\mathcal{F}$ consisting of open sublocales. On the other hand, if F is a member of $\mathcal{F}$ with $\mathrm{int}(F) = \bot$, then F is nowhere dense and therefore closed since X is submaximal. Thus $\mathrm{bd}(F) = \overline{F} \wedge \overline{X - F} = F \wedge X = F$. So by Proposition 3.8 we have that F is Boolean, and therefore $F \wedge S$ is closed in F for each $S \in \mathcal{F}$. Now the collection of all sublocales $F \wedge S$, for $S \in \mathcal{F}$, is a base for $\mathcal{F}$; so it remains to show that each of these sublocales is closed in X. But if $S \in \mathcal{F}$, then there exists a closed sublocale C of X such that $F \wedge S = F \wedge C$; and the latter sublocale is closed in X. □

As was shown by Schröder [13], the foregoing result characterizes submaximality for spatial locales.

We now justify the statement made in the abstract to the effect that the equivalence of localic submaximality to closedness of nowhere dense sublocales is a departure from spaces. To this end, we must show that a subspace of a topological space X is nowhere dense if and only if the sublocale it determines is nowhere dense in $\mathrm{Lc}(X)$. Call a frame homomorphism $h : L \to M$ *nowhere dense* if for each nonzero x in L there is a nonzero $y \leq x$ such that $h(y) = 0$. The justification is based on the following two propositions.

Proposition 3.9. *Let X be a topological space, S be a subset of X and $h : \mathfrak{O}X \to \mathfrak{O}S$ be the frame homomorphism given by $h(U) = U \cap S$. Then S is nowhere dense if and only if h is nowhere dense.*

Proof. If S is nowhere dense and U is a nonempty open set in X then $U \nsubseteq \mathrm{cl}(S)$. So for $x \in U \setminus \mathrm{cl}(S)$ there is an open neighborhood W of x such that $W \cap S = \emptyset$. Then $V = W \cap U$ is an open set with the stated property.

Conversely, if S is not nowhere dense then there is a nonempty set $U \subseteq \mathrm{cl}(S)$. Thus, for each nonempty $V \subseteq U$, V is a neighborhood of some point of U, and hence $V \cap U \neq \emptyset$, contrary to the hypothesis. □

Proposition 3.10. *An onto frame homomorphism $h : L \to M$ is nowhere dense iff $Fix(h_*h) \cap Fix(j) = \{1_L\}$, where j is the nucleus $(-)^{**}$.*

Proof. $(\Rightarrow)$: Let x be in the intersection of the two fix-sets. Then $x = x^{**}$ and $x = h_*h(x)$. We claim that $x^* = 0$. If not, there exists a nonzero $y \leq x^*$ such that $h(y) = 0$. Then $y \leq h_*(0) \leq h_*(h(x)) = x$. Consequently $y \leq x \wedge x^*$, contrary to y being nonzero.
$(\Leftarrow)$: If h is not nowhere dense, then there exists a nonzero $x \in L$ such that $h(y) \neq 0$ for each nonzero $y \leq x$. Now $x^* \in Fix(j)$. Let $z \in L$ be such that $h(z) \leq h(x^*)$. Then $h(z \wedge x) = h(x) \wedge h(z) \leq h(x) \wedge h(x^*) = 0$, implying that $z \wedge x = 0$ since $z \wedge x \leq x$. Thus $z \leq x^*$; and therefore $h_*(h(x^*)) \leq x^*$, whence $h_*h(x^*) = x^*$ since the other inequality always holds. So by the hypothesis we have $x^* = 1$, which is impossible since $x \neq 0$. □

We end by remarking that not all subframes of submaximal frames are submaximal. Indeed, every frame is isomorphic to a subframe of a Boolean frame by Corollary 2.6 in Johnstone [6], and Boolean frames are submaximal.

Concluding remark.

In spaces, submaximality is frequently considered in conjunction with resolvability, pseudo- and quasimaximality. These concepts can unfortunately not be adapted to a pointfree setting for the following reasons:

- *Resolvability* : Recall that a space is resolvable if it can be expressed as a disjoint union of two dense subsets. In locales we have that the meet (or localic intersection) of any two dense sublocales is dense, so that if the locale is nontrivial then there are no disjoint dense sublocales in it.
- *Pseudo-and quasimaximality* : Each of these is defined in terms of isolated points. Now "isolated points" cannot be of much interest in locales since many locales have very few or no "points" at all.

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References

1. A.V. Arhangel'skiĭ and P.J. Collins, *On submaximal spaces*, Topology Appl. **64** (1995), 219-241.
2. B. Banaschewski and A. Pultr, *Booleanization*, Cahiers Topologie Géom. Différentielle Catégoriques Vol. XXXVII-1 (1996), 41-60.
3. N. Bourbaki, *General Topology*, Addison Wesley, Mass., 1966.
4. J. Dontchev, *On submaximal spaces*, Tamgang J. Math., **26**(3)(1995), 243-250.
5. J. Isbell, *First steps in descriptive theory of locales*, Trans. Amer. Math. Soc. **327**(1) (1991),353-371.
6. P.T. Johnstone, *Stone Spaces*, Cambr. Univ. Press, 1982.
7. G.J. Kennedy and S.D. McCartan, *Submaximality and supraconnectedness are complementary topological invariants*, Topology Atlas preprint.
8. Y. Li, *Weak locale quotient morphisms and locally connected frames*, J. Pure Appl. Alg., **110** (1996), 101-107.
9. S.B. Niefield and K.I. Rosenthal, *Spatial sublocales and essential primes*, Topology Appl. **26** (1987), 263-269.
10. T. Plewe, *Quotient maps of locales*, Appl. Cat. Str., **8** (2000), 17-44.
11. T. Plewe, *Higher order dissolutions and Boolean coreflections of locales*, J. Pure Appl. Alg., **154** (2000), 273-293.

12. A. Pultr and A. Tozzi, *Equationally closed subframes and representation of quotient spaces*, Cahiers Topologie Géom. Différentielle Catégoriques **34**(3) (1993), 167-183.
13. J. Schröder, *On sub-, pseudo-, and quasimaximal spaces*, Comment. Math. Univ. Carolinae **39** (1) (1998), 197-206.

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