Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 29, No.2, 2005
Pages 527-537

ON SOLUBILITY OF SUBLOCALES

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Abstract. We show that a sublocale S of a spatial locale X is soluble (in the sense that there is a locale map $X[S] \to X$ universal among maps along which S pulls back to a closed sublocale) if and only if S is complemented in the lattice of sublocales of X. This partially answers a question left open in an earlier paper [5] by the author. The paper also includes a number of other results on solubility of sublocales.

Introduction

This paper is a sequel to [5], in which (on the way to a result characterizing open maps of locales) I considered the problem of 'dissolving' a set Σ of sublocales of a given locale X, that is, finding a universal way of forcing the sublocales in the set to become closed as sublocales of a new locale $X[\Sigma]$. Such a construction, if possible in general, would have many uses in locale theory (including a much simpler proof of the characterization of open maps than the one I eventually gave). There is a 'simple-minded' construction of $X[\Sigma]$ which looks as if it ought to work in general (and which was at one time widely believed among locale-theorists to do so), but which in fact fails in the cases of most interest: as shown by Banaschewski [1], it works for a singleton set $\Sigma = \{S\}$ if and only if S is complemented as a member of the lattice of sublocales of X. After reviewing the proof of this result in [5], I was obliged to leave open both the question whether there are sets Σ for which

2000 *Mathematics Subject Classification.* Primary 06D22; Secondary 18B30, 54B05.

Key words and phrases. Locale, sublocale, solubility.

something other than the 'simple-minded' construction of $X[\Sigma]$ will work, and the question whether there exists a Σ for which it can be proved that no construction will work.

The first of these questions is still open (though the results in this paper provide evidence that the answer is likely to be negative), but we now have a positive answer to the second: we shall show that if X is any spatial locale and $\Sigma = \{S\}$ for any non-complemented sublocale S of X, then $X[\Sigma]$ does not exist. Whilst it is satisfying to have a definite answer to this question, I have to confess that the form of the answer is not quite what I had hoped for: the restriction to spatial locales carries with it a need to employ classical logic, which 'goes against the grain' for those who (like me) value locale theory for its inherently constructive nature (see [3]). There is therefore work still to be done, in searching for a constructive proof of the main result of this paper (and one which will work for sublocales of non-spatial locales); but at present I have very little idea of how one might seek such a proof, and so I hope that the non-constructive proof presented here will be seen as an acceptable stop-gap.

The paper is organized in two sections. In the first, we briefly review the results on solubility which were presented in [5], and also present a couple of elementary results which were not included there because they were not relevant to the main theme of that paper; the results in this section deal with arbitrary locales, and are constructively valid. Section 2 presents the new results on sublocales of spatial locales; the Law of Excluded Middle is assumed throughout this section.

I have to conclude this Introduction with an apology to the audience who heard my talk at the Cape Town Topology conference, on which this paper is based. In the talk, I presented a purported counterexample to the conjectures which stand at the end of this paper (which had in fact been 'discovered' only a few days before the talk), and concluded that there were no plausible general conjectures about solubility remaining. That counterexample disintegrated when I attempted to write down the details; however, the manner of its disintegration led me to an appreciably stronger version of the main positive result (Theorem 2.4) than the one I presented in Cape Town. This has encouraged me to reinstate the conjectures referred to above.

1. Background and constructive results

Our notation and terminology for locales will be that of [4], Chapter C1: in particular, we distinguish notationally between a locale X and its frame $\mathcal{O}(X)$ of open sublocales (though not between a spatial locale and the sober space to which it corresponds). We write **Frm** for the category of frames and frame homomorphisms, and **Loc** for the dual category of locales. For a general locale X, we write X_p for the *spatial part* of X, i.e. the sublocale of X which is the union of its points — or equivalently, the space obtained by equipping the set of points of X (that is, locale maps $x\colon 1 \to X$) with the (sober) topology

$$\{\{x \in X_p \mid x^*U = 1\} \mid U \in \mathcal{O}(X)\} .$$

We write X_d for the *dissolution* of X, i.e. the locale defined by $\mathcal{O}(X_d) = N\mathcal{O}(X)$, where $N\mathcal{O}(X)$ is the frame of nuclei on $\mathcal{O}(X)$ (dual to the lattice $\mathrm{Sub}(X)$ of sublocales of X). It is well known that the canonical map $d\colon X_d \to X$ (dual to the frame map $\mathcal{O}(X) \to N\mathcal{O}(X)$ which sends an open sublocale U to the nucleus corresponding to its closed complement $\complement U$) has the property that d^*S is a closed sublocale of X_d for every sublocale S of X, and is universal with this property (i.e. if $f\colon Y \to X$ is any locale map such that f^*S is closed in Y for every $S \in \mathrm{Sub}(X)$, then f factors uniquely through $X_d \to X$). We may think of X_d as playing a role in locale theory analogous to that of the discrete modification of a space X (that is, the space obtained by retopologizing the underlying set of X with the discrete topology).

In point-set topology, we frequently wish to 'modify' a topology less drastically than this: that is, to declare certain subsets of X to be closed, but not all of them. Analogously, in locale theory we might wish to 'dissolve' merely a certain set of sublocales, in the following sense:

Definition 1.1. Let X be a locale. We say a set Σ of sublocales of X is *soluble* if there exists a locale map $d_\Sigma\colon X[\Sigma] \to X$ with the property that $d_\Sigma^*(S)$ is closed in $X[\Sigma]$ for every $S \in \Sigma$, and universal with this property. If Σ is a singleton $\{S\}$, then we say that the sublocale S is soluble (and write $X[S]$ for $X[\{S\}]$).

An elementary argument (sketched in [5]) shows that if $X[\Sigma]$ exists, then $\mathcal{O}(X[\Sigma])$ is necessarily isomorphic to a subframe of $\mathcal{O}(X_d) = N\mathcal{O}(X)$ containing all the closed nuclei together with the set (J_Σ, say) of nuclei which correspond to the sublocales in Σ. It is therefore tempting to conjecture that, if $X[\Sigma]$ exists, then $\mathcal{O}(X[\Sigma])$ should simply be the subframe of $N\mathcal{O}(X)$ generated by the nuclei just mentioned. In fact we do not have any counterexample to this conjecture; but since we do not have a proof either, we shall introduce the different notation $X\langle\Sigma\rangle$ for the locale corresponding to this subframe of $N\mathcal{O}(X)$, and we shall say that Σ is *exactly soluble* if $X\langle\Sigma\rangle$ has the universal property of $X[\Sigma]$.

For some time, there was a folk-belief amongst locale-theorists that any set of sublocales was soluble, and in fact exactly soluble. However, this turns out not to be so. In [5], we showed:

Lemma 1.2. *The following conditions on a set Σ of sublocales of X are equivalent:*

(i) *Σ is exactly soluble.*
(ii) *The canonical locale map $X\langle\Sigma\rangle \to X$ is a monomorphism in* **Loc**.
(iii) *The canonical locale morphism $X_d \to X\langle\Sigma\rangle$ (corresponding to the frame inclusion $\mathcal{O}(X\langle\Sigma\rangle) \to N\mathcal{O}(X)$) is a pullback-stable epimorphism in* **Loc**.
(iv) *$X_d \to X\langle\Sigma\rangle$ is a hereditary epimorphism in* **Loc** *(i.e. remains epic under pullback along inclusions).*
(v) *Every sublocale in Σ pulls back to a closed sublocale of $X\langle\Sigma\rangle$.*
(vi) *Every closed sublocale of $X\langle\Sigma\rangle$ is the pullback of some sublocale of X.*

Although several of these conditions appear 'self-evidently true', it turns out that they are not always satisfied. Indeed, in [5], we showed by means of a result of Banaschewski [1] that they hold for a singleton $\{S\}$ iff the sublocale S is complemented as an element of $\mathrm{Sub}(X)$. However, in that paper we were unable to exclude the possibility that in some cases $X[\Sigma]$ might exist without coinciding with $X\langle\Sigma\rangle$; nor were we able to produce any example of a set Σ for which we could prove the non-existence of $X[\Sigma]$. The first of these problems remains open (though we shall see that this behaviour cannot occur if X is spatial and Σ is a singleton); but the second has now been answered, and the main purpose of this paper is to place the answer on record.

However, before we begin investigating the particular cases in which we are now able to show that $X[\Sigma]$ does not exist, it seems appropriate also to place on record a couple of elementary facts about solubility which were not mentioned in [5]. The first may be seen as a justification for concentrating one's attention on solubility of single sublocales, rather than arbitrary sets of sublocales.

Lemma 1.3. *Let Σ_i $(i \in I)$ be a family of sets of sublocales of a locale X. If each Σ_i is soluble (resp. exactly soluble), then $\bigcup_{i \in I} \Sigma_i$ is soluble (resp. exactly soluble).*

Proof. The first assertion is immediate from the universal property of $X[\Sigma]$: if each $X[\Sigma_i]$ exists, then the 'wide pullback' of the $X[\Sigma_i] \to X$ (that is, their product in the slice category $\mathbf{Loc}/X$) is easily seen to have the universal property of $X[\bigcup_{i \in I} \Sigma_i]$. For the second, recall that limits in **Loc** are colimits in **Frm**; so if each $\mathcal{O}(X[\Sigma_i])$ is generated by the closed nuclei and the members of J_{Σ_i}, then $\mathcal{O}(X[\bigcup_{i \in I} \Sigma_i])$ is generated by the union of these sets, and hence coincides with $\mathcal{O}(X\langle\bigcup_{i \in I} \Sigma_i\rangle)$. □

Corollary 1.4. *If Σ is a set of sublocales of X such that each $S \in \Sigma$ is soluble (resp. exactly soluble), then Σ is soluble (resp. exactly soluble).* □

Corollary 1.4 does not entirely reduce the study of solubility of families of sublocales to that of solubility of individual sublocales, because its converse is not true: for eample, the family of all sublocales of X is always (exactly) soluble, but there are individual sublocales which are not soluble. Nevertheless, for the rest of this paper we shall concentrate mainly on solubility of single sublocales.

From Corollary 1.4 and the result of [5], already quoted, we may deduce that any set of complemented sublocales is exactly soluble. In passing, we remark that the standard proof that $N\mathcal{O}(X)$ is generated by open and closed nuclei (see [4], C1.1.17) allows us to express an arbitrary sublocale S as an intersection

$$S = \bigcap\{U \cup \complement(jU) \mid U \in \mathcal{O}(X)\}$$

where j is the nucleus on $\mathcal{O}(X)$ corresponding to S; and the sublocales of the form $U \cup \complement(jU)$ are all complemented.

Hence, for any Σ, we have an exactly soluble family

$$\widetilde{\Sigma} = \{U \cup \complement(jU) \mid U \in \mathcal{O}(X), j \in J_\Sigma\}$$

such that every sublocale in Σ pulls back to a closed sublocale of $X[\widetilde{\Sigma}]$. However, the latter will not in general enjoy the universal property we require for $X[\Sigma]$: for example, if Σ consists of closed sublocales (so that $X[\Sigma] \cong X$), it is easy to see that $\widetilde{\Sigma}$ will in general contain non-closed sublocales.

Another trivial deduction from the definition of $X[S]$ by a universal property is the following:

Lemma 1.5. *Suppose S is a soluble sublocale of X, and let $f : Y \to X$ be any locale map. Then f^*S is a soluble sublocale of Y, and in fact we have a pullback square*

$$\begin{array}{ccc} Y[f^*S] & \longrightarrow & Y \\ \Big\downarrow & & \Big\downarrow f \\ X[S] & \longrightarrow & X \end{array}$$

in **Loc**.

Proof. It is easy to see that the pullback of $X[S] \to X$ along f has the universal property of $Y[f^*S] \to Y$. □

It is easy to see that Lemma 1.5 could have been stated for families of sublocales, rather than single sublocales. However, the following easy but significant consequence requires the assumption that we are dissolving a single sublocale.

Corollary 1.6. *Suppose S is a soluble sublocale of X. Then the closed sublocale of $X[S]$ to which S pulls back is isomorphic to S itself.*

Proof. Applying 1.5 to the inclusion $S \rightarrowtail X$, we see that the pullback is isomorphic to $S[S]$. But since S is closed as a sublocale of itself, the latter is clearly isomorphic to S. □

A similar argument shows that if T is any sublocale of X disjoint from S (i.e. satisfying $S \cap T = \emptyset$), then it also pulls back to an isomorphic copy of itself as a sublocale of $X[S]$. In particular, if S has a complement in $\mathrm{Sub}(X)$, we may use this observation to reconstruct the description of $X[S]$ given in [5]. (But in general the result about sublocales disjoint from S is less useful than that about S itself, since there may be rather few sublocales disjoint from S.)

2. Sublocales of spatial locales

We now turn to the problem of determining which sublocales of a spatial locale are soluble. As mentioned in the Introduction, we shall feel free to use classical logic in this section; in particular, we rely heavily on the facts that discrete locales have no proper dense sublocales, and that closed sublocales of spatial locales are spatial — both of which are equivalent to the Law of Excluded Middle (cf. [4], C1.2.6(b)). We do not distinguish between sober spaces and spatial locales. However, we shall need to be aware of the fact that not every subspace of a sober space is sober (and hence not every subspace of a sober space corresponds to a sublocale); we recall from [4], C1.2.5, that there is a closure operation (called *subclosure*) on the subsets of an arbitrary space X, which if X itself is sober yields the sobrification operation on subspaces of X.

We shall need a couple of results on almost discrete spaces. We say that a space X is *almost discrete* if there is just one $x \in X$ such that $\{x\}$ is not open.

Lemma 2.1. *Any topology on a set may be expressed as an intersection of almost discrete topologies on the same set.*

Proof. Let $(X, \mathcal{O})$ be an arbitrary topological space. For each $x \in X$, define a new topology $\mathcal{O}_x$ as follows: each set not containing x is in $\mathcal{O}_x$, and a set containing x is in $\mathcal{O}_x$ iff it is a (not necessarily open) neighbourhood of x in $(X, \mathcal{O})$. It is clear that $\mathcal{O}_x$ is a topology on X, and that it is almost discrete (unless $\{x\}$ happens to be in $\mathcal{O}$, in which case $\mathcal{O}_x$ is discrete and we can omit it from the intersection). But a set belongs to $\mathcal{O}_x$ for all x iff it is an $\mathcal{O}$-neighbourhood of each of its points, iff it is in $\mathcal{O}$. □

It is easily seen that an almost discrete space is sober (an irreducible closed subset can contain at most two points, and if it contains two then one of them is the distinguished point). Thus Lemma 2.1 trivially implies that every topology may be expressed as an intersection of sober topologies.

The importance of almost discrete spaces for us lies in the fact that they have rather few sublocales. In fact we have:

Lemma 2.2. *If X is almost discrete, then every sublocale of X is spatial.*

Proof. First we note that X has a dense open subspace $X \setminus \{x\}$ (where x is the distinguished point) which is discrete, and hence must be its smallest dense sublocale X_b. Moreover, any sublocale of X must intersect $\{x\}$ in either $\{x\}$ itself or $\emptyset$; so there are only two dense sublocales, and they are both spatial. Now an arbitrary sublocale may be expressed as a dense sublocale of a closed sublocale; and any closed sublocale of X is (spatial and) either discrete or almost discrete, so in either case all its dense sublocales are spatial. □

We remark in passing that, for an arbitrary X, the spatiality of all sublocales of X is equivalent to the spatiality of X_d. For if all sublocales of X (including X itself) are spatial, then $\mathcal{O}(X_d) \cong \mathrm{Sub}(X)^{\mathrm{op}}$ may be identified with the poset of 'sub-open' subsets of X (that is, subsets whose complements are subclosed), and this is clearly a topology on X. Conversely, if X_d is spatial, then for any sublocale S the top and left edges of the pullback square

$$\begin{array}{ccc} S_d & \longrightarrow & S \\ \downarrow & & \downarrow \\ X_d & \longrightarrow & X \end{array}$$

represent S as an epimorphic image of a closed sublocale of X_d; so it is spatial.

Another result we shall need is a mild strengthening of [4], C1.2.13. Recall that, since the lattice $\mathrm{Sub}(X)$ of sublocales of X is a co-Heyting algebra, every sublocale S has a *supplement*, i.e. a smallest sublocale T such that $S \cup T = X$. We say a sublocale is *supplementary* if it occurs as a supplement (equivalently, if it is the supplement of its own supplement).

Lemma 2.3. *Every supplementary sublocale of a spatial locale is spatial.*

Proof. Let S be a sublocale of a spatial locale X, and let T be its supplement. We do not assume that S itself is spatial, and so we write S_p for its spatial part (i.e. the set of points $x:1 \to X$ which factor through $S \rightarrowtail X$, topologized as a subspace of X). For each point x of $X \setminus S_p$, we have $x^*S = \emptyset$ and hence $x^*T = 1$, i.e. x must factor through T. However, if we set T to be the smallest sublocale of X containing all these points — i.e. the subclosure of $X \setminus S_p$, topologized as a subspace of X — then $S \cup T$ contains all the points of X and so must be the whole of X. So this T is indeed the supplement of S; and it is spatial. □

The converse of 2.3 does not hold. Let $X = \mathbb{N} \cup \{\infty\}$, topologized as the sobrification of the cofinite topology on $\mathbb{N}$; i.e. a subset is open iff it is either empty or a cofinite set containing ∞. Then the supplement of $\{\infty\}$ is the whole of X, so $\{\infty\}$ is not a supplementary sublocale.

We are now ready for our main result:

Theorem 2.4. *For a sublocale S of a spatial locale X, the following are equivalent:*

(i) *S is complemented in* $\mathrm{Sub}(X)$.
(ii) *S is exactly soluble.*
(iii) *S is soluble.*

Proof. The equivalence of (i) and (ii) was proved in [5], and that (ii) ⇒ (iii) is obvious; so we have only to prove that (iii) implies (i). So suppose $X[S]$ exists; we shall write V for the open sublocale of $X[S]$ complementary to the closed sublocale to which S pulls back.

We note first that, since every sublocale of the one-point locale is closed, each point $x:1 \to X$ factors uniquely through $X[S] \to X$; moreover, the points of $X \setminus S_p$ must correspond bijectively to points of V. Hence we may regard the spatial part V_p of V as the set $X \setminus S_p$, topologized with some topology which contains the subspace topology (since the map $V_p \rightarrowtail V \rightarrowtail X[S] \to X$ is continuous). Note that we do not assume that V is spatial, nor that the image of $V \rightarrowtail X[S] \to X$ is contained in $X \setminus S_p$; nor (for the moment) do we assume that the latter subspace of X is sober (equivalently, subclosed).

Now let Y be a space obtained by equipping $X \setminus S_p$ with an almost discrete topology containing the subspace topology, as in 2.1, and let $f : Y \to X$ be the inclusion. Then the pullback f^*S is a sublocale of Y having no points; hence by 2.2 it must be the empty sublocale, and in particular closed. So f factors uniquely through $X[S] \to X$, and in fact through $V \to X$; moreover, since Y is spatial it factors through $V_p \rightarrowtail V$. But this shows that the topology on V_p is contained in any almost discrete topology containing the subspace topology on $X \setminus S_p$; so by 2.1 it must coincide with the subspace topology. In particular, the latter *is* sober, and so by the proof of 2.3 we may identify $X \setminus S_p$ with the supplement T of S in $\mathrm{Sub}(X)$.

Now consider the sublocale $S \cap T$ of X. Since this is contained in S, its pullback along $X[S] \to X$ is itself considered as a sublocale of the closed copy of S in $X[S]$ (cf. 1.6); in particular, it is disjoint from V and *a fortiori* from V_p. But $S \cap T$ is also a sublocale of T, and we have shown that the map $V_p \to T$ is an isomorphism; hence $S \cap T = \emptyset$, i.e. S is complemented as a sublocale of X. □

Since the proof of 2.4 makes heavy use of points, we cannot hope to deduce much from it about sublocales of non-spatial locales. By combining it with 1.5, we easily obtain

Corollary 2.5. *If S is a soluble sublocale of an arbitrary locale X, then $S \cap X_p$ is complemented as a sublocale of X_p.* □

However, since X_p is often far from being complemented as a sublocale of X, this does not seem likely to be very useful.

Nevertheless, the result of 2.4 seems sufficiently general to embolden us to end the present paper with two conjectures:

Conjecture 2.6. *If Σ is a soluble set of sublocales of an arbitrary locale X, then it is exactly soluble.*

Conjecture 2.7. *If Σ is a soluble set of sublocales of a locale X, then $X[\Sigma]$ coincides with $X[\Sigma_c]$, where Σ_c is the set of members of Σ which are complemented in* $\mathrm{Sub}(X)$*; equivalently, every sublocale in Σ pulls back to a closed sublocale of $X[\Sigma_c]$.*

The second conjecture would of course imply the first, since we have already noted that sets of complemented sublocales are exactly soluble. We note that Conjecture 2.7 is satisfied in all the cases known to us where a set including non-complemented sublocales is soluble. For example: if Σ is the whole of $\mathrm{Sub}(X)$, then $X[\Sigma] = X_d$ may be obtained as $X[\mathcal{O}(X)]$ (or even as $X[\mathcal{D}]$, where $\mathcal{D}$ is the set of dense open sublocales of X), and if Σ is the set of Y-fibrewise closed sublocales, for some locale map $f : X \to Y$, then

$$X[\Sigma] = X[\{f^*U \mid U \in \mathcal{O}(Y)\}]$$

(cf. [2]).

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