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ON SEPARATED AFFINE SETS AND EPIMORPHISMS

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ABSTRACT. Categories of separated affine sets over an algebra are investigated. An appropriate closure operator, the Zariski closure, gives a description of the epimorphisms and regular monomorphisms.

1. Introduction

An affine set over a given algebra (A, Ω), (Ω is any class of operations in the set A) is a pair $(X, A(X))$ with X a set and $A(X)$ a subalgebra of the powerset (Ω-)algebra A^X of all functions from X to A.

An affine map from $(X, A(X))$ to $(Y, A(Y))$ is a function $f: X \to Y$ such that $\beta \circ f$ belongs to $A(X)$ whenever β belongs to $A(Y)$.

$(X, A(X))$ is called separated (or T_0) if $A(X)$ separates the points of X.

ASet(Ω) (Sep**ASet**(Ω)) will denote the category of (separated) affine sets over (A, Ω) and affine maps. If $\Omega = \{1_X\}$ (i.e., every family of functions is a subalgebra) we simply write **ASet**.

Many very familiar objects as posets, closure spaces, topological spaces, semilattices, distributive lattices, vector spaces, fuzzy spaces, approach spaces, pointed sets, are particular instances of

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affine sets and for these objects the usual morphisms are precisely the affine maps (i.e., **ASet** realizes these categories).

Some useful categorical properties of **ASet** and *Sep***ASet** are established. A closure operator called Zariski closure is introduced and used to given an internal characterization of epimorphisms and regular monomorphisms of *Sep***ASet**. These properties are then inherited by all the categories $\mathbf{ASet}(\Omega)$ as subcategories of **ASet**. Every $\mathbf{ASet}(\Omega)$ has the same subobjects, quotients and coproducts (but not products) as **ASet** (which in categorical terminology is expressed by: $\mathbf{ASet}(\Omega)$ is hereditarily coreflective in **ASet**.)

Every Affine set in *Sep***ASet** can be arranged as a matrix with $cardX$ rows and $card(A(X))$ columns, with elements in A and with distinct columns (in fact $A(X)$ is a set). Moreover the matrix has distinct rows if and only if the affine set is separated. Thus *Sep***ASet** is self-dual since every transpose of a matrix with distinct rows and distinct columns has the same property.

From the above description we deduce that **ASet** is a full subcategory (the "topological part") of the category $\mathbf{Chu}_A$, of Chu spaces over A and homomorphisms, intensively studied by V. Pratt [P] and his school for applications to Logic and Computer Science. The categories of the form $\mathbf{ASet}(\Omega)$ were studied by Y. Diers [D1, D2] (from which most of the terminology is deduced) to provide, for every field K, a convenient category containing both, the affine algebraic sets over K (= the sets of all roots of some set of polynomials of $K[x_1, \ldots, x_n]$ in the affine space K_n) and the projective algebraic sets in the projective space $P_n(K)$.

2. Affine sets and affine maps

For the categorical terminology we refer to [AHS].

Let **Set** be the category of all sets and functions and let A be a given set.

Definition 2.1. The category, **ASet**, of affine sets modelled by A (or, over A) is defined as follows. The objects are the pairs $(X, A(X))$ with X a set and $A(X)$ a subset of the set A^X of all functions from X to A. An affine map from $(X, A(X))$ to $(Y, A(Y))$ is a function $f : X \to Y$ such that $\beta \circ f \in A(X)$ for each $\beta \in A(Y)$.

Theorem 2.2. **ASet** *is a topological category over* **Set** *via the obvious forgetful functor* $U : \mathbf{ASet} \to \mathbf{Set}$.

Proof. For every set X, family of affine sets $\{(Y_i, A(Y_i)) : i \in I\}$ and family of functions $\{f_i : X \to Y_i : i \in I\}$ put in $A(X)$ all the compositions $\beta_i \circ f_i, i \in I,\ \beta_i \in A(Y_i)$. Then $A(X)$ is the unique initial structure in X determined by the above data. In particular an affine map $f : (X, A(X)) \to (Y, A(Y))$ is initial if and only if every α in $A(Y)$ is of the form $\beta \circ f$ for some β in $A(Y)$. □

If a subset M of a set X is endowed with the initial structure induced by the inclusion in $(X, A(X))$ then we shall call it an affine subset of $(X, A(X))$.

Note that in **ASet**, if A has more than one element, a constant function need not be affine. This is equivalent to say that a one-point set admits more than one affine structure (in fact it has 2^A structures).

To avoid the above inconvenience we may assume that every $A(X)$ contains the constant functions. The corresponding, still topological, category will be denoted by **ASET**.

The fact that **ASet** is topological implies several pleasant properties. Some of them, that we will use, are listed in the following proposition.

Proposition 2.3.

(1) **ASet** *admits unique final structures. That is: for every set* Y*, family of affine sets* $\{(X_i, A(X_i)) : i \in I\}$ *and family of functions* $\{f_i : X_i \to Y : i \in I\}$ *put in* $A(Y)$ *all the* $\beta : Y \to A$ *such that* $\beta \circ f_i$ *belongs to* $A(X_i)$ *for every* $i \in I$*. Then* $A(Y)$ *is the unique final structure in* Y *induced by the above data;*

(2) *In* **ASet** *the epimorphisms are the surjective (= onto) affine maps and the monomorphisms are the injective (= one to one) affine maps;*

(3) *In* **ASet** *regular monomorphism = initial monomorphism (= embedding) = extremal monomorphism;*

(4) *In* **ASet** *regular epimorphism = final epimorphism (= quotient) = extremal epimorphism;*

(5) **ASet** *has limits (i.e., it is complete) and the limts are lifted via initiality from the category* **Set**.

In particular the product of a family $(X_i, A(X_i))_{i \in I}$ *of affine spaces is the cartesian product of the underlying sets endowed with all the functions obtained, for every* $i \in I$*, by composing the* i^{th} *projection followed by the elements of* $A(X_i)$*;*

(6) **ASet** *has colimits (i.e., it is cocomplete). The colimits are lifted via finality from the category* **Set***;*

(7) *The set of affine structures on a given set X forms a complete lattice with the partial order: a structure is finer than another if the identity map is affine. In particular every set admits a finest affine structure, the discrete structure, and a coarsest affine structure, the indiscrete structure;*

(8) *(onto, embedding) is a proper factorization structure for morphisms of* **ASet***. That is: (i) every affine map f admits a factorization $f = m \circ e$ with e surjective affine map and m injective and initial affine map and (ii) for every $g \circ e = m \circ f$ with e onto and m embedding there is a (unique) affine map d such that $d \circ e = f$ and $m \circ d = g$;*

(9) *(quotient, one to one) is a factorization structure for morphisms of* **ASet***;*

(10) *A full and isomorphism closed subcategory* **X** *of* **ASet** *is epireflective if and only if it is stable under the formation of products (that is: every product in* **ASet** *of affine sets in* **X** *belongs to* **ASet***) and embeddings (that is: if $m : M \to X$ is an embedding and X belongs to* **X** *then M is in* **X***). Furthermore* **X** *is quotient reflective if and only if it is closed under refinements (that is: if U is finer than an affine structure $A(X)$ and $(X, A(X))$ belongs to* **X** *then (X, U) is in* **X***);*

(11) *A full and isomorphism closed subcategory* **X** *of* **ASet** *is bicoreflective if and only if it is stable under the formation of coproducets (that is: every coproduct in* **ASet** *of affine sets in* **X** *belongs to* **X***) and quotients (that is: if $q : X \to Q$ is a quotient map and Y belongs to* **X** *then Q is in* **X***);*

(12) *Every bicoreflective subcategory of* **ASet** *is still a topological category over* **Set***.*

A subcategory **X** of **ASet** is said to be *hereditary* if it is stable under embeddings.

A general method to construct hereditary coreflective subcategories of **ASet** is by means of an algebra structure on the set A.

Let Ω be an algebra structure on A; that is: for a given class **T** of sets let

$$\Omega = \{\omega_T : A^T \to A \mid T \in \mathbf{T}\}$$

be a family of operations in A. For every set X, the set A^X becomes a Ω-algebra by pointwise extension. We will denote by $\mathbf{ASet}(\Omega)$ the full subcategory of **ASet** whose objects are those $(X, A(X))$ for which $A(X)$ is a subalgebra of the Ω-algebra A^X.

Note that $\mathbf{ASet}(\Omega) = \mathbf{ASet}$ for $\Omega = \{id_A\}$.

Proposition 2.4. *For every set A and algebra structure Ω of A, $\mathbf{ASet}(\Omega)$ is a hereditary coreflective subcategory of* **ASet**.

Proof. Since the intersection of subalgebras is a subalgebra, the $\mathbf{ASet}(\Omega)$-bicoreflection of an affine set $(X, A(X))$ is obtained by intersecting all subalgebras of A^X which contains $A(X)$. Moreover taking into account that an affine subset $(M, A(M))$ of an affine set $(X, A(X))$ is a subset M of X with $A(M)$ given by the restrictions to M of the elements of $A(X)$, it follows easily that $(M, A(M))$ is in $\mathbf{ASet}(\Omega)$ whenever $(X, A(X))$ is. □

The question whether every hereditary coreflective subcategory of **ASet** is of the form $\mathbf{ASet}(\Omega)$ has a negative answer. Indeed the discrete affine sets form a hereditary coreflective subcategory of **ASet** which is not of the form $\mathbf{ASet}(\Omega)$.

3. Separated sets and epimorphisms

Definition 3.1. An affine set $(X, A(X))$ is said to be separated if $A(X)$ separates the points of X. That is: for every pair of distinct points x and x' of X there exists $\alpha \in A(X)$ with $\alpha(x) \neq \alpha(x')$.

Sep$\mathbf{ASet}(\Omega)$ will denote the full subcategory consisting of all separated affine sets which belong to $\mathbf{ASet}(\Omega)$.

Proposition 3.2. *Sep*$\mathbf{ASet}(\Omega)$ *is quotient reflective in* $\mathbf{ASet}(\Omega)$.

Proof. If $(M, A(M))$ is an affine subset of a separated affine set $(X, A(X))$ then $A(X)$ separates the points of X, consequently $A(M)$, consisting of the restrictions to M of functions in $A(X)$ separates the points of M. Let now $(x_i), (y_i)$ be distinct points of a product of separated affine sets and assume that $x_k \neq y_k$. Then there exists an α in $A(X_k)$ with $\alpha(x_k) \neq \alpha(y_k)$ consequently $\alpha \circ p_k$ separates (x_i) from (y_i). Finally it is clear that every affine structure finer than a separated structure is separated. Now the proof directly follows from Proposition 2.3(10). □

Definition 3.3. The affine set $\mathbf{A} = (A, \{id_A\})$ (which is obviously separated) is called the Sierpinski affine set.

Lemma 3.4.

(1) *$f: (X, A(X)) \to \mathbf{A}$ is affine if and only if f belongs to $A(X)$.*

(2) *For every affine set $(X, A(X))$ the canonical affine map $\phi: (X, A(X)) \to \mathbf{A}^{A(X)}$ is initial. In particular if $(X, A(X))$ is separated then ϕ is an embedding.*

Proof. The proof of (1) is trivial and that of (2) follows from (1) and the nature of products in **ASet** (see Proposition 2.3(5)). □

The category *Sep***ASet** is simply cogenerated by the Sierpinski affine set in the sense that:

Proposition 3.5. *An affine set is separated if and only if it can be embedded in a product of copies of* **A**.

Proof. By Proposition 3.2 every affine subset of a product of copies of **A** is separated since **A** is separated. Conversely if $(X, A(X))$ is separated then the canonical map, which is initial by Lemma 3.4(2), is injective, consequently it is an embedding. □

The above Proposition applies to every *Sep***X** with **X** hereditary coreflective subcategory of **ASet** if **A** is substituted with its coreflection and the product is taken in **X** (or, equivalently, if the affine subsets are taken in the coreflection of the product).

Now we introduce a closure operator in the whole category **ASet** which is useful in characterizing the epimorphisms in *Sep***X** for every hereditary coreflective subcategory **X**.

Definition 3.6. For every affine set $(X, A(X))$ and subset M of X, the formula

$$z_{(X,A(X))}(M) = \bigwedge \{[\alpha, \beta] \mid \alpha, \beta \in A(X), [\alpha, \beta] \supset M\}$$

where $[\alpha, \beta]$ denotes the equalizer of α and β, defines a closure which is called the Zariski closure of M in $(X, A(X))$.

Clearly, the Zariski closure is an extensive (i.e. $M \subset z(M)$) and monotone (i.e. $M \subset N \Rightarrow z(M) \subset z(N)$) operator. Moreover, it follows from the stability under pullback of regular monomorphisms that the Zariski closure is continuous (that is: for every affine map $f : (X, A(X)) \to (Y, A(Y))$ and $M \in SubX, f(z(M)) \subset z(f(M))$). In conclusion the Zariski closure is a closure operator of **ASet** in the sense of Dikranjan and Giuli ([DG1]).

Finally z is idempotent ($z(z(M)) = z(M)$), in fact $[\alpha, \beta] \supset M$ is equivalent to $[\alpha, \beta] \subset z(M)$, and it is hereditary (i.e. $z_{(Y,Z(Y))}(M) = (z_{(X,A(X))}(M) \cap Y)$) for every affine subset $(Y, A(Y))$ of $(X, A(X))$; in fact $A(Y)$ is given restricting the functions in $A(X)$ to Y. In conclusion we have proved the following:

Proposition 3.7. *The Zariski closure is an idempotent and hereditary closure operator of* **ASet**.

A subset M of X is called *z-closed* in $(X, A(X))$ if $z_{(X,A(X))}(M) = M$; an affine map f is called *z-closed* if it sends z-closed subsets into z-closed subsets. Since z is hereditary, for an affine subset to be z-closed is equivalent to say that the inclusion is z-closed ([GT]). An affine map $f : (X, A(X)) \to (Y, A(Y))$ is called *z-dense* if $z_{(Y,A(Y))}(f(X)) = Y$.

Note that every affine map inversely preserves the *z-closed* subsets.

The idempotent property and (hereditariness, hence) weak hereditariness of z gives (see [DG1, Prop 1.3]).

Corollary 3.8. *(z-dense, z-closed embedding) is a factorization structure of* **ASet** *(and of* **X***, for every hereditary coreflective subcategory* **X** *of* **ASet***)*

For the Zariski closure operator of **ASet** it is not true that $z_{(X,A(X))}(\emptyset) = \emptyset$ for all affine sets $(X, A(X))$ (i.e., it is not *grounded*) and it is not true that $z_{(X,A(X))}(M \cup N) = z_{(X,A(X))}(M) \cup z_{(X,A(X))}(N)$ (a closure operator grounded and with the above property is called *additive*) as is explained below.

In the topological category **ASet** the indiscrete objects are the ones of the form $(X, \emptyset)$. In particular the property $z(\emptyset) = \emptyset$ is not fulfilled in nonempty indiscrete objects. The (clearly) hereditarily bicoreflective subcategory of **ASet** consisting of all $(X, A(X))$ with $A(X)$ containing all constant functions, that we denoted by **ASET**, has the induced Zariski closure which is grounded, but not additive (i.e., in general $z(M \cup N) \neq z(M) \cup z(N)$). Take $X = \{a, b, c\}$ and for given $a_1 \neq a_2$ in A let $A(X) = \{\alpha, \beta, \gamma\}$, where $\alpha(a) = \alpha(b) = \alpha(c) = a_1, \beta(A) = \beta(b) = a_1$ and $\beta(c) = a_2, \gamma(a) = \gamma(a) = \gamma(c) = a_1, \gamma(b) = a_2$. Then every point is closed but every two-point subset is not closed. Adding the constant map $\delta(a) = \delta(b) = \delta(c) = a_2$ we have that $\{b, c\}$ is not closed which shows that even in **ASET** the Zariski closure is not additive.

Proposition 3.9. *An affine set X is separated if and only if the diagonal $\triangle_X$ is z-closed in $X \times X$.*

Proof. $(X, A(X))$ is separated if and only if for every $x \neq y$ in X there exists $\alpha \in A(X)$ with $\alpha(x) \neq \alpha(y)$ if and only if for every $x \neq y$ in X there exists $\alpha \circ p_1(x, y) \neq \alpha \circ p_2(x, y)$ if and only if for every $x \neq y$ in X, (x, y) is not in the diagonal $\triangle_X$ of $X \times X$. It follows from Lemma 3.4(2) that *Sep***ASET** (see [G1]) is the largest epireflective, not bireflective, subcategory of **ASET** and that **ASET** is *universal* in the sense of Marny [M]. The same is true for every hereditary coreflective subcategory of **ASET**. □

Theorem 3.10. *The epimorphisms in Sep***ASet** *are the z-dense affine maps and the regular monomorphisms are the z-closed embeddings.*

Proof. If $f : (X, A(X)) \to (Y, A(Y))$ is not an epimorphism then there exists a separated affine set $(Z, A(Z))$ and two different affine maps $\alpha, \beta : (Y, A(Y)) \to (Z, A(Z))$ such that $\alpha \circ f = \beta \circ f$. Now, by Lemma 3.4(2), the compositions $\phi \circ \alpha$ and $\phi \circ \beta$ are different so that there exists projection $p : \mathbf{A}^{\mathbf{A(X)}} \to \mathbf{A}$ such that $p \circ \phi \circ \alpha$ not coincide with $p \circ \phi \circ \beta$. But the last two affine maps coincide when restricted to the image of f which says that f is not z-dense.

Conversely assume that f is not z-dense. Then there is a point x in the remainder of $f(X)$ in Y and two elements α and β in $A(Y)$ which coincide in $f(X)$ and with $\alpha(x) \neq \beta(x)$, in other words the composition of α with f coincide with the composition of β with f while $\alpha \neq \beta$ in Y: Consequently f is not an epimorphism in *Sep***ASet**.

To show that every regular monomorphism is a z-closed embedding note that a regular monomorphism being an equalizer can be expressed as an inverse image of a diagonal (which is z-closed by Proposition 3.9) consequently it is z-closed. The converse follows from the fact that the z-closed subsets are an intersection of equalizers with codomain **A**. □

4. Examples

(1) In the two point set $A = \{0,1\}$ let us consider the following operations: for arbitrary set T we let

$$\omega_T(a_t)_{t\in T} = \max_{t\in T} a_t$$

for finite set T we let

$$\omega'_T(a_t)_{t\in T} = \min_{t\in T} a_t.$$

Identifying the open sets in a topological space with their characteristic functions it is clear that **Top**, the category of topological spaces and continuous maps, is isomorphic to $\mathbf{ASet}(\Omega)$ ([HL], [D1]). **A** is in this case the Sierpinski space, and the separated affine sets are the usual T_0-spaces.

As observed in [G1] the Zariski closure in **Top** is the well known b-closure defined by

$z_X(M) = \{x \in X \mid U_x \cap M \cap \overline{x} \neq \emptyset$ for every nbhd of $x\}$.

Note that (see [G1]) in $\{0,1\}$ **Set** the Zariski closure and the b-closure do not coincide.

It is well known that there exist non-onto epimorphisms in $\mathbf{Top}_0$.

(2) A closure space is a set X endowed with a subset of subsets of X which is closed under arbitrary unions and contains the whole set X. A closure preserving map is a function inversely preserving the elements of the structure (i.e., a continuous map). The corresponding category **CS** of closure spaces and closure preserving maps (see e.g. [DGT], [DGL]) is obtained from the previous example by deleting the second family of operations except the 0-ary one. The Sierpinski closure space is as in **Top**, separated = T_0 and the Zariski closure coincides with the b-closure.

(3) In the unit interval $A = [0,1]$ let us consider the following operations : for arbitrary set T we let

$$\omega_T(a_t)_{t\in T} = \sup_{t\in T} a_t,$$

for finite set T we let

$$\omega'_T(a_t)_{t\in T} = \min_{t\in T} a_t.$$

Then it is clear that $\mathbf{Set}(A, \Omega)$ is isomorphic to the category **Fuz** of fuzzy topological spaces in the sense of [C]. Here **A** coincides with the fuzzy Sierpinski space $([0,1], \{0, 1, id\})$ considered in [S2]. The Zariski closure operator of **Fuz**, being the regular closure operator induced by the fuzzy Sierpinski space, coincides with the one considered in ([AC, Remark 2.8]).

The separated fuzzy spaces coincide with the usual fuzzy T_0 spaces.

(4) In the set $A = [0, \infty]$ let us consider the two kind of operations given in Example above and the additional operations:

for every $a \in [0, \infty]$ we consider the 1-ary operation

$$\omega'_a(x) = (x - a) \vee 0.$$

It is shown in [L] that $\mathbf{Set}(A, \Omega)$ coincides with the category **AP** of approach spaces defined by R. Lowen.

As far as the Author knows there is no satisfactory description of the Zariski closure (consequently of the epimorphisms) in the category of separated approach spaces.

(5) If A has at least two points then in the category **ASet** there are non onto epimorphisms between finite affine sets (not true e.g. in $\mathbf{Top}_0$). In fact every inclusion of a pointwise affine set (with the identity as structure) into **A** is forced to be an epimorphism.

References

[AHS] J. Adamek, H. Herrlich, G.E. Strecker, *Abstract and concrete categories*, John Wiley & Sons, New York, 1990.

[AC] I.W. Alderton, G. Castellini, *Epimorphisms in categories of separated fuzzy topological spaces*, Fuzzy Sets and Systems 56 (1993), 323-330.

[C] C.L. Chang, *Fuzzy topological spaces*, Jour. Math. Anal. Appl. 24 (1968), 182-189.

[DGL] D. Deses, E. Giuli, E. Lowen-Colebunders, *On complete objects in the category of T_0 closure spaces*, Applied General Topology 4 (2003), 25-34.

[D1] Y. Diers, *Categories of algebraic sets*, Applied Categorical Structures 4 (1996), no. 2-3, 329-341.

[D2] Y. Diers, *Affine algebraic sets relative to an algebraic theory*, Journal of Geometry 65 (1999), no. 1-2, 54-76.

[DG1] D. Dikranjan, E. Giuli, *Closure operators I*, Topology Appl. 27 (1987), 129-143.

[G1] E. Giuli, *On classes of T_0 spaces admitting completions*, Applied General Topology 4 (2003), 143-155.

[G2] E. Giuli, *The structure of affine algebraic sets*, Categorical Structures and their Applications (Berlin, 2003), 113-121, World Scientific Publishing Co., Singapore 2004.

[G3] E. Giuli, *Zariski closure, completeness and compactness*, Topology Appl. (to appear).

[GT] E. Giuli, W. Tholen, *Openness with respect to a closure operator*, Applied Categorical Structures 8 (2000), 487-502.

[HL] H. Herrlich, R. Lowen, *On simultaneously reflective and coreflective subconstructs*, in: Proceedings Symposium on Categorical Topology UCT 1994, (1999), 121-130.

[L] R. Lowen, *Approach Spaces: the missing link in the Topology-Uniformity-Metric Triad*, Oxford Mathematical Monographs, Oxford University Press (1997).

[M] Th. Marny, *On epireflective subcategories of topological categories*, Gen. Topology Appl. 10 (1979), 175-181.

[P] V.R. Pratt, *Chu spaces and their interpretation as concurrent objects*, Springer Lecture Notes in Computer Science 1000 (1995), 392-405.

[S2] A.K. Srivastava, *Fuzzy Sierpinski space*, Jour. Math. Anal. Appl. 103 (1993), 103-105.

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