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ON UNIFORMITIES AND UNIFORMLY CONTINUOUS FUNCTIONS ON FACTOR-SPACES OF TOPOLOGICAL GROUPS

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Abstract. Is it true that the left and the right uniformities on a topological group coincide as soon as every left uniformly continuous real valued function is right uniformly continuous? This question is known as Itzkowitz's problem, and it is still open. We show that the generalization of the problem to homogeneous factor-spaces of topological groups has a negative answer.

The left uniform structure of a topological group G, which we will denote by $\mathcal{U}_l(G)$, has a basis of entourages of the form

$$V_l = \{(x, y) \in G \times G : x^{-1}y \in V\},$$

where V is a neighborhood of the identity. Similarly, the right uniform structure will be denoted by $\mathcal{U}_r(G)$ and has a basis of the form

$$V_r = \{(x, y) \in G \times G : xy^{-1} \in V\}.$$

A real valued function on a topological group G is *left uniformly continuous* if it is uniformly continuous with respect to the left uniform structure. Similarly, for right uniform continuity. Plainly, every left uniformly continuous real valued function is right uniformly continuous and vice versa, whenever left and right uniform structures coincide. The converse is still unknown. This problem is known as Itzkowitz's problem.

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The question has been answered in the affirmative for particular classes of topological groups, including locally compact groups, metrisable groups, locally connected groups and some other classes of groups. (see [2, 3, 5, 8, 9, 10, 12]).

Itzkowitz's question can be stated not only for topological groups but for their homogeneous factor-spaces, because they also support two natural uniform structures (although not necessarily compatible with the factor topology), the left and the right ones, see [14]. Do these two uniform structures in G/H coincide as soon as every left uniformly continuous real valued function on G/H is right uniformly continuous, where H is a subgroup of the group G? The aim of this paper is to answer the question in the negative, by constructing a counterexample. Notice that on our example, both uniform structures on G/H are in fact compatible.

Let H be a subgroup of a topological group G. By $G/H = \{gH\colon g \in G\}$ we denote, as usual, the set of all left H-cosets of G. Let $\pi\colon G \to G/H$ denote the natural quotient mapping, $\pi(g) = gH$. We equip G/H with the quotient topology with regard to the the mapping π. The *left uniform structure*, $\mathcal{U}_l(G/H)$, on G/H is the finest uniformity such that the quotient mapping $\pi : (G, \mathcal{U}_l(G)) \mapsto G/H$ is uniformly continuous. Similarly, the *right uniform structure*, $\mathcal{U}_r(G/H)$, on G/H is the finest uniformity such that the mapping $\pi : (G, \mathcal{U}_r(G)) \mapsto G/H$ is uniformly continuous. It should be emphasized that if G/H is equipped with the indiscrete uniformity, then π is uniformly continuous. Thus, the definition always makes sense.

The uniform structure $\mathcal{U}_r(G/H)$ is also called the *standard uniformity* in [1]. It is worth noticing that the images of $\mathcal{U}_r(G)$ and $\mathcal{U}_l(G)$ under $\pi \times \pi$ are included in $\mathcal{U}_l(G/H)$ and $\mathcal{U}_l(G/H)$ respectively.

Proposition 1. *Let H be an open subgroup of a topological group G. The uniform structure $\mathcal{U}_l(G/H)$ is discrete.*

Proof. Since H is a neighborhood of the identity of G, the image of the entourage H_l of the diagonal under $(\pi \times \pi)$ is the set $\{(fH, gH)\colon f, g \in G, f^{-1}g \in H\} = \{(fH, gH) : f, g \in G, g \in fH\} = \{(fH, fH) : f \in G\}$. The latter set is the diagonal of $G/H \times G/H$. Hence, $\mathcal{U}_l(G/H)$ is just the discrete uniform structure. $\square$

Let X be a set. Let S_X denote the symmetric group on X, consisting of *all* self-bijections of the set X and equipped with the composition of bijections as the group law. Let γ be a partition of X. Define

$$\mathrm{St}_\gamma = \{f \in S_X : \forall B \in \gamma, f(B) = B\}.$$

Plainly, St_γ is a subgroup of S_X.

Proposition 2. *The subgroups* St_γ, *as* γ *runs over all partitions of* X *with* $|\gamma| \leq \mathfrak{c}$, *form a neighborhood basis for a Hausdorff group topology on* S_X, *which we will denote by* $\tau_\mathfrak{c}$.

Proof. Let $\Gamma = \{\gamma : \gamma \text{ is a partition of X with } |\gamma| \leq \mathfrak{c}\}$. Since, for any $\gamma \in \Gamma$, St_γ is a group, $\mathrm{St}_\gamma^2 = \mathrm{St}_\gamma = \mathrm{St}_\gamma^{-1}$.

Let $\gamma \in \Gamma$ and $g \in \mathrm{St}_X$. Define a cover β of X as follows: $B \in \beta$ if and only if $B = g(A)$, for some A in γ. Evidently β is in Γ and $g^{-1}\mathrm{St}_\beta g \subseteq \mathrm{St}_\gamma$.

If $\gamma, \beta \in \Gamma$, then define a cover α of X as follows: A set D is in α if and only if $D = A \cap B$ for some $A \in \gamma$ and $B \in \beta$. Clearly, α is in Γ and $\mathrm{St}_\alpha \subseteq \mathrm{St}_\gamma \cap \mathrm{St}_\beta$.

Now the topology can be defined by taking the set

$$\Omega = \{\mathrm{St}_\gamma : \gamma \text{ is a partition of } X, |\gamma| \leq \mathfrak{c}\},$$

as a basis of open neighborhoods of the identity.

If ι is the identity of S_X and $f \in S_X$ not equal to ι, then there exists $x \in X$ such that $f(x) \neq x$. Put $\gamma = \{\{x\}, \{f(x)\}, X \setminus \{x, f(x)\}\}$. Then clearly, $\gamma \in \Gamma$ and $f \notin \mathrm{St}_\gamma$. This implies $\bigcap_{\gamma \in \Gamma} \mathrm{St}_\gamma = \iota$. Hence, S_X is Hausdorff. □

Remark 3.

(1) The $\tau_\mathfrak{c}$ topology on S_X is finer than the pointwise convergence topology. Indeed, for any finite subset M of X, let $\gamma = \{\{x\} : x \in M\} \cup \{X \setminus M\}$. Clearly, $\gamma \in \Gamma$ and St_γ consists of all permutations leaving fixed each element of M.

(2) If $|X| \leq \mathfrak{c}$, then one obtains just the discrete topology on S_X. Indeed $\gamma = \{x\}_{x \in X}$ is a partition of X, of cardinality $\leq \mathfrak{c}$ so St_γ is just the identity of S_X.

Let γ be any partition of a set X, with $|\gamma| \leq \mathfrak{c}$. Let $V_\gamma = \bigcup_{A \in \gamma} A \times A$. It is easy to see that the collections of all sets of the V_γ is a basis for a uniform structure on X. Let us denote this uniform structure by $\mathcal{U}_\mathfrak{c}(X)$.

Proposition 4. *If a set X and the real line $\mathbb{R}$ are equipped with the uniformity $\mathcal{U}_{\mathfrak{c}}(X)$ and the additive uniformity, respectively, then every real valued function f on X is uniformly continuous.*

Proof. Given a function $f : X \mapsto \mathbb{R}$, define a partition γ of X as the collection of all sets $f^{-1}(x), x \in \mathbb{R}$. If now $x, y \in X$ and $(x, y) \in V_\gamma$ then $f(x) = f(y)$. Hence, f is uniformly continuous. □

Lemma 5. *Let a be an arbitrary but fixed element of a set X and let γ be a partition of X. If $b, c \in X$ then the following are equivalent.*

(1) *There are $g, f \in S_X$ such that $f^{-1}(\gamma) = g^{-1}(\gamma)$ and $f(a) = b, g(a) = c$.*
(2) *There exists $A \in \gamma$ such $b, c \in A$.*

Proof. We will only prove (2) $\Rightarrow$ (1) as the other implication is obvious. Let $f \in S_X$ such $f(a) = b$. Define $g : X \mapsto X$ by

$$g(x) = \begin{cases} c, & \text{if } x = a, \\ b, & \text{if } x = f^{-1}(c), \\ f(x), & \text{otherwise.} \end{cases}$$

Clearly, $g \in S_X$. Notice that $f^{-1}(c) = g^{-1}(b), f^{-1}(b) = g^{-1}(c)$ and for each $x \in X \setminus \{b, c\}, f^{-1}(x) = g^{-1}(x)$. It follows that that for each $A \in \gamma, f^{-1}(A) = g^{-1}(A)$. That is $f^{-1}(\gamma) = g^{-1}(\gamma)$. □

For an $a \in X$, let St_a denote the subgroup of S_X consisting of elements of S_X that stabilize a. In our notation this is equal to St_γ, where $\gamma = \{a\} \cup \{X \setminus \{a\}\}$. Notice that every such subgroup is open in the topology of pointwise convergence, and therefore in the topology $\tau_{\mathfrak{c}}$ as well.

Theorem 6. *Let $a \in X$ be arbitrary. Denote $H = St_a$. If S_X is endowed with the topology $\tau_{\mathfrak{c}}$, then*

$$\mathcal{U}_r(S_X/H) = \mathcal{U}_{\mathfrak{c}}(S_X/H) \cong \mathcal{U}_{\mathfrak{c}}(X).$$

Proof. The group S_X acts on both X and S_X/St_a by $(f, x) \mapsto f(x)$ and $(f, gSt_a) \mapsto fgSt_a$, respectively. Define a map $\Phi : S_X/St_a \mapsto X$ by $\Phi(fSt_a) = f(a)$. This map can be easily shown to be well defined and the image of St_a under Φ is a.

The map Φ is equivariant, because if $g \in S_X$ and $fSt_a \in S_X/St_a$ then

$$\Phi(gfSt_a) = gf(a) = g(f(a)) = g(\Phi(fSt_a)).$$

Moreover, Φ is a bijection: for every $x \in X$ there is a $f \in S_X$ such that $f(a) = x$, together with the fact that every $h \in St_a$ stabilizes the point a, imply $\Phi(fSt_a) = x$. Also if $\Phi(fSt_a) = \Phi(gSt_a)$, for some $f, g \in S_X$, then $f(a) = g(a)$ so that $g^{-1}f(a) = a$. Therefore, $g^{-1}f \in St_a$ and hence $fSt_a = gSt_a$.

Now for any partition γ of X with $|\gamma| \leq \mathfrak{c}$, the set

$$\begin{aligned}\{(fH, gH) : fg^{-1} \in \mathrm{St}_{\gamma}\} &= \{(fH, gH) : f^{-1}(\gamma) = g^{-1}(\gamma)\} \\ &\cong \{(f(a), g(a)) : f^{-1}(\gamma) = g^{-1}(\gamma)\}.\end{aligned}$$

By Lemma 5, the above set is equal to the set

$$\{(x, y) : (\exists A \in \gamma) x, y \in A\} = \bigcup_{A\in\gamma} A \times A = V_\gamma. \qquad \square$$

Example 7. The group S_X on a set X with cardinality greater than $\mathfrak{c}$ equipped with the topology in $\tau_{\mathfrak{c}}$ with its subgroup $H = \mathrm{St}_a$ provides a negative answer to the generalization of Itzkowitz's question to factor-spaces.

Namely, every right uniformly continuous function on the factor-space S_X/St_a is left uniformly continuous, yet the right and the left uniformities on S_X/St_a are different. Indeed, Proposition 1 implies that any real valued function on S_X/H is uniformly continuous with respect to $\mathcal{U}_l(S_X/H)$. At the same time Proposition 4 and Theorem 6 assure that any such function is uniformly continuous with respect to $\mathcal{U}_r(S_X/H)$. But Proposition 1 and Theorem 6 show that these uniform structures are different.

Remark 8. Notice that both the left and the right uniformities on S_X/H are compatible (they generate the discrete topology on S_X/H, which is the quotient topology). This follows from Proposition 1 and Theorem 6.

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