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MINIMAL BASES, IDEAL EXTENSIONS, AND BASIC DUALITIES

MARCEL ERNÉ

Abstract. By a B-space, we mean a topological space with a minimal base (which is then even the least base). While the sober B-spaces are precisely the algebraic domains with the Scott topology, those T_0-B-spaces in which every monotone net having a join converges to that join are just the algebraic posets. An elementary construction shows that every topological space with a prescribed base is not only topologically dense but also meet-dense in a minimal B-space whose least base induces the original base. A similar construction provides a universal B-reflection for each base space. The sobrification of a B-space is the (Scott-topologized) ideal completion of its base set, consisting of all points having least neighborhoods. Combination of both reflections amounts to a universal B-sobrification.

We establish several equivalences and dualities between certain categories of spaces and categories of ordered sets, among them a duality between T_0-base spaces and so-called ideal spaces that induces a duality between T_0-B-spaces and ideal extensions, but also the Lawson duality of continuous and algebraic domains. As a result, algebraic posets provide convenient "computational models" not only for T_0-B-spaces, but even for arbitrary T_0-(base) spaces.

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Contents

0. Introduction: The Role of Minimal Bases

From the pioneering work of Alexandroff we know that the systems of all upper sets (or lower sets) in quasi-ordered sets (*qosets*) are precisely those topologies in which arbitrary intersections of open sets are open, today usually called *Alexandroff topologies* or *A-topologies* (Alexandroff himself termed them "discrete" [3]). Antisymmetry of the quasi-order is reflected by the T_0-axiom. By the *core* of a point in an arbitrary topological space we mean the intersection of its neighborhoods. We shall speak of an *A-space* if its topology is an Alexandroff topology. This is equivalent to saying that every core is open, and consequently the cores form the least base. (The "A-spaces" in the sense of Eršov [26] have a different, more general meaning and are closer to our *C-spaces* discussed in Section 3.)

If a space X has a minimal base then that base must already be contained in all others and consist of all open cores; but that does not mean that *all* cores would be open. Spaces with a minimal base are called *B-spaces* [17] or *monotope* (cf. [36]), and points having an open core (that is, a least neighborhood) are referred to as *base points*; we denote by $\mathrm{B}X$ the set of all base points, and we adopt the convention that *no topological base contains the empty set as a member.*

In the present note, we are particularly interested in "algebraic" representations of B-spaces and related types of spaces. By an *algebraic poset* we mean a partially ordered set (*poset*) in which every element is a directed supremum of compact elements, where c is *compact* if it belongs to every (directed) ideal having a join above c (in [18], compactness has a slightly stronger meaning). But, as in [10, 30, 44, 45], *we do not require up-completeness*; in other words, not all directed subsets need possess suprema. We shall give both a purely topological characterization and an ideal-theoretical description of such general algebraic posets: introducing appropriate morphism classes, we establish an isomorphism between the category of algebraic posets and that of so-called weak monotone convergence B-spaces on the one hand, and an equivalence (but also a duality) between algebraic posets and *algebraic ideal extensions* on the other hand. The resulting duality between weak monotone convergence B-spaces and algebraic ideal extensions is induced by a duality between T_0-base spaces and so-called *ideal spaces*, which in turn comes from an elementary self-duality for T_0-*cover spaces.* Furthermore, we refine the pointfree characterization of B-spaces to an equivalence between the category of T_0-B-spaces and the category of so-called *consistently prime-based superalgebraic lattices.*

B-spaces are much more general than A-spaces: while every subspace of an A-space is again an A-space, we shall represent *every base space*, that is, every topological space with a given base $\mathcal{B}$, as a topologically and order-theoretically dense subspace of some B-space whose least base is isomorphic to $\mathcal{B}$ via relativization. Moreover, we show that any base space has a unique minimal B-extension and also a categorical B-reflection (with respect to suitable morphisms), which in the T_0 case is (homeomorphic to) the minimal B-extension. The sobrification of an arbitrary B-space is a sober B-space, hence an algebraic domain carrying the Scott topology; the underlying poset is the ideal completion of the base point set (with the specialization order). Combining the aforementioned basic construction of the B-reflection with the sobrification, one arrives at a "B-sobrification" of the original base space. In particular, every T_0-(base) space is a strictly dense subspace of a sober B-space, in other words, of an algebraic domain. A direct approach to that result is obtained via the open filter space. Since the base of the subspace is isomorphic to the least base of the "algebraic model", and since bases are often much more economic than the whole topologies, this approach opens facilities to correlate topological properties of the represented space or its base with properties of the model, a method nowadays often used in domain theory and related fields. In fact, order-topological structures of that kind are helpful tools in various areas of theoretical computer science (see, for example, [2, 5, 10, 41, 44, 45]).

Surprisingly, even for arbitrary T_0-base spaces, a purely order-theoretical description is possible in terms of certain enriched algebraic posets. Though our "algebraic models" fail to be up-complete in general, they are *conditionally up-complete posets* (*cups*), i.e. every upper bounded directed subset has a join. It turns out that the algebraic cups are precisely the down-sets of algebraic domains. Under the equivalence between algebraic posets and weak monotone convergence B-spaces, the algebraic cups correspond to the conditional monotone convergence spaces (in which every upper bounded monotone net converges to its supremum). In particular, the algebraic domains correspond to the monotone convergence spaces with minimal bases; indeed, a B-space is sober iff it is a monotone convergence space. On the other hand, the equivalence between conditionally monotone convergence B-spaces and algebraic cups extends to one between arbitrary T_0-base spaces and *conditionally up-complete algebraic posets with point generators* (or *caps* for short); these are pairs (A, M) consisting of an algebraic cup A and a cofinal subset M such that every compact element of A is a meet of elements from M. Assigning to any such pair (A, M) the set M equipped with the base inherited from the minimal base of the Scott topology on A, one obtains an equivalence between the category of caps and the category of T_0-base spaces. On the other hand, passing to the set K of all compact elements, endowed with the system of all ideals $\{c \in K : c \leq m\}$ $(m \in M)$, one arrives at a duality between caps and ideal spaces. In [21], we apply our constructions to T_1-spaces, where the "algebraic modellization" is still more convenient.

1. Algebraic and Continuous Posets

Let us start with a short account of the vocabulary needed for the translation between order-theoretical and topological notions (see also [17, 18, 30]). The *down-sets* (or *lower sets*) Y of a poset or qoset $(X, \leq)$ are characterized by the condition $x \leq y \in Y \Rightarrow x \in Y$; thus, they are the unions of *principal ideals*

$$(y] = \downarrow y = \{x \in X : x \leq y\} \quad (y \in X)$$

and form both a closure system and a topology, the *lower Alexandroff topology* or *Alexandroff completion* $\mathcal{A}(X, \leq)$. The *up-sets* (or *upper sets*) constitute the *upper Alexandroff topology* $\alpha(X, \leq) = \mathcal{A}(X, \geq)$, whose least base consists of all *principal filters* (or *principal dual ideals*)

$$[x) = \uparrow x = \{y \in X : x \leq y\} \quad (x \in X).$$

The *closed* sets with respect to that topology are precisely the down-sets. Hence, the closure of a set Y in the Alexandroff space

$(X, \alpha(X, \leq))$ is the *down-closure*

$$\downarrow Y = \{x \in X : x \leq y \text{ for some } y \in Y\}.$$

Generally, we denote by $\mathcal{C}X$ ($\mathcal{O}X$) the lattice of all closed (open) sets of an arbitrary topological space X. The *specialization order* is given by

$$x \leq y \Leftrightarrow \mathcal{O}_x \subseteq \mathcal{O}_y\,, \text{ where } \mathcal{O}_x = \{U \in \mathcal{O}X : x \in U\}.$$

Notice that for any base $\mathcal{B}$ of X, we have

$$x \leq y \Leftrightarrow \mathcal{B}_x \subseteq \mathcal{B}_y\,, \text{ where } \mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}.$$

Order-theoretical statements about spaces refer to the specialization order. Thus, $\mathcal{C}X$ is always contained in $\mathcal{A}(X, \leq)$ (closed sets are down-sets), and equality characterizes A-spaces. The point closures are the principal ideals $(x]$, while the *core* of a point x, i.e. the intersection of all neighborhoods of x, is the principal filter $[x)$. The set of all *base points* is

$$\mathrm{B}X = \{x \in X : [x) \in \mathcal{O}X\}.$$

If Y is a subset of a qoset or space, is will be convenient to write

$$Yx = Y \cap (x] \text{ and } xY = [x) \cap Y.$$

By an *ideal* of a poset (or qoset) P, we mean here (as in [29] and [30], but deviating from [14], [35] and [28]) a directed down-set; a *filter* of P is an ideal of the dually ordered set. The ideals of P form the *(directed) ideal completion* $\mathrm{I}P$. The subposet of all principal ideals is designated by $\mathrm{i}P$. As usual, by a *join* (*meet*) we mean a least upper (greatest lower) bound; but notice that joins and meets are unique only in the case of partial orders or T_0-spaces. A tricky application of Zorn's Lemma (see [13] or [18]) shows:

Lemma 1. *If all (upper bounded) non-empty well-ordered subsets of a poset have a join then so do all (upper bounded) directed subsets.*

A poset in which all directed subsets have a join is called *up-complete*, *upper Dedekind-complete, directed complete* or a *dcpo* (sometimes also a *cpo*; in [9] and [18], every cpo has a least element). By a *conditionally up-complete poset* or *cup* we mean a poset whose upper bounded directed subsets have joins. Note that *every down-set of a dcpo is a cup.* It remains open whether, conversely, any cup is a down-set in some dcpo (but see Theorem 1).

There is almost no dissension about the notions of continuous and algebraic (complete) lattices [29], but in the more general setting of posets, several degrees of generalization have been considered. Sometimes the requirement of up-completeness is included in the definitions (see, for example, [29] and [32]–[43]), sometimes not (see e.g. [10, 30, 44, 45]). For a discussion and comparison of various reasonable notions of continuity for posets, the reader may consult [11] and [19]. In the present context, it appears useful to follow the more general trace, where up-completeness is not assumed *a priori*. Thus, we mean by a *continuous poset* one in which for every element y there is a least ideal having a join above y, called the *way-below ideal of* y and denoted by $\Downarrow y$. One writes $x \ll y$ for $x \in \Downarrow y$ and calls the elements x with $x \ll x$ *compact*. $\mathrm{K}P$ will designate the subposet of all compact elements of a poset P. An *algebraic poset* is then characterized by the property that each element is the join of a directed set of compact elements. In that case, the way-below ideal is the down-set generated by these compact elements, and consequently, the poset must be continuous. Up-complete continuous posets are also called *(continuous) domains* [30], but some authors use the name *"domain"* for more special, more general or even unrelated notions (cf. [2]). Concerning algebraic domains, see e.g. [16, 30, 35, 40, 45], and for generalizations to so-called Z-algebraic posets, [18] and [47].

The *Scott topology* σP on an arbitrary poset $P = (X, \leq)$ consists of all up-sets U such that any directed set having a supremum that belongs to U must already meet U. In a continuous poset, the *way-below relation* $\ll$ is idempotent (*"interpolation property"*), and the sets $\Uparrow x = \{y \in X : x \ll y\}$ form a base for the Scott topology. As in [29], ΣP will designate the topological space $(X, \sigma P)$. Thus, Σ gives rise to a concrete functor (keeping the underlying maps of the morphisms fixed) from the category of posets and maps preserving directed joins to the category of T_0-spaces. The following facts are folklore of domain theory (and proof ideas are to be found in [30]):

Proposition 1. *If C is a down-set of an algebraic (respectively, continuous) domain A, then C is an algebraic (respectively, continuous) cup with*

$$x \ll y \text{ in } C \;\Leftrightarrow\; x \ll y \text{ in } A \quad \text{for all } x, y \in C.$$

In particular, $\mathrm{K}C = C \cap \mathrm{K}A$ *and* $\sigma C = \sigma A|_C$.

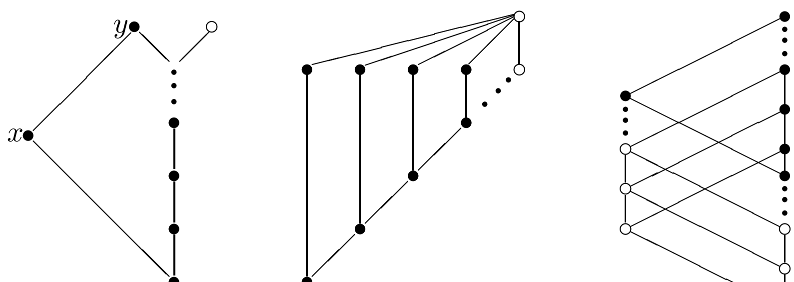
The statements in the previous proposition are rather subtle:

(1) If $C = Ay$ is a principal ideal in an algebraic poset A and C is a complete lattice in its own right, it need not even be continuous; furthermore, an element $x \in C$ that is compact in A may fail to be compact in C.

(2) A principal ideal in an algebraic domain is again one, but even complete algebraic lattices may possess down-sets that are not domains.

(3) Although all intervals of complete algebraic lattices are algebraic, an interval or principal *filter* in an algebraic domain need not even be continuous [16, 30].

Respective counterexamples are sketched in the subsequent diagrams, where the subposets are indicated by bold black dots.



(1) (2) (3)

Theorem 1. *The algebraic cups are the down-sets of algebraic domains.*

Proof. For any algebraic cup A, there is a well-defined embedding

$$\iota : A \to \mathrm{IK}A,\ x \mapsto Kx = \mathrm{K}A \cap (x]$$

in the algebraic domain $\mathrm{IK}A$ of all ideals of the subposet $K = \mathrm{K}A$. The image $\iota(A)$ is a *down-closed ideal extension*, that is, a down-set of the ideal completion $\mathrm{IK}A$: indeed, for $I \in \mathrm{IK}A$, the inclusion $I \subseteq Kx$ entails that I has the upper bound x, hence a join a in A, so I coincides with Ka, by compactness of the elements in K and directedness of I. Replacing the ideals Kx by their preimages x, one obtains an algebraic domain that contains A as a down-set. $\square$

Specific algebraic domains are the *noetherian posets*, in which all ascending sequences become stationary (*Ascending Chain Condition*). Using the Principle of Dependent Choices (see e.g. [27]), a weak form of the Axiom of Choice, one observes:

Corollary 1. *Noetherian posets are characterized by each of the following properties:*

- *every non-empty subset has a maximal element,*
- *every directed subset has a greatest element,*
- *every non-empty well-ordered subset has a compact join,*
- *the poset is up-complete, and every element is compact.*

We call a poset *pre-noetherian* if each of its elements is compact.

At the end of this preliminary section, let us recall a convenient characterization of *completely distributive lattices* L (cf. Raney [46]): for every element $x \in L$, there is a least down-set with join above x. By reasons of analogy, completely distributive (complete) lattices are also called *supercontinuous*, and algebraic completely distributive lattices are called *superalgebraic*, being join-generated by supercompact (that is, completely join-prime) elements; by definition, an element p is *supercompact* or *completely join-prime* iff it is contained in any down-set whose join dominates p.

2. Algebraic Posets as Ideal Extensions

For algebraic posets, one would expect an ideal-theoretical characterization, in the same vein as the algebraic domains are, up to isomorphism, the *ideal completions* $\mathrm{I}P$, consisting of all ideals of arbitrary posets P, which in turn are isomorphic to the subposet $\mathrm{K}A$ of all compact elements of $A = \mathrm{I}P$ (the principal ideals of P). But there is a crucial difference between that classical result and the situation of arbitrary posets: if up-completeness is dropped, algebraic posets need not be determined (up to isomorphism) by their subposet of compact elements, and a representing ideal system will not be the whole ideal completion. Therefore, we need the more general notion of *ideal extensions* of a poset (cf. [14]). These are collections of ideals that contain at least all principal ideals; we call such an ideal extension $\mathcal{I}$ *algebraic* if for any directed subsystem of $\mathcal{I}$ possessing a join in $\mathcal{I}$, this join is the set-theoretical union.

Among all ideal extensions, the algebraic ones may be characterized by the property that the principal ideals are precisely the compact members. Special algebraic ideal extensions are the *down-closed* ideal extensions, i.e. those which are down-sets of the ideal completion. By Theorem 1 and its proof, they correspond to the algebraic cups. Note that an up-complete algebraic ideal extension must already be the whole ideal completion. The following result is probably known to workers in domain theory, so we omit the proof; for a similar result about "unionized" subset systems, see [47].

Lemma 2. *A poset A is algebraic iff it is isomorphic to an algebraic ideal extension of some poset (that is isomorphic to* $\mathrm{K}A$*).*

In order to extend the known equivalence between ideal completions and algebraic domains (see e.g. [16] or [18]) to arbitrary ideal extensions $\mathcal{I}$, it is helpful to regard them as pairs $(P,\mathcal{I})$, where P is the underlying poset. (Notice that P is completely determined by $\mathcal{I}$, being the union $\bigcup\mathcal{I}$ together with the order given by $x \leq y$ iff $y \in I \in \mathcal{I}$ implies $x \in I$.) As morphisms between ideal extensions $(P,\mathcal{I})$ and $(P',\mathcal{I}')$ we take *isotone quasi-closed maps*, i.e. maps $\varphi : P \to P'$ such that $x \leq y$ implies $\varphi(x) \leq \varphi(y)$, and $I \in \mathcal{I}$ implies $\downarrow \varphi(I) \in \mathcal{I}'$.

On the other hand, we consider so-called *directed based sets* (A, B), consisting of a quasi-ordered set A and a subset B such that each Bx is directed and $x \leq y \Leftrightarrow Bx \subseteq By$ for all $x, y \in A$. In case A is a poset, the latter condition means join-density of B in A, and we speak of a *directed based poset*. As morphisms between directed based (po-)sets (A, B) and (A', B'), it is reasonable to take isotone (i.e. order preserving) maps $\phi : A \to A'$ that *preserve the bases* (i.e. $\phi(B) \subseteq B'$) and *interpolate*: for $a \in A$ and $b' \in B'$ with $b' \leq \phi(a)$, there is a $b \in B$ such that $b \leq a$ and $b' \leq \phi(b)$. In the full subcategory of algebraic posets, regarded as directed based posets $(A, \mathrm{K}A)$, the morphisms preserve not only compactness but also directed joins. For any morphism $\phi : A \to A'$ between directed based posets (A, B) and (A', B'), the restriction $\varphi : B \to B'$ with $\varphi(b) = \phi(b)$ is well-defined, isotone, and satisfies $\downarrow \varphi(Ba) = B'\phi(a)$ for each $a \in A$. Thus, φ is in fact a morphism between the ideal extensions $\mathrm{E}(A, B) = (B, \{Ba : a \in A\})$ and $\mathrm{E}(A', B') = (B', \{B'a' : a' \in A'\})$. By join-density of B in A, the poset A is isomorphic to $\{Ba : a \in A\}$.

Moreover, the morphism ϕ may be reconstructed from φ by the equation $\phi(a) = \bigvee \varphi(Ba)$ (using interpolation and join-density of B').

Conversely, any morphism φ between ideal extensions $(P, \mathcal{I})$ and $(P', \mathcal{I}')$ gives rise to a map

$$\phi = \mathrm{I}^{+}\varphi : \mathcal{I} \to \mathcal{I}',\ I \mapsto {\downarrow}\varphi(I),$$

and it is easy to verify that ϕ is a morphism between the directed based posets $(\mathcal{I}, \mathrm{i}P)$ and $(\mathcal{I}', \mathrm{i}P')$. Furthermore, the restricted map $\phi|_{\mathrm{i}P} : \mathrm{i}P \to \mathrm{i}P'$, sending Pa to $P'\varphi(a)$, is essentially the same as the original φ. In all, we have the following:

Theorem 2. *Mapping each directed based poset (A, B) to the ideal extension $(B, \{Ba : a \in A\})$ and restricting each morphism to the bases, one obtains an equivalence* E *between the category of directed based posets and the category of ideal extensions. The inverse equivalence is established by the functor* I^{+}*.*

For the full subcategory of algebraic posets with maps preserving compactness and directed joins, we conclude (using Theorem 1):

Corollary 2. *Under the above equivalence, algebraic posets correspond to algebraic ideal extensions, algebraic cups to down-closed ideal extensions, and algebraic domains to ideal completions.*

Though an algebraic poset is neither determined by the lattice of its Scott-closed sets nor by the subposet of its compact elements, these two structures determine each other up to isomorphism.

Proposition 2. *Via relativization, the Scott completion $\mathcal{C}\Sigma A$ of an algebraic poset A is isomorphic to the Alexandroff completion $\mathcal{A}\mathrm{K}A$ of the subposet $\mathrm{K}A$ of all compact elements. Hence, $\mathrm{K}A$ is isomorphic to the poset of all supercompact elements of $\mathcal{C}\Sigma A$.*

Proof. Put $K = \mathrm{K}A$. Let us show that an isomorphism between $\mathcal{C}\Sigma A$ and $\mathcal{A}K$ is provided by the relativization map $C \mapsto C \cap K$. In the reverse direction, one associates with every down-set $D \in \mathcal{A}K$ the σ-closure $\overline{D}$ in A. Then, for any $C \in \mathcal{C}\Sigma A$ and $a \in C$, one gets $a = \bigvee Ka \in \overline{C \cap K}$, because Ka is directed, and the inclusion $\overline{C \cap K} \subseteq C$ is obvious. On the other hand, any down-set D in K is clearly contained in $\overline{D} \cap K$, and each $x \in \overline{D} \cap K$ belongs to D since that down-set meets the open core $[x)$. Thus, the closure map is in fact inverse to the relativization map. □

3. A-, B- and C-Spaces

In some sense, the "continuous" or "local" counterparts of the "rather discrete" B-spaces (having minimal bases) are the so-called *core spaces* or *C-spaces* [11, 17], in which every point has a neighborhood base consisting of (not necessarily open) cores.

For easy reference, let us list various known characterizations of A-, B- and C-spaces (cf. [17]).

Proposition 3. *Let X be a topological space.*

A. *The following are equivalent:*

(a) *X is an A-space.*

(b) *$\mathcal{C}X$ is a topology.*

(c) *$\mathcal{O}X$ is a closure system.*

(d) *The closure operator preserves arbitrary unions.*

(e) *Each point has a least neighborhood (i.e. all cores are open).*

(f) *X has a least base, consisting of all cores.*

(g) *The open sets are the up-sets of a (unique) quasi-order.*

B. *The following are equivalent:*

(a) *X is a B-space.*

(b) *$\mathcal{C}X$ is a (super)algebraic lattice.*

(c) *$\mathcal{O}X$ is a superalgebraic lattice.*

(d) *The closure operator is given by $x \in \overline{Y} \Leftrightarrow \mathrm{B}X \cap (x] \subseteq {\downarrow} Y$.*

(e) *Each point has a neighborhood base of open cores.*

(f) *X has a base consisting of cores.*

(g) *The open sets are the up-sets generated by subsets of $\mathrm{B}X$.*

C. *The following are equivalent:*

(a) *X is a C-space.*

(b) *$\mathcal{C}X$ is a (super)continuous lattice.*

(c) *$\mathcal{O}X$ is a supercontinuous lattice.*

(d) *The closure operator preserves intersections of down-sets.*

(e) *X is locally compact and has a base of open filters.*

(f) *X is locally supercompact.*

(g) *The open sets are the up-sets $Y\rho$ of an idempotent relation ρ.*

Corollary 3. *A topological space X is a B-space iff it has an A-subspace Y with $\mathcal{O}X \simeq \mathcal{O}Y$ (namely the space of all base points).*

Part A of Theorem 3 is essentially due to Alexandroff [3]. Notice that the closed sets of a B-space X form an algebraic lattice, but not necessarily an algebraic closure system, which would mean that X is an A-space. In Part C, supercompactness of a set C means that any open cover of C has already a member that contains C, and a *locally supercompact space* is one in which every point has a neighborhood base of supercompact sets. The cores are precisely the supercompact up-sets. (A weaker notion of supercompactness, occurring in other topological contexts, is not relevant to our purposes.) Implicitly, some of the equivalences in Parts B and C arise in the work of Hoffmann [32, 33, 34]. But, in [34], an incorrect link was taken over from [4] (corrected in [35]). For the first comprehensive characterization of C-spaces, see [11], and [24] for the more general case of arbitrary closure spaces. The only implication in Part C that probably requires a choice principle is (c)$\Rightarrow$(e) :

Lemma 3. *Given Dependent Choices, every supercontinuous frame is spatial; in particular, C-spaces then have a base of open filters.*

Examples. (A) Every semigroup, endowed with the closure system of all ideals, is an A-space.

(B) Every ordinal with the Scott topology is a B-space, but not an A-space unless it is finite.

(C) Powers of the real unit interval with the Scott topology are C-spaces in which no proper core is open, so they are not B-spaces.

Thus, every A-space is a B-space, and every B-space is a C-space, but neither of these implications may be inverted generally.

In [17], purely order-theoretical descriptions have been given for A-, B- and C-spaces (cf. Proposition 3 (g)). Since these characterizations will be of interest for some of our later investigations, we recall here briefly the main definitions and facts. A relation ρ on a set X is *interpolating* if

$$x \,\rho\, z \text{ implies that there is a } y \text{ with } x \,\rho\, y \,\rho\, z,$$

and *basically reflexive* if

$$x \,\rho\, z \text{ implies that there is a } y \text{ with } x \,\rho\, y \,\rho\, y \,\rho\, z.$$

Note that a relation is idempotent iff it is both transitive and interpolating. By an *auxiliary order* or *auxiliary relation* [29, 30] for a quasi-order $\leq$ there is meant a transitive relation ρ such that the sets $\rho\, y = \{x : x\,\rho\, y\}$ are ideals with $x \leq y \;\Leftrightarrow\; \rho\, x \subseteq \rho\, y$. A map φ between auxiliary ordered sets (X, ρ) and (X', ρ') is *interpolating* if for $x'\rho'\,\varphi(z)$ there is a y with $x'\rho'\,\varphi(y)$ and $y\,\rho\, z$. Thus, ρ is interpolating iff so is the identity map on X. Interpolating maps are of basic relevance to the theory of computation, where the elements x with $x\,\rho\, y$ are interpreted as "states of approximation" for the object y to be computed. The preceding definition is in accordance with that from Section 2, because there is a one-to-one correspondence between basically reflexive auxiliary orders and B-spaces (by taking the elements with $x\,\rho\, x$ as base points; cf [24]).

Theorem 3. *Assigning to each topological space X its interior relation ρ, given by $x\,\rho\, y$ iff y is an inner point of the core of x, one obtains one-to-one correspondences between*

- *A-spaces and reflexive auxiliary orders, i. e. quasi-orders,*
- *B-spaces and basically reflexive auxiliary orders,*
- *C-spaces and interpolating (idempotent) auxiliary orders.*

These correspondences give rise to isomorphisms between the respective categories of spaces and of auxiliary orders with interpolating isotone maps. The associated quasi-orders are the specialization orders (hence antisymmetric iff the T_0-axiom holds).

4. Monotone Convergence Spaces

Aiming for a topological description of pre-noetherian, algebraic or continuous (but not necessarily up-complete) posets, we need a few more order-topological notions. For any topological space X, we mean by the *patch topology* the join of the given topology and the *lower topology*, which has the cores as subbasic *closed* sets. In particular, the *Lawson topology* [29, 30, 40] is the patch topology of the Scott topology. On $\mathbb{R}^n$, it coincides with the Euclidean topology. Therefore, the subsequent considerations generalize known facts from real analysis. By definition, a net $\nu : D \to X$

- is *monotone (increasing)* if $m \leq n$ implies $\nu(m) \leq \nu(n)$,
- is *upper bounded (ub)* by $x \in X$ if $\nu(n) \leq x$ for all $n \in D$,
- has a *least upper bound* if this holds for its range $\nu(D)$.

Lemma 4. *For a monotone net $\nu : D \to X$ in a space X and for any point $x \in X$, the following conditions are equivalent:*

(a) *x is a (least) upper bound and a limit of ν.*
(b) *x is an upper bound and an accumulation point of ν.*
(c) *x is a limit of ν in the space with the patch topology.*
(d) *The principal ideal $(x]$ is the closure of $\nu(D)$ in X.*

In T_0-spaces, these facts imply that x is the unique supremum of ν.

Proof. (a)$\Rightarrow$(b) is clear.

(b)$\Rightarrow$(c). Any neighborhood U of x that is open in the original topology contains some $\nu(m)$, and for all $n \in D$ with $n \geq m$, it follows that $\nu(m) \leq \nu(n) \in \uparrow U = U$. For $x \in X \setminus [y)$, we have $\nu(D) \subseteq (x] \subseteq X \setminus [y)$. Thus, ν converges to x both in the original and in the lower topology, hence also in the patch topology.

(c)$\Rightarrow$(d). By (c), x is *a fortiori* a limit of ν in the original space. Hence, any closed set containing the range $\nu(D)$ must also contain the limit x and, being a down-set, the whole point closure $(x]$. It remains to verify that, conversely, $\nu(D)$ is contained in $(x]$. If not, choose an $m \in D$ with $\nu(m) \not\leq x$; then x lies in the patch-open set $U = X \setminus [\nu(m))$. Thus, there is an $n \in D$ with $\nu(n) \in U$ and $n \geq m$, so that $\nu(n) \geq \nu(m)$, in contrast to $\nu(n) \in U = X \setminus [\nu(m))$.

(d)$\Rightarrow$(a). By (d), x is an upper bound of $\nu(D) \subseteq \overline{\nu(D)}$, and if y is any other upper bound of $\nu(D)$, it follows that $(x] = \overline{\nu(D)} \subseteq (y]$, i.e. $x \leq y$. For each $U \in \mathcal{O}_x$, we find an $m \in D$ with $\nu(m) \in U$ (since $x \in \overline{\nu(D)}$). By monotonicity of ν and directedness of D, this already entails that ν converges to x. $\square$

A T_0-space is called

– *sober* if the point closures are the only irreducible closed sets,
– a *monotone convergence (mc) space* if every monotone net in the space has a join (supremum) and converges to it,
– a *conditional monotone convergence (cmc) space* if every ub monotone net in the space has a join and converges to it,
– *a weak monotone convergence space* or simply a *weak space* if any monotone net having a join converges to that join.

In each of these definitions, "monotone net" may be replaced with "directed subset" (regarded as a net) or with "ideal". From Lemma 4, one immediately deduces the fact that the mc spaces are those T_0-spaces in which the closure of any directed subset is a point closure, i.e. the *d-spaces* or *temperate spaces* in the sense of Wyler [48]. Similarly, the cmc spaces are those T_0-spaces in which the closure of any upper bounded directed subset is a point closure; and the weak mc spaces are just those T_0-spaces whose topology is weaker than the Scott topology (w.r.t. specialization), whence they simply are referred to as *weak spaces.* (Conditional) monotone convergence spaces, also called *order-consistent spaces* in [29], are well known from real analysis; moreover, they play quite a role in domain theory (see [30] and the references therein).

Corollary 4. *Among the T_0-spaces, the cmc spaces are characterized by each of the following conditions:*

(a) *Every ub monotone net has a join and converges to it.*
(b) *Every ub monotone net converges in the patch topology.*
(c) *Every ub directed set has a join, and the space is weak.*
(d) *Every ub directed set has a join and meets each neighborhood of that join.*

Analogous characterizations hold for mc spaces. The (conditional) mc spaces are precisely the (conditionally) up-complete weak spaces.

None of the implications

sober $\Rightarrow$ *mc space* $\Rightarrow$ *cmc space* $\Rightarrow$ *weak (mc) space*

may be inverted in general (but see Proposition 5).

Examples.

(1) Any complete chain (e.g. the unit interval) with the Scott topology is a sober C-space.
(2) While every dcpo whith the Scott topology is an mc space, Johnstone's example [39] shows that a dcpo with the Scott topology is not always sober.
(3) The real line with the Scott topology is a cmc space but not up-complete, hence not an mc space.
(4) The rational line with the Scott topology is a weak space but not conditionally up-complete, hence not a cmc space.

(5) An infinite set with the cofinite topology is an mc space but not sober, and its topology is strictly coarser than the Scott topology of the specialization order (which is discrete).

The topological characterization of pre-noetherian, algebraic and continuous posets is prepared by calling a poset P an

- *A-poset* if the Scott and Alexandroff topology agree on P,
- *B-poset* if the Scott topology on P has a least base,
- *C-poset* if the Scott topology on P is locally supercompact.

Proposition 4. *For each element y of an arbitrary poset P, put $P^\sigma_y = \{x \in P : \exists U \in \sigma P\,(y \in U \subseteq [x))\}$. Then P is an*

- *A-poset iff $P^\sigma_y = (y]$ for each $y \in P$,*
- *B-poset iff each y is the directed join of $P^\sigma_y = \mathrm{K}P \cap (y]$,*
- *C-poset iff each P^σ_y is directed with join y.*

The A-posets are the pre-noetherian posets, the B-posets are the algebraic posets, and the C-posets are the continuous posets.

Proof. In order to check that if P is a C-poset then each P^σ_y is directed with join y, consider a finite subset F of P^σ_y and choose open sets U_x with $y \in U_x \subseteq [x)$ $(x \in F)$. Since $X = \Sigma P$ is a C-space, we find an open U and a point u such that

$$y \in U \subseteq [u) \subseteq \bigcap\{U_x : x \in F\} \subseteq \bigcap\{[x) : x \in F\}.$$

Hence, u is an upper bound of F in P^σ_y, proving directedness of P^σ_y. For $y \not\leq z$, the set $V = X \setminus (z]$ is an open neighborhood of y, and as before we find a point $x \in P^\sigma_y$ with $x \not\leq z$, which shows that y is the join of P^σ_y. Conversely, if P^σ_y is directed with join y then y has a neighborhood base of cores in ΣP, because for $y \in V \in \sigma P$ there is an $x \in P^\sigma_y \cap V$, hence an open subset U with $y \in U \subseteq [x) \subseteq V$.

Concerning B-posets, note that an element c is compact iff $[c)$ is an open core in the Scott topology; consequently, for an algebraic poset, the set of all open cores is the least base of the Scott topology. The remaining statements about A-, B- and C-spaces are easily verified (in ZF without choice). In order to see that a continuous poset is a C-poset, use the fact that the sets $\Uparrow x = \{y \in P : x \ll y\}$ are Scott open, by the interpolation property (see [29, 30]). $\square$

The concrete *specialization functor* Σ^- associates with any topological space the underlying set equipped with the specialization order (and keeps the underlying maps fixed). This functor gives rise to various categorical isomorphisms:

Theorem 4. *The Scott functor* Σ *and the specialization functor* Σ^- *induce mutually inverse isomorphisms between the categories of*

– *weak A-spaces and A-posets (pre-noetherian posets),*
– *weak B-spaces and B-posets (algebraic posets),*
– *weak C-spaces and C-posets (continuous posets).*

The spaces on the left hand are cmc spaces iff the corresponding posets on the right hand are cups, and the spaces are mc spaces iff the corresponding posets are up-complete (domains).

Proof. We emphasize again that no choice principles are needed for the subsequent arguments. For any poset P, it is clear that ΣP is a weak space with $\Sigma^-\Sigma P = P$. If P is a C-poset then ΣP is a C-space, by Proposition 4.

Conversely, if X is an arbitrary C-space then its topology is finer than the Scott topology on $P = \Sigma^- X$: indeed, for each $y \in X$, the set of all points x such that the core $[x)$ is a neighborhood for y is directed with join y, by the neighborhood base property; hence, every Scott-open neighborhood of y must contain such cores $[x)$ and is therefore open in the given C-space topology. Thus, a weak C-space X carries the Scott topology, i.e. $X = \Sigma\Sigma^- X$, and $\Sigma^- X$ is a C-poset. In all, this establishes the claimed isomorphism between the category of C-posets (with maps preserving directed joins) and the category of weak C-spaces. The cases of A- and B-spaces are easily accomplished.

The isomorphisms between the respective categories of (conditionally) up-complete posets and (conditional) monotone convergence spaces are now immediate conseqences of Corollary 4. □

Theorem 4 shows, among other interesting correlations between ordered and topological structures, that the topology of any weak monotone convergence space with a minimal base is entirely determined by its specialization order: it must be the associated Scott topology.

5. Sober A-, B- and C-Spaces

Examples (2) and (5) in the previous section have shown that the class of sober spaces is properly contained in that of all monotone convergence spaces. Nevertheless, one can prove (using the interior relation ρ from Theorem 3):

Proposition 5. *A C-space (in particular, an A- or B-space) is sober iff it is a monotone convergence space.*

Proof. Let C be an irreducible closed set in a C-space X that is also a monotone convergence space. Then the set

$$\rho C = \{x \in X : \exists y \in C \ (x \,\rho\, y)\} = \{x \in X : C \cap [x)^\circ \neq \emptyset\}$$

is directed: given a finite $F \subseteq \rho C$, irreducibility of C yields a $y \in C \cap \bigcap\{[x)^\circ : x \in F\}$, and as y has a neighborhood base of cores, we find some $w \in \bigcap\{[x)^\circ : x \in F\}$ with $y \in C \cap [w)^\circ \neq \emptyset$. Thus, w is an upper bound of F in ρC. By up-completeness, ρC has a join z, and by weakness, it follows that C is the closure of $\{z\}$: indeed, $x \in \rho C$ implies $C \cap [x) \neq \emptyset$, *a fortiori* $x \in C$, and consequently $z = \bigvee \rho C$ lies in C. On the other hand, every neighborhood of any $y \in C$ intersects ρC, whence $C \subseteq \overline{\rho C} \subseteq (z] = \overline{\{z\}}$. □

Now, from Theorem 4, we immediately infer (cf. [17]):

Theorem 5. *Via the specialization functor and the Scott functor,*

- *the sober A-spaces correspond to the noetherian domains,*
- *the sober B-spaces correspond to the algebraic domains,*
- *the sober C-spaces correspond to the continuous domains.*

Concerning the case of B-spaces ("monotope spaces"), see also Hofmann and Mislove [36]. For generalizations to $\mathcal{Z}$-sober spaces, $\mathcal{Z}$-algebraic posets and $\mathcal{Z}$-continuous posets, see [18] and [19].

Whereas soberness is extremely non-invariant under lattice isomorphisms (for any sober space there is a non-sober one having an isomorphic lattice of open sets), it turns out that within the class of T_0-A-spaces, soberness may be characterized by a lattice-invariant property. By a *superatomic* lattice, we mean a superalgebraic lattice in which every non-trivial interval has an atom; in superalgebraic lattices, the latter is equivalent to the existence of (unique) irredundant meet-decompositions into completely meet-irreducible elements (see e.g. [8]). The following fact was shown in [17]:

Lemma 5. *A T_0-A-space is sober iff its topology is superatomic.*

For any complete lattice L, the $\vee$-*spectrum* $\check{\Sigma}L$ is the set $\check{\mathrm{P}}L$ of all $\vee$-primes endowed with the hull-kernel topology, whose open sets are all sets of the form $\check{\mathrm{P}}L \setminus (a]$ $(a \in L)$. The set $\hat{\mathrm{P}}L$ of all $\wedge$-primes and the ($\wedge$-)*spectrum* $\hat{\Sigma}L$ are defined dually; hence, $\hat{\Sigma}L$ is the $\vee$-spectrum of the dual lattice. It is well known that both spectra are always sober spaces, and that the category of spatial frames is dual to the category of sober spaces via the spectrum functor and the open set lattice functor in the reverse direction [29, 30, 38].

Summarizing the previous facts, we arrive at the following basic results of domain theory, parts of which are due to Hoffmann [34], Hofmann and Mislove [36], Lawson [40] and others (see also [30]). As we have seen, the approach via A-, B- and C-spaces is very natural and convenient (cf. [17] and [18]).

Corollary 5. *For a topological space X with topology τ, the following conditions are equivalent:*

(1A) *$\Sigma^{-}X$ is a noetherian poset whose Scott topology is τ.*
(2A) *X is an mc space and $\mathcal{O}X$ is superatomic.*
(3A) *X is a sober A-space.*

Similarly, the following statements are equivalent:

(1B) *$\Sigma^{-}X$ is an algebraic domain whose Scott topology is τ.*
(2B) *X is an mc space and $\mathcal{O}X$ is superalgebraic.*
(3B) *X is a sober B-space.*

Moreover, the following statements are equivalent:

(1C) *$\Sigma^{-}X$ is a continuous domain whose Scott topology is τ.*
(2C) *X is an mc space and $\mathcal{O}X$ is supercontinuous.*
(3C) *X is a sober C-space.*

Via the open set functor and the spectrum functor, the following pairs of categories are duals of each other:

- *sober A-spaces (A-posets) and superatomic lattices*
- *sober B-spaces (B-posets) and superalgebraic lattices*
- *sober C-spaces (C-posets) and supercontinuous lattices.*

6. Sobrification and Duality

Abstractly (that is, categorically), the *sobrification* $\Sigma^{\circ}X$ of a space X is the sober reflection of that space; concretely, it may be constructed by taking the $\wedge$-spectrum $\hat{\Sigma}\mathcal{O}X$ of the open set lattice [29, 30, 38], or equivalently, the $\vee$-spectrum $\check{\Sigma}\mathcal{C}X$ of the closed set lattice [32]. The latter representation is often more convenient, because in that version, the sobrification map sends each point to its closure. The following observation is due to Hoffmann [32]:

Theorem 6. A. *The sobrification of an A-space is the ideal completion of the associated quasi-ordered set with the Scott topology.*

This result may be extended, *mutatis mutandis*, to B-spaces and even to C-spaces. The original space and its sobrification always have isomorphic lattices of closed (or open) sets. We conclude from Proposition 3. B that the sobrification of a B-space is a sober B-space, hence an algebraic domain with the Scott topology, by Theorem 5. As expected, that algebraic domain arises as the ideal completion of a subposet of the original space.

Theorem 6. B. *For any B-space X with base point set* $\mathrm{B}X$*, the sobrification is the ideal completion* $\mathrm{IB}X$*, endowed with the Scott topology. In the T_0 case, the sobrification map induces a bijection between the base points of X and the compact elements of* $\mathrm{IB}X$*.*

Proof. Mutually inverse isomorphisms between the closed set lattice $\mathcal{C}X$ and the down-set lattice $\mathcal{A}\mathrm{B}X$ are given by mapping each closed set C to its trace $C \cap \mathrm{B}X$ and, in the reverse direction, each down-set of $\mathrm{B}X$ to its closure (cf. Proposition 2). Since the irreducible closed sets are the $\vee$-prime members of $\mathcal{C}X$ and the ideals of $\mathrm{B}X$ are the $\vee$-prime members of $\mathcal{A}\mathrm{BX}$, the relativization map yields an isomorphism between $\check{\mathrm{P}}\mathcal{C}X$ and $\mathrm{IB}X$. Consequently, by Theorem 5, we have a homeomorphism between $\check{\Sigma}\mathcal{C}X$ and $\Sigma\mathrm{IB}X$. Now, the open cores are precisely the supercompact elements of the open set lattices. It follows that the sobrification map induces a bijection between the base points of the original space and those of its sobrification, i.e. the compact elements of the domain $\mathrm{IB}X$. $\square$

For the case of an algebraic poset, regarded as a B-space with the Scott topology, this amounts to a result due to Mislove [45]:

Corollary 6. *For any algebraic poset P, the sobrification of ΣP is the Scott space ΣA of an algebraic domain A whose compact elements coincide (up to the embedding) with those of the poset P.*

There is a nice analogue of Theorem 6.B for C-spaces, which occurs implicitly in [17] and [42]. By Theorem 3, every C-space is completely determined by its interior relation $x\,\rho\,y \;\Leftrightarrow\; y \in [x)^{\circ}$. Generally, for any idempotent relation ρ, a *rounded down-set* or *ρ-down-set* is a set of the form $\rho Y = \{x : x\,\rho\,y \text{ for some } y \in Y\}$; if it is an ideal (w.r.t. the quasi-order $\leq$ given by $x \leq y \Leftrightarrow \rho x \subseteq \rho y$), one speaks of a *rounded ideal* [30, 31, 42]); *rounded up-sets* and *rounded filters* are defined dually. Now, we observe (cf. [6]):

Lemma 6. *For any C-space X, the rounded up-sets are the open sets, and the rounded filters are the $\vee$-prime open sets. On the other hand, the assignments $C \mapsto \rho C$ and $D \mapsto \overline{D}$ yield mutually inverse isomorphisms between the supercontinuous lattice of closed sets and the lattice of rounded down-sets. Under these isomorphisms, the irreducible closed sets correspond to the rounded ideals.*

Hence, the rounded ideals may serve as the points of the sobrification, whose topology is supercontinuous. By Proposition 3 C and Theorem 5, it is the Scott topology. Thus, one has (cf. [42]):

Theorem 6. C. *Up to homeomorphism, the sobrification of any C-space X is the poset $\mathrm{I}^{\circ}X$ of all rounded ideals, equipped with the Scott topology. Thus, $\mathrm{I}^{\circ}X$ is a continuous domain.*

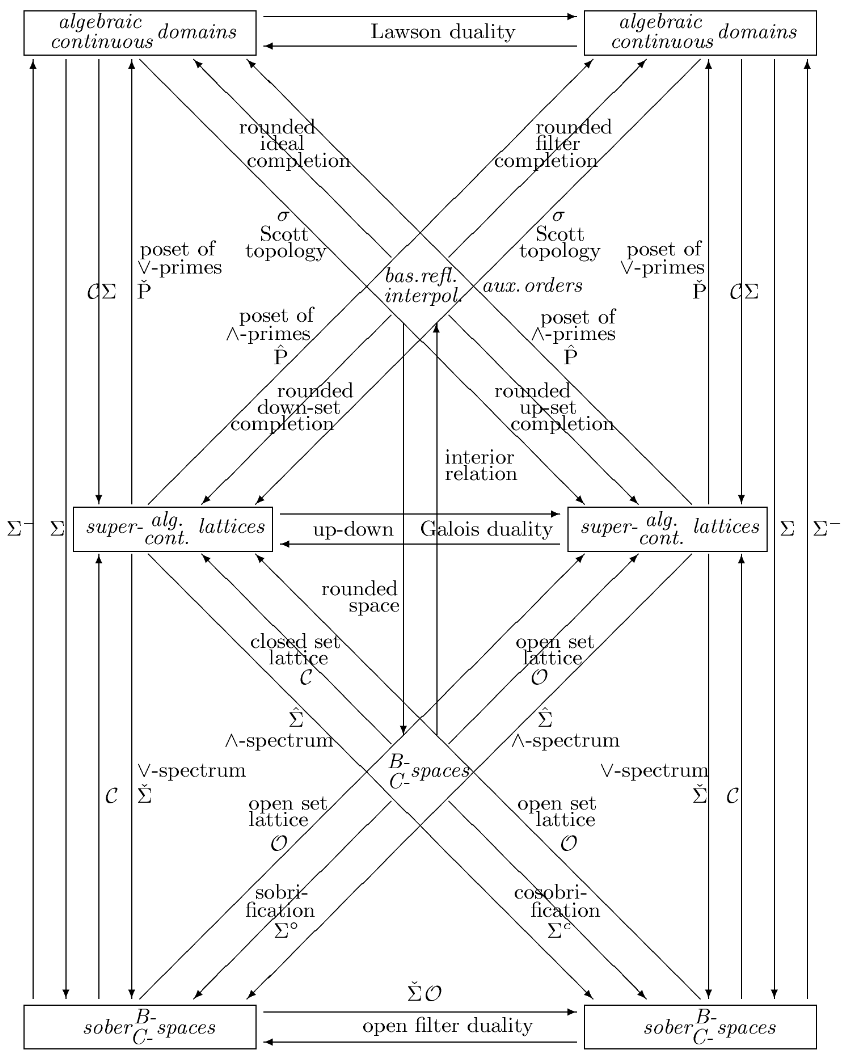
With any topological space X, one may associate not only its sobrification $\Sigma^{\circ}X \simeq \check{\Sigma}\mathcal{C}X \simeq \hat{\Sigma}\mathcal{O}X$, but also the *"cosobrification"* $\Sigma^{c}X \simeq \check{\Sigma}\mathcal{O}X \simeq \hat{\Sigma}\mathcal{C}X$, i.e. the $\vee$-($\wedge$-)spectrum of the open (closed) set lattice. By an *A^*-space*, we mean a B-space whose poset of $\vee$-prime open sets satisfies the Ascending Chain Condition. From the general Stone duality developed in [18] and [20], one derives:

Proposition 6. *The contravariant cosobrification functor Σ^{c}, sending any continuous map φ to the preimage map $\varphi^{\leftarrow}$, induces*

- *a duality between sober A-spaces and sober A^*-spaces*
- *a self-duality for sober B-spaces (hence for algebraic domains)*
- *a self-duality for sober C-spaces (hence for continuous domains).*

The self-duality of continuous domains is the *Lawson duality* [40].

Equivalences, dualities and reflections for B- and C-spaces



7. B-extensions

While the class of A-spaces is closed under the formation of subspaces and rather special from the purely topological point of view, the existence of minimal bases is considerably less restrictive and extremely non-hereditary: as we shall see below, *every* topological space arises as a dense subspace of a B-space. Here, density has not only a topological but also an order-theoretical meaning. To be more precise: given a *base space* $(X, \mathcal{B})$, that is, a topological space X with a distinguished base $\mathcal{B}$, we mean by a *B-extension* for $(X, \mathcal{B})$ any B-space X' such that

(B1) the least base of X' induces $\mathcal{B}$ by relativization,
(B2) each base point of X' is a meet (that is, a greatest lower bound w.r.t. specialization) of a non-empty subset of X.

Condition (B1) entails that X is a subspace of X', and (B2) assures topological density of X in X', because every open core meets X. Recall that meets need not be unique in the absence of the T_0-axiom. Below we provide an alternate description of B-extensions by means of *quasi-isomorphisms*, i.e. surjections φ between quasi-ordered sets such that $x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y)$. Of course, such a map is an isomorphism whenever its domain is a poset.

Lemma 7. *A space X' is a B-extension of a base space $(X, \mathcal{B})$ iff X' is a B-space containing X as a subspace such that*

(B3) *the map* $b \mapsto bX = \{x \in X : b \leq x\}$ *yields a quasi-isomorphism between* $\mathrm{B}X'$ *and* $\mathcal{B}$*, ordered by dual inclusion.*

The meets in (B2) *are unique iff the map in* (B3) *is an isomorphism. This certainly holds if X' is a T_0-space.*

Proof. Recall that $b \leq x$ means $b \in \overline{\{x\}}$. Clearly, (B1) and (B2) together imply (B3), which in turn entails (B1). If for some b in $\mathrm{B}X'$, the set bX were to have a lower bound x' in X' such that $x' \not\leq b$, then there would exist an open core cX' with $x' \in cX'$ but $b \notin cX'$, i.e. $c \leq x'$ but $c \not\leq b$. However, assuming (B3), this would lead to $bX \not\subseteq cX$, contradicting the previous assumptions $c \leq x'$ and $bX \subseteq x'X$. By contraposition, each $b \in \mathrm{B}X'$ is a greatest lower bound of bX, and by our general convention to exclude the empty set from any base, bX is non-empty.

If b' is another greatest lower bound of bX (which is not forbidden *a priori*, because specialization need not be a partial order), then $bX' = b'X'$, hence $b' \in \mathrm{B}X'$ and $bX = b'X$; but then the map in (B3) cannot be an isomorphism. $\square$

Condition (B3) allows one to translate topological properties of the base $\mathcal{B}$ into order-theoretical properties of the base point set $\mathrm{B}X'$, and *vice versa*.

We call a B-extension *minimal* if no proper subspace is a B-extension of the given base space.

Proposition 7. *A minimal B-extension X' of $(X, \mathcal{B})$ is characterized by the additional condition*

(B4) *Each point b' in the remainder $X' \setminus X$ is a T_0-base point; i.e. the core $b'X'$ is open and distinct from any other core.*

Hence, X is meet-dense in any minimal B-extension X' of $(X, \mathcal{B})$, and if X is T_0, then so is X'.

Proof. Let X' be a B-extension and $b' \in X' \setminus X$. If b' is not a base point or $b'X' = cX'$ for a $c \neq b'$ then $X'' = X' \setminus \{b'\}$ turns out to be a smaller B-extension with $\mathrm{B}X'' = \mathrm{B}X' \setminus \{b'\}$: indeed, $b \in \mathrm{B}X''$ implies $bX'' = U \cap X'' = U \setminus \{b'\}$ for some $U \in \mathcal{O}X'$, and it follows that $bX' = U$ (if $bX' = U \cup \{b'\}$ then $b \in U$ and $b \leq b'$ entail $b' \in U$ and $bX' = U$ again). In any case, $b \in \mathrm{B}X'$. Conversely, $b \in \mathrm{B}X'$ and $b \neq b'$ implies $bX' \in \mathcal{O}X'$ and $b \in X''$, hence $bX'' = bX' \cap X'' \in \mathcal{O}X''$, i.e. $b \in \mathrm{B}X''$.

Now, it is clear that X'' has a base of open cores (induced by the minimal base of X'), and X'' is a B-extension of X since by (B3), the map $b \mapsto bX$ is a quasi-isomorphism between $\mathrm{B}X' \setminus \{b'\} = \mathrm{B}X''$ and $\mathcal{B}$.

On the other hand, assume that $X' \setminus X$ is a subset of $\mathrm{B}X'$, and let X'' be any B-extension of $(X, \mathcal{B})$ contained in X'. For $b' \in X' \setminus X$, we have $b' \in \mathrm{B}X'$, so (B1) for X' entails $b'X \in \mathcal{B}$, and (B1) for X'' yields a $b'' \in \mathrm{B}X''$ with $b'X = b''X$. By (B2), it follows that $b'' \leq b'$; as $b'' \in X'' \subseteq X'$, we have $b'' \in X$ or $b'' \in X' \setminus X \subseteq \mathrm{B}X'$. In any case, $b' \leq b'' \leq b'$ (using (B3) for X'), whence $b'X' = b''X'$, and the condition that b' be a T_0-base point forces $b' = b''$. We conclude that $X' = (X' \setminus X) \cup X \subseteq X''$, and consequently $X' = X''$. $\square$

Theorem 7. *Every base space $(X, \mathcal{B})$ has a minimal B-extension $X_{\mathcal{B}}$, which is unique up to homeomorphism.*

Proof. Let X° denote the set of all cores, put $X^\circ_{\mathcal{B}} = X^\circ \cup \mathcal{B}$, and choose a bijection $\gamma : Y_{\mathcal{B}} \to \mathcal{B}_0 = \mathcal{B} \setminus X^\circ$, where $Y_{\mathcal{B}}$ is a set disjoint from X. (To make the construction unique, one may take $Y_{\mathcal{B}} = \{(C, X) : C \in \mathcal{B}_0\}$ and $\gamma(C, X) = C$; then the Axiom of Foundation ensures $X \cap Y_{\mathcal{B}} = \emptyset$.) Furthermore, put $X_{\mathcal{B}} = X \cup Y_{\mathcal{B}}$ and extend γ to a surjection from $X_{\mathcal{B}}$ onto $X^\circ_{\mathcal{B}}$ by setting $\gamma(x) = [x)$ for $x \in X$. Define a quasi-order on $X_{\mathcal{B}}$ by

$$x \leq y \text{ in } X_{\mathcal{B}} \text{ iff } \gamma(x) \supseteq \gamma(y) \text{ in } X^\circ_{\mathcal{B}}.$$

We denote by B the preimage $\gamma^{\leftarrow}(\mathcal{B}) = \{b \in X_{\mathcal{B}} : \gamma(b) \in \mathcal{B}\}$ and claim that the sets Bx ($x \in X_{\mathcal{B}}$) are directed. Consider a finite $F \subseteq Bx$; for each $b \in F$, we have $b \leq x$, that is, $\gamma(x) \subseteq \gamma(b)$. Thus, $\gamma(x) \subseteq U = \bigcap\{\gamma(b) : b \in F\} \in \mathcal{O}X$, and since $\gamma(x)$ is either the core $[x)$ or a member of $\mathcal{B}$, the base property of $\mathcal{B}$ yields in any case some $c \in B$ with $\gamma(x) \subseteq \gamma(c) \subseteq U$; in other words, c is an upper bound of F in Bx. Moreover, for $x \not\leq y$ in $X_{\mathcal{B}}$, there is a $b \in B$ with $b \leq x$ but $b \not\leq y$: if $\gamma(x) \in \mathcal{B}$, simply take $b = x$; otherwise, x must belong to X, and then $x \not\leq y$ means $\gamma(y) \not\subseteq [x) = \bigcap \mathcal{B}_x$, hence $\gamma(y) \not\subseteq C$ for some $C = \gamma(b) \in \mathcal{B}_x$, i.e. $\gamma(x) \subseteq \gamma(b)$. Thus, we have $x \leq y \;\Leftrightarrow\; Bx \subseteq By$, and consequently

$$\mathcal{B}' = \{bX_{\mathcal{B}} : b \in B\}$$

is the minimal base of a B-space $X_{\mathcal{B}}$ whose specialization order is the above relation $\leq$ (see [24] and Proposition 9). Moreover, $\mathcal{B}'$ induces the given base $\mathcal{B}$ on X: in fact, each $C \in \mathcal{B}$ is of the form $\gamma(b)$ for some $b \in B$, and it follows that $bX = \gamma(b) = C$, since for $x \in X$, we have

$$x \in C \;\Leftrightarrow\; [x) \subseteq C \;\Leftrightarrow\; \gamma(x) \subseteq \gamma(b) \;\Leftrightarrow\; b \leq x \;\Leftrightarrow\; x \in bX.$$

From this and Lemma 7, we see at once that $X_{\mathcal{B}}$ is a B-extension of $(X, \mathcal{B})$. By definition, each point b in the remainder $X_{\mathcal{B}} \setminus X$ is contained in $Y_{\mathcal{B}}$, hence in the base point set B; moreover, if $[b) = [c)$ for some $c \in X_{\mathcal{B}}$ then either $c \in Y_{\mathcal{B}}$, whence $\gamma(b) = bX = cX = \gamma(c)$ and therefore $b = c$ (by injectivity of γ on $Y_{\mathcal{B}}$), or $c \in X$, in which case we obtain $\gamma(b) = bX = [c) \in X^\circ$, while $b \in Y_{\mathcal{B}}$ excludes $\gamma(b) \in X^\circ$. By Proposition 7, this establishes minimality of $X_{\mathcal{B}}$ among all B-extensions of $(X, \mathcal{B})$.

It remains to verify the uniqueness statement. Let X' be any minimal B-extension for $(X, \mathcal{B})$. Define a map $\mu : X' \to X_{\mathcal{B}}$ by $\mu(x) = x$ for $x \in X$ and $\mu(b) = \gamma^{-1}(bX)$ for $b \in X' \setminus X \subseteq \mathrm{B}X'$. Then μ is well-defined and bijective, by definition of γ and the fact that the points of $X' \setminus X$ bijectively correspond to the members of $\mathcal{B}_0$ via $b \mapsto bX$ (Lemma 7 and Proposition 7). For $b \in \mathrm{B}X'$, we have either $b \in X$ and $\mu(b) = b \in \mathrm{B}X \subseteq \mathrm{B}X_{\mathcal{B}}$, or $b \in X' \setminus X$ and $\mu(b) = \gamma^{-1}(bX) \in \mathrm{B}X_{\mathcal{B}}$ (since $\mu(b)X_{\mathcal{B}} \in \mathcal{B}' \subseteq \mathcal{O}X'$). Conversely, if $\mu(x) \in \mathrm{B}X_{\mathcal{B}}$ then either $\mu(x) = x \in \mathrm{B}X \subseteq \mathrm{B}X'$ or $x \in X' \setminus X \subseteq \mathrm{B}X'$. Furthermore, by Proposition 7,

$$x' \leq y' \Leftrightarrow y'X \subseteq x'X \Leftrightarrow \gamma(\mu(y')) \subseteq \gamma(\mu(x')) \Leftrightarrow \mu(x') \leq \mu(y').$$

Thus, μ is an order isomorphism inducing a bijection between the base point sets and is therefore a homeomorphism between X' and $X_{\mathcal{B}}$. □

The advantage of B-extensions is evident: B-spaces have a rather simple computational structure, and order-theoretical properties of B-extensions reflect topological properties of the original base space, by virtue of the quasi-isomorphism between the least base of the B-extension and the prescribed base of the subspace. For instance, we have as an immediate consequence of Theorem 7:

Corollary 7. *Every topological space X is a dense subspace of a B-space X' having the same weight. In particular, X is second countable iff so is X'.*

Example. Consider the reals $\mathbb{R}$ with the countable base

$$\mathcal{B} = \{\,]r, \infty[: r \in \mathbb{Q} \text{ (i.e. } r \text{ is rational) } \}$$

for the Scott topology. The B-extension $\mathbb{R}_{\mathcal{B}}$ is obtained by "doubling" each rational number $r \in \mathbb{Q}$ to a covering pair $r < r'$ and taking the principal filters $r'\mathbb{R}_{\mathcal{B}}$ as members of the minimal base. Adding further "copies" r'' and putting $r' \leq r'' \leq r'$ yields B-spaces X' so that the inclusion map from $\mathbb{R}$ into X' admits many extensions to $\mathbb{R}_{\mathcal{B}}$ (sending r' to itself or alternatively to r''). Of course, the spaces X' obtained that way are not T_0.

8. The B-reflection and the B-sobrification

The last example demonstrates that the spaces $X_{\mathcal{B}}$ cannot serve as universal "B-reflections" for arbitrary base spaces, by lack of the required uniqueness property of extensions – but a slight modification will provide a T_0-B-reflection for arbitrary base spaces $(X, \mathcal{B})$.

As in the previous section, we denote by X° the collection of all cores $[x)$ with $x \in X$. We order the union $X^\circ_{\mathcal{B}} = X^\circ \cup \mathcal{B}$ by *dual* inclusion and define a topology on $X^\circ_{\mathcal{B}}$ by declaring as basic open sets the principal filters (!)

$$[B) = \{U \in X^\circ_{\mathcal{B}} : U \subseteq B\} \quad (B \in \mathcal{B}).$$

It is easy to see that these sets actually form a base for a topology. Observing that each element of $X^\circ_{\mathcal{B}}$ is an intersection of members of $\mathcal{B}$, we have for $U, V \in X^\circ_{\mathcal{B}}$ the equivalence

$$V \subseteq U \;\Leftrightarrow\; \forall\, B \in \mathcal{B}\, (U \in [B) \;\Rightarrow\; V \in [B)).$$

Hence, the specialization order of the space $X^\circ_{\mathcal{B}}$ is dual inclusion. Thus, $X^\circ_{\mathcal{B}}$ is T_0 and has a base of open cores. Clearly, the map

$$\iota = \iota_{X,\mathcal{B}} : X \to X^\circ_{\mathcal{B}},\ x \mapsto [x)$$

is one-to-one iff the original space is T_0. For each $B \in \mathcal{B}$, we have

$$\iota^{\leftarrow}([B)) = \{x \in X : [x) \subseteq B\} = B,$$

showing that the map ι induces an isomorphism between the bases $\mathcal{B}$ and $\{[B) : B \in \mathcal{B}\}$ (and that ι is a topological embedding in the T_0 case). Moreover, X° is meet-dense in $X^\circ_{\mathcal{B}}$ since each basic set is a union, hence a meet (w.r.t. specialization) of cores. By definition, the points of $X^\circ_{\mathcal{B}} \setminus X^\circ$ are T_0-base points. Thus,

> *for each base space $(X, \mathcal{B})$, the space $X^\circ_{\mathcal{B}}$ is a minimal B-extension of the T_0-reflection X°.*

In case X was T_0, the map γ in the construction of $X_{\mathcal{B}}$ is easily seen to be a homeomorphism onto $X^\circ_{\mathcal{B}}$.

We are now going to show that, with respect to suitable morphisms, the space $X^\circ_{\mathcal{B}}$ may be regarded as the *T_0-B-reflection* of $(X, \mathcal{B})$. The morphism class to be chosen in order to make the categorical machinery run might look a bit exotic at first glance. What we have to consider are *basic open and continuous* maps (which occur also in similar contexts of order-topological adjunctions or reflections; see e.g. [18] and [37]). Basic continuity of a map φ between base spaces $(X,\mathcal{B})$ and $(X',\mathcal{B}')$ means that the preimage $\varphi^{\leftarrow}(B')$ of any basic open set $B' \in \mathcal{B}'$ belongs to the base $\mathcal{B}$ (which implies topological continuity, but not conversely); and basic openness means that for each $C \in \mathcal{B}$, there is a least $B' \in \mathcal{B}'$ with $\varphi(B) \subseteq B'$. Thus, a map $\varphi : (X,\mathcal{B}) \to (X',\mathcal{B}')$ is basic open and continuous iff the restriction $\varphi^{\leftarrow} : \mathcal{B}' \to \mathcal{B}$ is well-defined and has a right adjoint $\varphi^{+} : \mathcal{B} \to \mathcal{B}'$ (where the bases are ordered by *dual* inclusion; thus, $B \subseteq \varphi^{\leftarrow}(B') \Leftrightarrow \varphi^{+}(B) \subseteq B'$). Obviously, basic open and continuous maps are stable under composition; hence, they may serve as morphisms for a category of base spaces. A map is called *core continuous* if preimages of open cores are open cores.

Lemma 8. *For a map $\varphi : X \to X'$ between B-spaces (regarded as base spaces with the minimal bases), the following are equivalent:*

- (a) *φ is basic open and continuous.*
- (b) *φ is core continuous and sends base points to base points.*
- (c) *φ is interpolating and induces a right adjoint map between $\mathrm{B}X$ and $\mathrm{B}X'$.*

Proof. (a)$\Rightarrow$(b). By definition, φ is continuous as a map between base spaces iff it is core continuous. For $b \in \mathrm{B}X$ and $b' \in \mathrm{B}X'$, we get (using the fact that continuous maps are isotone):

$$\varphi^{+}([b)) \subseteq [b') \Leftrightarrow [b) \subseteq \varphi^{\leftarrow}([b')) \Leftrightarrow b' \leq \varphi(b) \Leftrightarrow [\varphi(b)) \subseteq [b').$$

Since each core in X' is an intersection of open cores, it follows that $\varphi^{+}([b))$ must coincide with $[\varphi(b))$, and that $\varphi(b)$ belongs to $\mathrm{B}X'$.
(b)$\Rightarrow$(c). By core continuity, preimages of principal filters in $\mathrm{B}X'$ are principal filters in $\mathrm{B}X$, and φ is interpolating (Theorem 3).
(c)$\Rightarrow$(a). Let $\psi : \mathrm{B}X' \to \mathrm{B}X$ be the left adjoint of $\varphi : \mathrm{B}X \to \mathrm{B}X'$. Thus, $\psi(b') \leq b \Leftrightarrow b' \leq \varphi(b)$, and the interpolation property yields

$$\varphi^{\leftarrow}([b')) = \{x \in X : b' \leq \varphi(x)\} =$$
$$\{x \in X : \exists b \in \mathrm{B}X\,(b' \leq \varphi(b), b \leq x)\} = [\psi(b'))$$

for each $b' \in \mathrm{B}X'$, assuring that φ is (core) continuous, hence isotone. For each $b \in \mathrm{B}X$, the open core $[\varphi(b))$ is the least basic open set containing $\varphi([b))$, showing that φ is basic open. $\square$

Now, we are ready for the main reflection theorem:

Theorem 8. *The category of T_0-B-spaces with base point preserving and core continuous maps is reflective in the category of base spaces with basic open and continuous maps, and bireflective in the category of T_0-base spaces. For each base space $(X, \mathcal{B})$, a reflection is given by the core map*

$$\iota_{X,\mathcal{B}} : X \to X^{\circ}_{\mathcal{B}},\ x \mapsto [x).$$

Proof. (1) The map $\iota = \iota_{X,\mathcal{B}}$ is basic open and continuous, since

$$\iota^{\leftarrow} : \{[B) : B \in \mathrm{B}X^{\circ}_{\mathcal{B}} = \mathcal{B}\} \to \mathcal{B},\ [B) \mapsto B = \bigcup [B)$$

is an isomorphism (thus, ι is a *"quasi-homeomorphism"* [30]).

(2) Now, to the universal property. Let φ be a basic open and continuous map from $(X, \mathcal{B})$ to a T_0-B-space X', regarded as a base space with minimal base $\{[b') : b' \in \mathrm{B}X'\}$. There is a well-defined map $\hat{\varphi} : X^{\circ}_{\mathcal{B}} \to X'$ with $\hat{\varphi}([x)) = \varphi(x)$ for $x \in X$ and $\hat{\varphi}(B) = \min \varphi^{+}(B)$ for $B \in \mathcal{B}$. Note that for open cores $[x) \in \mathcal{B}$, we have $\hat{\varphi}([x)) = \varphi(x) = \min \varphi^{+}([x))$, since for $b' \in \mathrm{B}X'$,

$$\varphi^{+}([x)) \subseteq [b') \Leftrightarrow [x) \subseteq \varphi^{\leftarrow}([b')) \Leftrightarrow x \in \varphi^{\leftarrow}([b')) \Leftrightarrow \varphi(x) \in [b').$$

By definition, $\hat{\varphi}$ sends base points to base points.

Let us show that $\hat{\varphi}$ is core continuous. For $b' \in \mathrm{B}X'$, the preimage $C = \varphi^{\leftarrow}([b'))$ lies in $\mathcal{B}$, and $\hat{\varphi}^{\leftarrow}([b'))$ is the core $[C)$ in $X^{\circ}_{\mathcal{B}}$, since for $B \in \mathcal{B}$,

$$B \subseteq C \Leftrightarrow \varphi^{+}(B) \subseteq [b') \Leftrightarrow b' \leq \hat{\varphi}(B) \Leftrightarrow B \in \hat{\varphi}^{\leftarrow}([b')),$$

and for $x \in X$,

$$[x) \subseteq C \Leftrightarrow x \in \varphi^{\leftarrow}([b')) \Leftrightarrow \varphi(x) = \hat{\varphi}([x)) \in [b') \Leftrightarrow [x) \in \hat{\varphi}^{\leftarrow}([b')).$$

(3) Any basic continuous map Φ between $X^{\circ}_{\mathcal{B}}$ and a T_0-base space $(X', \mathcal{B}')$ is uniquely determined by its values on $\iota(X)$. Indeed, using the unique map $\Psi : \mathcal{B}' \to \mathcal{B}$ with $\Phi^{\leftarrow}(B') = [\Psi(B'))$, we can show that for each $U \in X^{\circ}_{\mathcal{B}}$, the image $\Phi(U)$ is the (unique!) meet of the set $\Phi(\iota(U)) = \{\Phi([x)) : x \in U\}$:

$$\begin{aligned}
x' \leq \Phi(U) &\Leftrightarrow \forall B' \in \mathcal{B}'\ (x' \in B' \Rightarrow \Phi(U) \in B') \\
&\Leftrightarrow \forall B' \in \mathcal{B}'\ (x' \in B' \Rightarrow U \subseteq \Psi(B')) \\
&\Leftrightarrow \forall x \in U\ \forall B' \in \mathcal{B}'\ (x' \in B' \Rightarrow \iota(x) \subseteq \Psi(B')) \\
&\Leftrightarrow \forall x \in U\ \forall B' \in \mathcal{B}'\ (x' \in B' \Rightarrow \Phi(\iota(x)) \in B') \\
&\Leftrightarrow \forall x \in U\ (x' \leq \Phi(\iota(x))).
\end{aligned}$$

(4) The uniqueness part of the universal property immediately follows from (3), which also shows that in case $(X, \mathcal{B})$ is T_0, the map ι is not only a monomorphism but also an epimorphism, hence a bimorphism. $\square$

We know from Theorem 6. B that the base points of a T_0-B-space X are those of its sobrification $\Sigma^\circ X = \check{\Sigma}\,\mathcal{C}X = \Sigma\mathrm{IB}X$; in particular, the sobrification embedding preserves base points and is core continuous. Moreover, if we have any base point preserving and core continuous map $\varphi : X \to X'$ into a sober B-space, then the unique extension to $\Sigma^\circ X$ is base point preserving and core continuous, too. Composing the sober reflection with the B-reflection (and ordering the bases by dual inclusion), we arrive at

Proposition 8. *For any base space $(X, \mathcal{B})$, the ideal completion $\mathrm{I}\mathcal{B}$ with the Scott topology is the B-sobrification, that is, the universal sober B-reflection in the category of base spaces with basic open and continuous maps. In particular, X is a dense subspace of the sober B-space $\Sigma\mathrm{I}\mathcal{B}$ (hence of an algebraic domain) whose least base induces the given base $\mathcal{B}$.*

Clearly, $\Sigma\mathrm{I}\mathcal{B}$ will in general not serve as a minimal B-extension, but it is minimal among all *sober* B-extensions (in the T_0 case).

By virtue of the isomorphism between the respective bases, we finally obtain a result of particular relevance to the theory of computational models for spaces:

Corollary 8. *A T_0-space has a countable base iff it is a dense subspace of an ω-algebraic (countably based) domain endowed with the Scott topology.*

Stronger results are available for T_1-spaces (with distinguished bases) and for zero-dimensional spaces. The investigation of models for such spaces is deferred to a separate note [21], and algebraic models for metrizable spaces (and some of their generalizations) are presented in a common note with A. Pultr [22].

9. Lattice-Theoretical Representation of Spaces

From the work of Hoffmann [34] and Lawson [40] (see also [38]) we know that the algebraic (respectively, continuous) domains are precisely the prime spectra of superalgebraic (respectively, supercontinuous) lattices (cf. Corollary 5). Are there similar characterizations for general algebraic (continuous) posets? A quick inspection shows that such a strict correspondence as in the up-complete case is impossible, because many non-isomorphic algebraic (or continuous) posets may have isomorphic lattices of Scott-closed sets. However, the following concept, introduced in [15], leads to a solution (cf. Theorem 2): by a *based lattice* we mean a pair (C, P) constituted by a complete lattice C and a join-dense subset (join-base) P of C; if P consists of join-prime (respectively completely join-prime) elements then we speak of a ($\vee$-)*prime-based* (respectively, *completely prime-based*) lattice; we say (C, P) is *consistently prime-based* if, moreover, for any directed subset D of P having a join in P, this is also the join of D in C, and (C, P) is *cup prime-based* if, in addition, P is a cup. By definition, the *spatial coframes* are just the first components C of prime-based lattices (C, P), and the superalgebraic ones are those of completely prime based lattices. A *morphism* between prime-based lattices preserves joins and induces a map between the join-bases. We cite from [15]:

Lemma 9. *Sending each based lattice* (C, P) *to the pair* $(P, \mathcal{C})$ *where* $\mathcal{C} = \{Pa : a \in C\}$, *one obtains an equivalence between the category of based lattices (with maps preserving joins and join-bases) and the category of (not necessarily topological)* T_0*-closure spaces.*

For our purposes, several subequivalences between categories of topological spaces (regarded as closure spaces) and their lattice-theoretical counterparts are of interest:

Theorem 9. *Under the above functorial equivalence, the following pairs of subcategories are equivalent:*

prime-based lattices	*and*	T_0*-spaces,*
completely prime-based lattices	*and*	T_0*-A-spaces,*
prime-based superalgebraic lattices	*and*	T_0*-B-spaces,*
prime-based supercontinuous lattices	*and*	T_0*-C-spaces,*
consistently prime-based lattices	*and*	*weak spaces.*

Proof. For any T_0-space X, the map

$$\eta_X : X \to \mathrm{i}X = \{(x] : x \in X\},\ x \mapsto (x]$$

is an order isomorphism between X, ordered by specialization, and $\mathrm{i}X$, ordered by inclusion. The pair $(C,P) = (\mathcal{C}X, \mathrm{i}X)$ is a prime-based lattice, and via η_X, the original space X is homeomorphic to the space $\mathrm{i}X$ with closed set lattice $\mathcal{C}\mathrm{i}X = \{Pa : a \in C\}$. On the morphism level, the equivalence functor acts by restriction to the selected join-bases of $\vee$-prime elements. Continuity of the restricted maps is easily checked. The statements about A-, B- and C-spaces follow at once from the results in Section 3.

It remains to confirm the claim about weak spaces. As before, put $(C,P) = (\mathcal{C}X, \mathrm{i}X)$, this time for a weak space X. Then, for each directed subset $\eta_X(D)$ of P with join $\eta_X(x) = (x] = \bigvee_P \eta_X(D)$ in P, we have $x = \bigvee D$ in X, and as X is a weak space, $(x] = \overline{D} = \bigvee_C \eta_X(D)$ in C (see Lemma 4 and Corollary 4). Thus, (C,P) is consistently prime-based. Conversely, if that is assumed and D is a directed subset of X with $x = \bigvee D$, then $\eta_X(x) = \bigvee \eta_X(D)$ in P and in C, i.e. $(x] = \overline{D}$. Hence, X is a weak space. $\square$

Combining Theorem 4 with Theorem 9, we obtain:

Corollary 9. *The restriction functor $(C,P) \mapsto P$ yields an equivalence between consistently prime-based superalgebraic (resp. supercontinuous) lattices and algebraic (resp. continuous) posets. An equivalence functor in the reverse direction is obtained by assigning to any algebraic or continuous poset P the pair $(\mathcal{C}\Sigma P, \mathrm{i}P)$.*

Applying the equivalences in Theorem 9 to such pairs (C,P) whose second component is the subposet of *all* $\vee$-prime elements of C, one arrives at the known equivalences between spatial (co-) frames and sober spaces, and also between superalgebraic (respectively, supercontinuous) lattices and algebraic (respectively, continuous) domains (see [17, 30, 40], and [25] for a generalization to $\mathcal{Z}$-continuous posets and $\mathcal{Z}$-supercompactly generated lattices).

Since B-spaces may be viewed as a generalization of algebraic posets, it appears desirable to represent them in a purely order-theoretical fashion. Again, single posets do not suffice, but one needs *directed based sets* (A,B) and isotone interpolating maps as morphisms, as introduced in Section 2. Now, one easily finds the desired representation (cf. [17, 24] and Theorem 3):

Proposition 9. *Assigning to any B-space X the pair $(\Sigma^{-}X, \mathrm{B}X)$, one obtains a concrete isomorphism between the category of B-spaces (with continuous maps) and the category of directed based sets (with isotone interpolating maps). The reverse categorical isomorphism sends any directed based set (A, B) to the underlying set of A topologized by taking $\{bA : b \in B\}$ as a base for the open sets.*

Algebraic posets, regarded as pairs $(A, \mathrm{K}A)$, nicely fit into this framework: together with maps preserving compactness and directed joins as morphisms, they form a full subcategory of the category of directed based posets with isotone interpolating maps. Indeed, the categorical isomorphism in Proposition 9 extends that between weak B-spaces and algebraic posets (Theorem 4), in particular that between sober B-spaces and algebraic domains (Theorem 5).

Compare Theorem 2 with Proposition 9: composing the duality in the former with the categorical isomorphism in the latter, one obtains a duality between T_0-B-spaces and ideal extensions. In the next section, that duality will arise in a more general framework.

10. Dualities for Base Spaces and Ideal Spaces

In Section 2, we have established an equivalence between algebraic posets and algebraic ideal extensions. For a *duality* between algebraic posets (or weak B-spaces) and algebraic ideal extensions, we need different morphism classes. Here, a much more comprehensive point of view, including not only weak B-spaces but *all* T_0-base spaces on the topological side, will be useful. The following general construction is a slight modification of the basic duality presented in [20]. By a *cover* of a set X we mean a collection $\mathcal{B}$ of non-empty subsets with union X; if, moreover, for $x \neq y$ in X there is a $B \in \mathcal{B}$ with $x \in B$, $y \notin B$ or $x \notin B$, $y \in B$, we speak of a *T_0-cover*. The pair $(X, \mathcal{B})$ is referred to as a (T_0-)*cover space*. For $x \in X$, consider the generalized *neighborhood system* $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. The *(upper) specialization order*, defined by $x \leq y \Leftrightarrow \mathcal{B}_x \subseteq \mathcal{B}_y$, is antisymmetric iff the cover space satisfies the T_0-axiom. Notice that each $B \in \mathcal{B}$ is an up-set with respect to that specialization order. A *(cover) continuous map* between cover spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ is a map $\varphi : X \to X'$ with $\varphi^{\leftarrow}(B') \in \mathcal{B}$ for each $B' \in \mathcal{B}'$.

Lemma 10. *The category of T_0-cover spaces is self-dual. The duality is established by the contravariant functor sending any* T_0-*cover space* $(X, \mathcal{B})$ *to the "dual" T_0-cover space*

$$\mathrm{D}(X, \mathcal{B}) = (\mathcal{B}, \mathcal{B}_X) \quad \textit{with} \quad \mathcal{B}_X = \{\mathcal{B}_x : x \in X\}$$

and any continuous map φ between T_0-*cover spaces* $(X, \mathcal{B})$ *and* $(X', \mathcal{B}')$ *to the preimage map* $\varphi^{\leftarrow} : \mathcal{B}' \to \mathcal{B}$. *The open set lattice* $\{\bigcup \mathcal{X} : \mathcal{X} \subseteq \mathcal{B}\}$ *of* $(X, \mathcal{B})$ *is order-dual to the open set lattice of* $\mathrm{D}(X, \mathcal{B})$, *by virtue of the map* $U \mapsto \{B \in \mathcal{B} : B \not\subseteq U\}$.

For the proof, see [20]. Note that the specialization order of the dual T_0-cover space $\mathrm{D}(X, \mathcal{B})$ is set inclusion. One may also define the dual for arbitrary cover spaces and finds that *the "double dual"* $\mathrm{DD}(X, \mathcal{B})$ *is the T_0-reflection of the original cover space* $(X, \mathcal{B})$.

By a T_0-*base* we mean a T_0-cover $\mathcal{B}$ such that each of the neighborhood systems $\mathcal{B}_x$ is a filter base. In other words, the T_0-bases are precisely the bases of T_0-topologies, and the corresponding pairs $(X, \mathcal{B})$ may be regarded as T_0-base spaces, or as *topologically based posets*, i.e. posets with a T_0-base consisting of up-sets. Recall that a cover continuous map is also continuous as a map between the corresponding spaces, and that we exclude the empty set from any base. Therefore, cover continuity also entails topological *density*. To avoid that restriction, one has to relax the continuity condition and to allow $\varphi^{\leftarrow}(B')$ to be a member of $\mathcal{B}$ *or empty*, for each $B' \in \mathcal{B}'$. The subsequent considerations then have to be adapted appropriately. The category of T_0-B-spaces may be embedded in the category of T_0-base spaces, by passing from the topologies to their least bases; however, in doing so, one has to take as morphisms between B-spaces the *core continuous* maps (see Section 8).

On the "algebraic side", we call a T_0-cover $\mathcal{B}$ *ideal* and the pair $(X, \mathcal{B})$ an *ideal space* if each $B \in \mathcal{B}$ is an ideal with respect to the *dual* specialization order. There is an obvious one-to-one correspondence between ideal extensions and ideal covers containing all cores.

Theorem 10. *The duality of T_0-covers induces dualities between*

(1)	*T_0-base spaces*	*and*	*ideal spaces,*
(2)	*T_0-B-spaces*	*and*	*ideal extensions,*
(3)	*weak B-spaces*	*and*	*algebraic ideal extensions,*
(4)	*cmc B-spaces*	*and*	*down-closed ideal extensions,*
(5)	*sober (mc) B-spaces*	*and*	*ideal completions.*

Proof. In view of the general self-duality for T_0-cover spaces, the validity of these statements has to be checked on the object level only.

(1) A T_0-cover $\mathcal{B}$ is a T_0-base iff each $\mathcal{B}_x$ is an ideal of the poset $\mathcal{B}$.

(2) If $\mathcal{B}$ is the least base of a T_0-B-space X then it consists of open cores $B = [b)$; therefore, the dual ideal cover $\{\mathcal{B}_x : x \in X\}$ contains all principal ideals $(B] = \{C \in \mathcal{B} : B = [b) \subseteq C\} = \mathcal{B}_b$ $(b \in \mathrm{B}X)$. Conversely, if $(P, \mathcal{I})$ is an ideal extension then the dual T_0-base $\{\mathcal{I}_p : p \in P\}$ consists of open cores:

$$\mathcal{I}_p = \{J \in \mathcal{I} : (p] \subseteq J\} = [I) \text{ for } I = (p] \in \mathcal{I}.$$

(3) Suppose $\mathcal{B} = \{\,[b) : b \in P = \mathrm{K}A\}$ is the least base of a weak B-space associated with an algebraic poset A (see Theorem 4). Then the ideal extension $(\mathcal{B}, \{\mathcal{B}_a : a \in A\})$ is isomorphic to the algebraic ideal extension $(P, \{Pa : a \in A\})$ via the isomorphism $b \mapsto [b)$ (cf. Lemma 2 and Corollary 2).

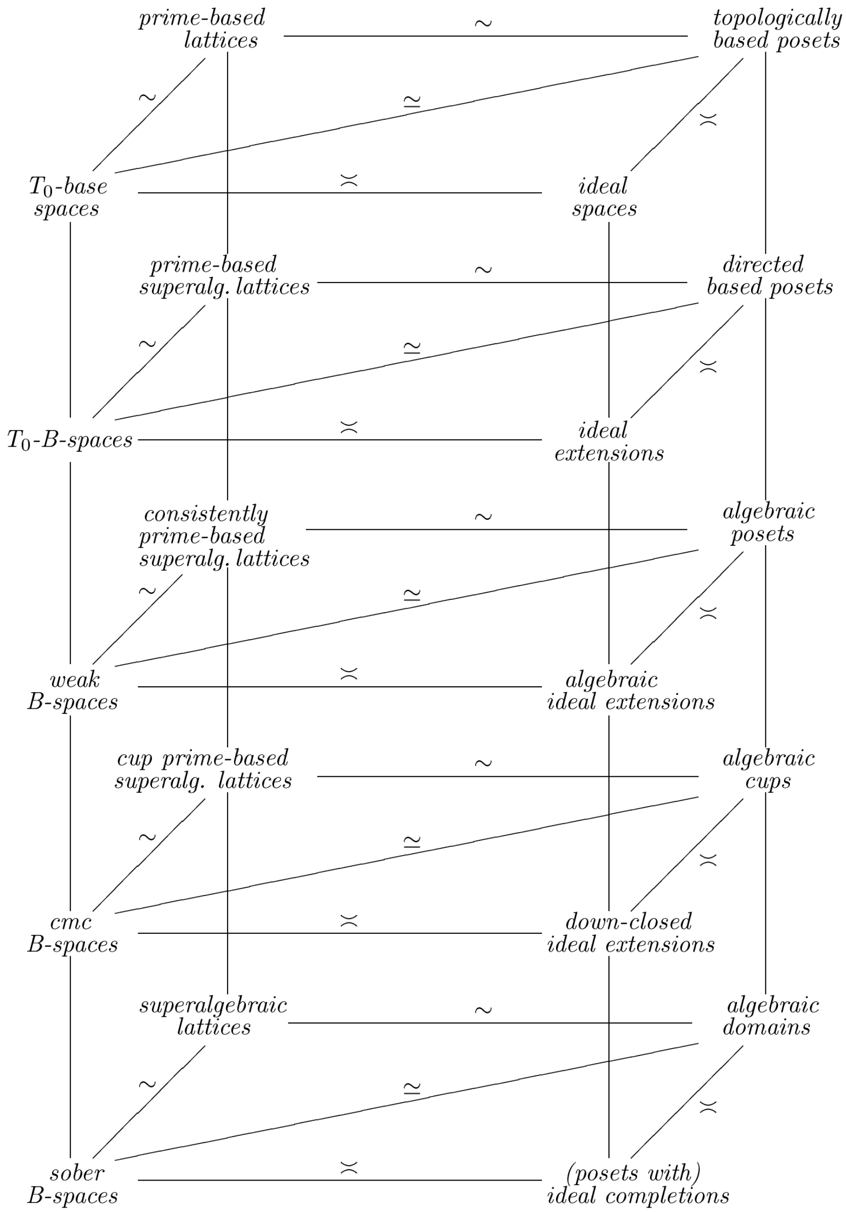
On the other hand, if $(P, \mathcal{I})$ is any algebraic ideal extension, we may assume (again by Lemma 2) that $P = \mathrm{K}A$ for some algebraic poset A and $\mathcal{I} = \{Pa : a \in A\}$. Then the T_0-base space $(\mathcal{I}, \{\mathcal{I}_p : p \in P\})$ is isomorphic to the weak B-space associated with A, by virtue of the isomorphism that sends a to Pa.

(4) Analogous arguments hold for cmc B-spaces and down-closed ideal extensions.

(5) Finally, if $\mathcal{B}$ is the least base of a sober B-space X associated with an algebraic domain A then the algebraic ideal extension $\{\mathcal{B}_x : x \in X\}$ is isomorphic to A and therefore up-complete, hence an ideal completion. And conversely, the dual T_0-cover of an ideal completion is the least base of a (Scott-topologized) algebraic domain, hence of a sober B-space. $\square$

Corollary 10. *The category of algebraic posets (cups, domains) with core continuous maps is dual to the category of algebraic (down-closed, up-complete) ideal extensions with continuous maps.*

In the diagram on the next page, we indicate by $\simeq$ categorical isomorphisms, by $\sim$ categorical equivalences, by $\asymp$ categorical dualities, and by vertical lines full categorical inclusions.


prime-based lattices
topologically based posets
T_0-base spaces
ideal spaces
prime-based superalg. lattices
directed based posets
T_0-B-spaces
ideal extensions
consistently prime-based superalg. lattices
algebraic posets
weak B-spaces
algebraic ideal extensions
cup prime-based superalg. lattices
algebraic cups
cmc B-spaces
down-closed ideal extensions
superalgebraic lattices
algebraic domains
sober B-spaces
(posets with) ideal completions

The reader might guess that the dualities mentioned in Proposition 6 are somehow induced by the duality between T_0-base spaces and ideal spaces – and that is the case indeed, as we shall see below.

An *ideal base space* $(X, \mathcal{B})$ is both an ideal space and a base space; in other words, $\mathcal{B}$ is a base consisting of open filters (i.e. $\vee$-prime open sets) of some T_0-space. By Theorem 10,

the self-duality of T_0-cover spaces induces one of ideal base spaces.

A space having a base of open filters is called a *D-space*, also *strongly locally connected* or a *"space having a dual"* (Hoffmann [32]), in view of the fact that its lattice of open sets is dual to the lattice of closed sets of another space. Thus, a D-space is characterized by the condition that not only the lattice of open sets, but also that of closed sets is a spatial frame. The Principle of Dependent Choices ensures that every supercontinuous lattice is a spatial frame, and consequently, that every C-space is a D-space (see Lemma 3; if one wants to avoid choice principles rigorously, one has to consider C-D-spaces instead of C-spaces in order to make all conclusions sound.)

Example. A D-space that fails to be a C-space is obtained as follows: consider a countable power P of the half-open unit interval $]0, 1]$. Then ΣP is a D-space, since σP is the product topology of the Scott topologies on the factors, which are strongly connected (having the dense point 1) and D-spaces (cf. [11, 32, 35]). Though being up-complete, P is not continuous – in fact, the way-below relation is empty, and consequently, ΣP cannot be a C-space (cf. Corollary 5).

Apparently, D-spaces are in one-to-one correspondence with those base spaces $(X, \mathcal{B})$ whose base $\mathcal{B}$ consists of *all* $\vee$-prime members of the generated topology; we might call them *D-base spaces*. On the other hand, the "dual" $\mathrm{D}(X, \mathcal{B}) = (\mathcal{B}, \mathcal{B}_X)$ of a T_0-D-(base) space is a D-base space iff X is *sober*. Indeed, under the isomorphism between the closed set lattice of X and the open set lattice of its dual (Lemma 10), the $\vee$-prime closed sets of X correspond to the $\vee$-prime open sets of the dual; hence, X is sober (i.e. the point closures are the only $\vee$-prime closed sets) iff $\mathcal{B}_X$ is the set of all $\vee$-prime open sets of $(\mathcal{B}, \mathcal{B}_X)$. Summing up the preceding thoughts, we arrive at

Proposition 10. *Let X be any topological space and $\mathcal{B} = \check{\mathrm{P}}\mathcal{O}X$ the set of all open filters.*

(1) *X is a D-space (i.e. $\mathcal{B}$ is a base) iff the open set lattice $\mathcal{O}X$ is dually isomorphic to $\mathcal{O}\Sigma^c X$ (under the bijective map $\mathrm{B}_X : \mathcal{O}X \to \mathcal{O}\Sigma^c X,\ U \mapsto \{B \in \mathcal{B} : B \not\subseteq U\}$).*

(2) *A D-space X is sober iff the map $\beta_X : X \to \Sigma^c\Sigma^c X, x \mapsto \mathcal{B}_x$ is a homeomorphism.*

(3) *The self-duality of T_0-cover spaces induces a self-duality of sober D-spaces, which in turn induces that of sober B-spaces and that of sober C-(D-)spaces.*

11. Algebraic Cups with Point Generators

The equivalence between weak B-spaces and algebraic posets and the duality to algebraic ideal extensions (see Corollary 10) extends to some further dualities and equivalences that are particularly helpful in constructing "algebraic models", not only for T_0-B-spaces but also for arbitrary T_0-(base) spaces.

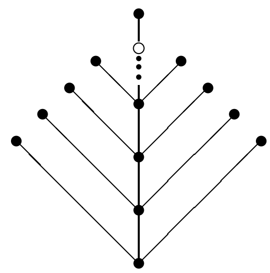
In order to obtain the desired algebraic counterparts of T_0-base spaces, we need a refinement of the notion of algebraic cups (see Theorem 4 again). Thus, we introduce so-called *caps* as pairs (A, M) consisting of a *conditionally* up-complete *algebraic* poset A and a *point-generator* M, that is, a cofinal subset (i.e. $A = \downarrow M$) such that each compact element of A is a meet of elements of M.

Examples. (1) Of course, every algebraic cup A may be regarded as a cap (A, A).

(2) By Zorn's Lemma, every algebraic domain A (but not every algebraic cup!) has a least meet-dense subset M, consisting of all completely meet-irreducible elements. The pair (A, M) is then a cap.

(3) Let X be a Boolean space, or any T_1-space having a compact-open base. Then $(\mathcal{O}X, \{X \setminus \{x\} : x \in X\})$ is a cap. Since $\mathcal{O}X$ is an algebraic lattice, this is a special instance of (2).

(4) In a tree-like algebraic cup with the subsequent diagram, the maximal elements form a point generator that is *not* meet-dense in the whole cup (which is a dcpo).



(5) "Generic" examples of caps are obtained as follows. Given any ideal space $(X, \mathcal{I})$, put

$$\downarrow\mathcal{I} = \{I \in \mathrm{I}(X, \geq) : I \subseteq J \ \text{ for some } \ J \in \mathcal{I}\}.$$

Recall that ideals here refer to the *dual* specialization order $\geq$ of the cover space $(X, \mathcal{I})$. Then $(\downarrow\mathcal{I}, \mathcal{I})$ is a cap: indeed, $\downarrow\mathcal{I}$ is a down-set in the algebraic domain $\mathrm{I}(X, \geq)$, the compact members of $\downarrow\mathcal{I}$ are precisely the principal ideals (use the cover condition $X = \bigcup\mathcal{I}$), and any such principal ideal is an intersection of ideals belonging to $\mathcal{I}$, by the T_0-axiom. We shall see soon that *every* cap is isomorphic to one of this type (cf. Theorem 1).

In the spirit of general Galois connections (see e.g. [23] or [29]), we consider the following type of morphisms between caps (A, M) and (A', M') with base point sets $K = \mathrm{K}A$ and $K' = \mathrm{K}A'$: a *partial Galois morphism* is a pair (φ, ψ) of maps $\varphi : M \to M'$ and $\psi : K' \to K$ such that

$$\psi(k') \leq m \ \Leftrightarrow \ k' \leq \varphi(m)$$

for all $k' \in K'$ and $m \in M$. In that situation, we say ψ is the *lower partial adjoint* of φ, which in turn is the *upper partial adjoint* of ψ. As for "full" Galois connections, one easily verifies that either partial adjoint is isotone and determines the other uniquely (by the density properties). Moreover, the class of partial lower, respectively, upper adjoints is closed under composition. A topological interpretation of such partial adjoints is given in

Lemma 11. *Let (A, M) and (A', M') be caps with base point sets $K = \mathrm{K}A$ and $K' = \mathrm{K}A'$. Then:*

(1a) $\mathrm{I}(A, M) = (K, \{Km : m \in M\})$ *is an ideal space.*

(2a) $\mathrm{T}(A, M) = (M, \{kM : k \in K\})$ *is a* T_0*-base space.*

(1b) *A map* $\psi : K' \to K$ *has a partial upper adjoint* $\varphi : M \to M'$ *iff it is a continuous map between the ideal spaces* $\mathrm{I}(A', M')$ *and* $\mathrm{I}(A, M)$.

(2b) *A map* $\varphi : M \to M'$ *has a partial lower adjoint* $\psi : K' \to K$ *iff it is a continuous map between* $\mathrm{T}(A, M)$ *and* $\mathrm{T}(A', M')$.

The proof of these statements is straightforward. Now, we are prepared for the promised dualities and equivalences.

Theorem 11. *The category of caps with partial upper adjoints is*

(0) *dual to the category of caps with partial lower adjoints via Galois connection, passing to the partial adjoints,*

(1) *dual to the category of ideal spaces, by associating with any cap* (A, M) *the ideal space* $\mathrm{I}(A, M) = (K, \{Km : m \in M\})$ *and passing to the partial lower adjoints,*

(2) *equivalent to the category of* T_0*-base spaces, by sending a cap* (A, M) *to the* T_0*-base space* $\mathrm{T}(A, M) = (M, \{kM : k \in K\})$ *and keeping the underlying maps of the morphisms fixed.*

Proof. Part (0) is clear by the previous remarks.

Concerning (1), we refer to Example (5) above and observe:

(1c) *An arbitrary ideal space* $(P, \mathcal{I})$ *is isomorphic to* $\mathrm{I}(\downarrow\mathcal{I}, \mathcal{I})$ *by virtue of the principal ideal map* $x \mapsto \bigcap\{I \in \mathcal{I} : x \in I\}$.

This together with (1b) in Lemma 11 shows that the concrete functor I, sending (A, M) to $(K, \{Km : m \in M\})$, is dense, full and faithful, hence an equivalence (see [1]) between the category of caps with partial lower adjoints and that of ideal spaces. Composition with the duality from (0) yields the claimed duality in (1).

(2) By Parts (2a) and (2b) of Lemma 11, T gives rise to a well-defined full and faithful functor from caps to T_0-base spaces. To prove density, we consider the poset $\hat{\mathrm{I}}\mathcal{B}$ of all filters in $\mathcal{B}$ with nonempty intersection and show:

(2c) $(\hat{\mathrm{I}}\mathcal{B}, \mathcal{B}_X)$ *is a cap, and* $\mathrm{T}(\hat{\mathrm{I}}\mathcal{B}, \mathcal{B}_X)$ *is isomorphic to* $(X, \mathcal{I})$.

Indeed, for filters $\mathcal{F} \subseteq \mathcal{B}$, the condition $x \in \bigcap \mathcal{F}$ means $\mathcal{F} \subseteq \mathcal{B}_x$. Hence, $\hat{\mathrm{I}}\mathcal{B}$ is the down-set generated by $\mathcal{B}_X = \{\mathcal{B}_x : x \in X\}$ in the algebraic domain $\mathrm{I}\mathcal{B}$, and consequently, $\hat{\mathrm{I}}\mathcal{B}$ is an algebraic cup with cofinal subset $\mathcal{B}_X$ (Theorem 1). Each compact member of $\hat{\mathrm{I}}\mathcal{B}$, that is, each principal ideal $(B] = \{C \in \mathcal{B} : B \subseteq C\}$, is the intersection of all $\mathcal{B}_x$ containing B. By the T_0-axiom, we have a bijection $\beta_X : X \to \mathcal{B}_X,\ x \mapsto \mathcal{B}_x$, which provides an isomorphism between $(X, \mathcal{B})$ and

$$\mathrm{T}(\hat{\mathrm{I}}\mathcal{B}, \mathcal{B}_X) = (\mathcal{B}_X, \{\{\mathcal{B}_x : (B] \subseteq \mathcal{B}_x\} : B \in \mathcal{B}\}).$$

Thus, T is full, faithful and dense, hence an equivalence. □

We complement (1) by the remark that in the opposite direction, a duality functor between ideal spaces and caps is obtained by sending each ideal space $(P, \mathcal{I})$ to the cap $(\downarrow\mathcal{I}, \mathcal{I})$. Composition of that functor with the equivalence in (2) returns the basic duality in Theorem 10.

A perhaps more natural choice of morphisms between two caps (A, M) and (A', M') would be to take those maps $\phi : A \to A'$ which are core continuous and preserve point generators ($\phi(M) \subseteq M'$). As topological counterparts to such "full" morphisms, we need a specific class of morphisms, the F-*continuous* maps: these are continuous maps $\varphi : (X, \mathcal{B}) \to (X', \mathcal{B}')$ such that for each $B \in \mathcal{B}$, the system $\{B' \in \mathcal{B}' : \varphi(B) \subseteq B'\}$ is a filter in $\mathcal{B}$. The latter condition is certainly fulfilled for all basic open maps, but it also holds automatically whenever each $\mathcal{B}_x$ is closed under finite intersections. As specific morphisms between ideal spaces, we take those continuous maps which have the additional property that inverse images of arbitrary (equivalently, of principal) ideals are again ideals. We call such maps I-*continuous*. It is readily checked that a map φ between caps is a full morphism iff it is an F-continuous map between the corresponding T_0-base spaces, which means that the map $\varphi^{\leftarrow}$ between the "dual" ideal spaces is I-continuous. Thus, Theorem 11 is supplemented by

Proposition 11. *The category of caps and full morphisms is equivalent to the category of T_0-base spaces with* F-*continuous maps and dual to the category of ideal spaces with* I-*continuous maps.*

Corollary 11. *The duality of T_0-cover spaces restricts to a duality between T_0-base spaces with* F-*continuous maps and ideal spaces with* I-*continuous maps.*

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Faculty for Mathematics and Physics, University of Hannover, D-30167 Hannover, Germany

E-mail address: `erne@math.uni-hannover.de`