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HOFMANN-MISLOVE POSETS

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Abstract. In this paper we attempt to find and investigate the most general class of posets which satisfy a properly generalized version of the Hofmann-Mislove theorem. For that purpose, we generalize and study some notions (like compactness, the Scott topology, Scott open filters, prime elements, the spectrum etc.), and adjust them for use in general posets instead of frames. Then we characterize the posets satisfying the Hofmann-Mislove theorem by the relationship between the generalized Scott closed prime subsets and the generalized prime elements of the poset. The theory becomes classic for distributive lattices. The topologies induced on the generalized spectra in general need not be sober.

1. Introduction and Terminology

The Hofmann-Mislove theorem is one of the most often applied results in theoretical computer science and computer science motivated topology. It says that there is a 1-1 correspondence between Scott open filters of a frame and compact saturated sets of its abstract points which can be naturally represented as the prime elements of the frame. Yet recent developments in the topic indicate that the frame structure of a poset or the sobriety of a topological space impose an unwanted limitation on the classic theory in

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some cases. The author can maintain this assertion by his personal experience with investigating the properties of the de Groot dual, or in solving the question of D. E. Cameron, whether every compact topology is contained in some maximal compact topology [2]. In both cases, some modification of the Hofmann-Mislove theorem is useful. However, the author believes that the utility of the presented generalization is not limited only to these two topics. Some other applications, perhaps more close to theoretical computer science, might be found in the future. In this paper we attempt to reach the boundaries of possible generalizations which, however, still leave the core and the main principles of the Hofmann-Mislove theorem untouched. The presented generalized version reduces to the classic theory if the considered poset is a distributive lattice. The topological formulation of the Hofmann-Mislove theorem will be studied in more detail in a separate paper.

In this section we explain most of the used terminology, except for some primitives and notions that we touch only marginally. These notions are not essential for understanding the paper. However, for a thorough introduction and detailed explanation of the topics the reader is referred to the books and monographs [3], [7] and [8].

Let P be a set, $\leq$ a reflexive and transitive, but not necessarily antisymmetric binary relation on P. Then we say that $\leq$ is a *preorder* on P and $(P, \leq)$ is a *preordered set*. For any subset A of a preordered set $(P, \leq)$ we denote $\uparrow A = \{x \mid \ x \geqslant y \text{ for some } y \in A\}$ and $\downarrow A = \{x \mid \ x \leqslant y \text{ for some } y \in A\}$. An important example of a preordered set is given by the *preorder of specialization* of a topological space (X, τ), which is defined by $x \leq y$ if and only if $x \in \operatorname{cl}\{y\}$. This preorder is a partial order in the usual sense if and only if the space (X, τ) is T_0. For any $x \in X$ it is obvious that $\downarrow \{x\} = \operatorname{cl}\{x\}$. A set is said to be *saturated* in (X, τ) if it is an intersection of open sets. One can easily verify that a set $A \subseteq X$ is saturated in (X, τ) if and only if $A = \uparrow A$, that is, if and only if A is an upper set with respect to the preorder of specialization of (X, τ). Thus for every set $B \subseteq X$, the set $\uparrow B$ is called the *saturation* of B. Compactness is understood without any separation axiom. The family of all compact saturated sets in (X, τ) is a closed base for a topology τ^d, which is called *de Groot dual* of the original topology τ. A topological space is said to be *sober* if it is T_0 and every irreducible closed set is a closure of a (unique) singleton.

Let ψ be a family of sets. We say that ψ has the *finite intersection property*, or briefly, that ψ has *f.i.p.*, if for every $P_1, P_2, \ldots, P_k \in \psi$ it follows $P_1 \cap P_2 \cap \cdots \cap P_k \neq \varnothing$. Let (X, τ) be a topological space, and let $\Phi, \Psi \subseteq 2^X$. Recall that a set $S \subseteq X$ is *compact with respect* (or *relative*) *to the family* Φ, if for every subfamily $\varphi \subseteq \Phi$ such that $\{S\} \cup \varphi$ has f.i.p. it follows $S \cap (\bigcap \varphi) \neq \varnothing$. We say that the family Φ is Ψ*-up-conservative* (Ψ*-down-conservative*, respectively) if for every $A \in \Phi$ and $B \in \Psi$ it follows $\uparrow (A \cap B) \in \Phi$ ($\downarrow (A \cap B) \in \Phi$, respectively). We say that Ψ is *upper-closed* (*lower-closed*, respectively) if for every $A \in \Psi$ it follows $A = \uparrow A$ ($A = \downarrow A$, respectively). The family Ψ is said to be *up-compact* (*down-compact*, respectively) if every $A \in \Psi$ is compact with respect to the family $\{\uparrow \{x\} \mid x \in X\}$ ($\{\downarrow \{x\} \mid x \in X\}$, respectively). The family Ψ is said to be *up-complete* (*down-complete*, respectively) if $\{\uparrow \{x\} \mid x \in X\} \subseteq \Psi$ ($\{\downarrow \{x\} \mid x \in X\} \subseteq \Psi$, respectively).

Let $(X, \leq)$ be a *partially ordered set*, or briefly, a *poset*. If $(X, \leq)$ has, in addition, finite meets, then any element $p \in X$ is said to be *prime* if $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$ for every $x, y \in X$. The set P of all prime elements of $(X, \leq)$ is called the *spectrum* of $(X, \leq)$. We say that the poset $(X, \leq)$ is *directed complete*, or *DCPO*, if every directed subset of X has a least upper bound – a supremum. A subset $U \subseteq X$ is said to be *Scott open*, if $U = \uparrow U$ and whenever $D \subseteq X$ is a directed set with $\sup D \in U$, then $U \cap D \neq \varnothing$. One can easily check that the Scott open sets of a DCPO form a topology. This topology we call the *Scott topology*. Thus a set $A \subseteq X$ is closed in the Scott topology if and only if $A = \downarrow A$ and if $D \subseteq A$ is directed, then $\sup D \in A$. It follows from Zorn's Lemma that in a DCPO, every element of a Scott closed subset S is below some maximal element of S. It is easy to see that the closure of a singleton $\{x\}$ in the Scott topology is $\downarrow \{x\}$, thus the original order $\leq$ of X can be recovered from the Scott topology as the preorder of specialization.

Let us describe some other important topologies on posets. The *upper topology* [3], which is also referred as the *weak topology* [7] or the *lower interval topology* [6] has the collection of all principal lower sets $\downarrow \{x\}$, where $x \in X$, as the subbase for closed sets. The preorder of specialization of the lower interval topology coincides with the original order of $(X \leq)$. Hence, the saturation of a subset

$A \subseteq X$ with respect to this topology is $\uparrow A$. Similarly, the *lower topology*, also referred as the *weak*^{d} *topology* [7] or the *upper interval topology* [6], arises from a subbase for closed sets which consists of all principal upper sets $\uparrow\{x\}$, where $x \in X$. Note that the weak^d topology is not the de Groot dual of the weak topology in general; the weak^d topology is the weak topology with respect to the inverse partial order. The preorder of specialization of the upper interval topology is a binary relation inverse to the original order of $(X, \leq)$. Consequently, the saturation of a subset $A \subseteq X$ with respect to this topology is $\downarrow A$. The topology on the spectrum P of a directed complete $\wedge$-semilattice $(X, \leq)$, induced by the upper interval topology, is called the *hull-kernel topology* [7].

Let $(X, \leq)$ be a poset. The set $F \subseteq X$ is said to be filtered, if every finite subset of F has a lower bound in F. Since the empty set is included, it has a lower bound in F which is, therefore, nonempty. If, in addition, $F = \uparrow F$ then F is called a filter on $(X, \leq)$. In this setting, a filter base in a set X can be defined as a filtered set $\varphi \subseteq 2^X$ in the poset $(2^X, \subseteq)$, such that $\varnothing \notin \varphi$.

2. Filtered Compactness and the generalized Scott topology

We will start with an example which illustrates the relationship between the Scott topology and compactness in terms of de Groot dual. In [5] the author proved that for a given topological space (X, τ) with the family of compact saturated sets $\mathcal{K}$ it holds $\tau = \tau^{dd}$ if and only if (X, τ) has an up-compact, $\mathcal{K}$-down-conservative closed subbase. We need this result for the example.

Example 2.1. Let $(X, \leq)$ be a frame, ω be the upper-interval topology on X, σ be the Scott topology on X. We leave to the reader to show that the compact saturated sets in (X, ω) are exactly the Scott closed sets, so $\sigma = \omega^d$ (the reader can, e.g., adjust the proof of Proposition 2.3). For every $a \in X$, the principal filter $\uparrow\{a\}$ is up-compact and, for every Scott closed $K \subseteq X$, $\uparrow(\uparrow\{a\} \cap K) = \varnothing$ or $\uparrow(\uparrow\{a\} \cap K) = \uparrow\{a\}$, so the family of principal filters extended by the empty set is down-conservative with respect to the family of the Scott closed sets. By the previously mentioned result, $\omega = \omega^{dd}$. Hence, ω and σ are the de Groot duals of each other. $\square$

However, this relation between the upper-interval topology and the Scott topology need not remain true in any of the both directions if we replace the frames by more general posets. This fact naturally leads to the following, slightly adjusted notion of compactness.

Definition 2.1. *Let X be a set, $\Phi \subseteq 2^X$. We say that $K \subseteq X$ is filtered compact with respect to the family Φ if $K \cap (\bigcap \varphi) \neq \varnothing$ for every filter base $\varphi \subseteq \Phi$ such that each of its elements meet K. In a poset $(X, \leq)$ we say that $K \subseteq X$ is up-filtered compact, if it is filtered compact with respect to the family $\{\uparrow \{x\} \mid x \in X\}$ of principal upper sets.*

Throughout the paper we work especially with up-filtered compactness of general posets, but we may be interested what this notion means in terms of topological spaces and how it differs from usual compactness. The following analogue of Alexander's subbase theorem describes filtered compactness in terms of convergence of more general families than filter bases consisting of members of the given closed subbase.

Proposition 2.1. *Let (X, τ) be a topological space, $\mathcal{C}$ the family of all closed sets, $\mathcal{C}_0 \subseteq \mathcal{C}$ a closed subbase. The following statements are equivalent for a subset $K \subseteq X$:*

(i) *K is filtered compact with respect to $\mathcal{C}_0$.*
(ii) *For every filter base $\varphi \subseteq \mathcal{C}$ whose every element meets K such that $\varphi \cap \mathcal{C}_0$ is a filter base, $K \cap (\bigcap \varphi) \neq \varnothing$.*
(iii) *For every family $\varphi \subseteq \mathcal{C}$ such that $\varphi \cup \{K\}$ has f.i.p. and $\varphi \cap \mathcal{C}_0$ is a filter base, $K \cap (\bigcap \varphi) \neq \varnothing$.*

Proof. It is clear that (iii) $\to$ (ii) $\to$ (i). Suppose (i). We say that a family $\varphi \subseteq \mathcal{C}$ has property $\mathcal{P}$ if $\varphi \cup \{K\}$ has f.i.p. and $\varphi \cap \mathcal{C}_0$ is a filter base. Let $\mathcal{L}$ be a chain of closed families having property $\mathcal{P}$, linearly ordered by set inclusion. It is easy to check that $\bigcup \mathcal{L}$ again has $\mathcal{P}$. Let $\varphi \subseteq \mathcal{C}$ be a family with $\mathcal{P}$. By Zorn's Lemma, φ is contained in some maximal family having $\mathcal{P}$, say ψ. We put $\psi_0 = \psi \cap \mathcal{C}_0$. It follows from (i) that there exists some $p \in K \cap (\bigcap \psi_0)$. Suppose that $p \notin \bigcap \varphi$. Then there exists $F \in \varphi$ such that $p \notin F$. But $F \in \mathcal{C}$, so there exists a set A, for every $\alpha \in A$ a finite set I_α, and for every $i \in I_\alpha$ a closed set $C_i \in \mathcal{C}_0$, such that $F = \bigcap_{\alpha \in A} F_\alpha$, where $F_\alpha = \bigcup_{i \in I_\alpha} C_i$. There exists some $\beta \in A$ such that $p \notin F_\beta$.

We have $F \subseteq F_\beta$ and $\psi \cup \{K\}$ has f.i.p., so $\varnothing \neq K \cap F \cap P_1 \cap P_2 \cap \dots \cap P_k \subseteq K \cap F_\beta \cap P_1 \cap P_2 \cap \dots \cap P_k$ for every $P_1, P_2, \dots, P_k \in \psi$. Hence, there exists $m \in I_\beta$ such that C_m has the same property as F_β, i.e., for every $P_1, P_2, \dots, P_k \in \psi$ we have $K \cap C_m \cap P_1 \cap P_2 \cap \dots \cap P_k \neq \varnothing$. We put $\psi' = \psi \cup \{C_m\} \cup \{C_m \cap (\bigcap_{j=1}^k P_j) \mid P_j \in \psi_0, j = 1, \dots, k\}$. Then ψ' has property $\mathcal{P}$, so from the maximality of ψ it follows $\psi' = \psi$ and we have $C_m \in \psi \cap \mathcal{C}_0$. Then $p \in C_m \subseteq F_\beta$, which is a contradiction. Hence, $p \in K \cap (\bigcap \varphi)$, which yields (iii). □

Corollary 2.1. *Let (X, τ) be a topological space, $\tau_0 \subseteq \tau$ an open subbase of τ. The following statements are equivalent for a subset $K \subseteq X$:*

(i) *K is filtered compact with respect to $\mathcal{C}_0 = \{X \smallsetminus U \mid U \in \tau_0\}$.*
(ii) *For every directed open cover $\mathcal{O} \subseteq \tau$ such that $\mathcal{O} \cap \tau_0$ is directed, there exists $U \in \mathcal{O}$ containing K.*
(iii) *For every open cover $\mathcal{O} \subseteq \tau$ such that $\mathcal{O} \cap \tau_0$ is directed, there exists finite $\mathcal{O}' \subseteq \mathcal{O}$ covering K.*

In the following example we will show that in general, compactness and filtered compactness are different properties. Note that the construction of the topological space is due to B. Burdick [1], who used it as an example of a space whose iterations of the de Groot dual (including the original topology) can generate four different topologies.

Example 2.2. Let (X, τ) be the first uncountable ordinal $X = \omega_1$ equipped with the topology $\tau = \{\langle 0, \alpha) \smallsetminus F \mid 0 \leqslant \alpha \leqslant \omega_1, F \text{ is finite}\}$. Then any closed set in (X, τ) has the form $C = \langle \alpha, \omega_1) \cup F$, where $0 \leqslant \alpha \leqslant \omega_1$ and F is finite. It is easy to see that (X, τ) is a T_1 space. Suppose that C is a non-empty closed set which is not a singleton. If $\alpha = \omega_1$, then $C = F$ has at least two elements and it is easy to decompose it into two strictly smaller non-empty closed sets. If $\alpha < \omega_1$, we have $C = \{\alpha\} \cup \langle \alpha + 1, \omega_1) \cup F$. In any case, C is not irreducible. Then (X, τ) is sober. We leave to the reader to check that τ^d is the cocountable topology.

Now we will continue directly with the previously constructed space (X, τ), but alternatively, instead of (X, τ) one can use also any sober space whose de Groot dual is not compact. Since the family Φ of all compact saturated sets is a closed base for (X, τ^d), by Alexander's subbase theorem X is not compact with respect to Φ.

However, in a sober space, any filter base consisting of non-empty compact saturated sets has a non-empty intersection (see [4], Corollary 2), so X is filtered compact with respect to Φ. □

Note that if the closed subbase $\mathcal{C}_0$ of (X, τ) is closed under binary intersections, filtered compactness with respect to $\mathcal{C}_0$ coincides with compactness. If the poset $(X, \leq)$ has binary joins, the family $\{\uparrow\{x\} \mid x \in X\}$ is closed under binary intersections. Hence, in this case, up-filtered compact means the same as compact with respect to the upper interval topology. Similarly as compactness, up-filtered compactness of a set is equivalent to up-filtered compactness of its saturation.

Proposition 2.2. *Let $(X, \leq)$ be a poset. Then $K \subseteq X$ is up-filtered compact if and only if $\downarrow K$ is up-filtered compact.*

Proof. Let $U \subseteq X$, $U = \uparrow U$. It is easy to observe that $K \cap U \neq \varnothing$ if and only if $\downarrow K \cap U \neq \varnothing$. Applying this observation to $U = \uparrow\{a\}$ or $U = \bigcap \varphi$, where $\varphi = \{\uparrow\{a\} \mid a \in A\}$ is a filter base from the definition of up-filtered compactness (cf. Definition 2.1 and its notation), one can complete the proof. □

We would like to have a similar relationship between the upper interval topology and the Scott topology as it is demonstrated for frames in Example 2.1. In a DCPO, as one can prove, the Scott closed sets are exactly the up-filtered compact saturated sets with respect to the upper interval topology. However, how to extend the Scott topology to posets which are not directed complete? There are, at least, two outmost possibilities. For instance, one can define that a lower set is Scott closed if its each directed subset has a supremum, which is contained in the lower set. In this case we can get very few Scott closed sets. Another, rather extreme possibility is to define that the lower set contains suprema of its directed subsets only if the suprema exist. This definition may generate too large a family of Scott closed sets. In DCPO's both cases coincide with the original definition of the Scott closed set, but, unfortunately, for general posets they need not work properly. We can demonstrate it by an example.

Example 2.3. Let $X = \mathbb{R} \setminus \{0, 2\}$ and let $\leq$ be the natural linear order of the real numbers. We put $A = \{x \mid x \in X, x \leq -1\}$, $B = \{x \mid x \in X, x \leq 1\}$ and $C = \{x \mid x \in X, x \leq 2\}$.

Then only the set A matches the first, the strongest possibility. All the three sets A, B, C match the second, the weakest possible definition of a Scott closed set. The upper interval topology on $(X, \leq)$ is the family $\tau = \{\varnothing, X\} \cup \{(-\infty, a) \cap X \mid a \in X\}$. Clearly, all the three sets A, B, C are saturated. However, the sets A, B are compact in this topology, but C is not compact. Hence, choosing the first possibility, there would be more compact saturated sets than the Scott closed sets, while choosing the second possibility would cause too many Scott closed sets and some of them would not be compact. □

In the previous example, a compromise solution which works well is represented by the set B. In general, it is given by the following definition.

Definition 2.2. *Let $(X, \leq)$ be a poset. We say that $A \subseteq X$ is a Scott closed basic set, if $A = \downarrow A$ and each directed $D \subseteq A$ has an upper bound in A. A set $B \subseteq X$ is said to be a Scott open basic set, if $X \smallsetminus B$ is a Scott closed basic set.*

For our convenience, we will use the shortcuts 'SCB set' for 'Scott closed basic set' and 'SOB set' for 'Scott open basic set'. We leave to the reader to check that the family of the Scott closed basic sets is closed under finite unions. However, in a general case it need not be closed under intersections, as we can see from the following example. Hence, the Scott open basic sets form a base for the open sets of some topology, but it itself need not be a topology in general.

Example 2.4. Let $(\mathbb{N}, \leq)$ be the set of natural numbers with their natural order, and let $a, b \notin \mathbb{N}$, $a \neq b$. We put $X = \mathbb{N} \cup \{a, b\}$. For any $x, y \in X$ we put $x \leqslant y$ if and only if any of the following cases is fulfilled:

(i) $x, y \in \mathbb{N}$, $x < y$,
(ii) $x \in \mathbb{N}$, $y \in \{a, b\}$,
(iii) $x = y$.

We leave to the reader to check that $\leqslant$ is a reflexive, antisymmetric and transitive relation. Let $A = \mathbb{N} \cup \{a\}$, $B = \mathbb{N} \cup \{b\}$. Then A, B are SCB sets, but in $A \cap B$ its directed subset $\mathbb{N} \subseteq A \cap B$ has no upper bound, so $A \cap B$ is not an SCB set. □

Definition 2.3. *The topology generated by the family of SOB sets we call the generalized Scott topology.*

Proposition 2.3. *Let $(X, \leq)$ be a poset. Then $K \subseteq X$ is an SCB set if and only if K is saturated in the upper interval topology and up-filtered compact.*

Proof. Let $K \subseteq X$ be an SCB set. The specialization preorder of the upper interval topology is the opposite of the given partial order $\leq$, so K is saturated (in the upper interval topology). Let $\varphi = \{\uparrow\{a\} \mid a \in A\}$ be a filter base such that $K \cap \uparrow\{a\} \neq \varnothing$ for every $a \in A$. Since φ is a filter base then if $a, b \in A$ there exists $c \in A$ such that $\uparrow\{c\} \subseteq \uparrow\{a\} \cap \{b\}$, that is, $c \geq a, b$. So A is directed. Further, if $a \in A$, then there is some $x \in K \cap \uparrow\{a\}$. It follows that $a \leq x$ and since K is a lower set, we have $a \in K$. Hence $A \subseteq K$. Then A has an upper bound $u \in K$ since K is an SCB set. But then $u \in K \cap (\bigcap_{a \in A} \uparrow\{a\})$. It means that K is up-filtered compact.

Conversely, let $K \subseteq X$ be up-filtered compact and saturated in the upper interval topology. Then there exists $F \subseteq X$ such that $K = \bigcap_{a \in F}(X \smallsetminus \uparrow\{a\})$ and, consequently, K is a lower set. Let $A \subseteq K$ be directed. Then $\Phi = \{\uparrow\{a\} \mid a \in A\}$ is a closed filter base and all its elements clearly meet K. Since K is up-filtered compact, there exists $u \in K \cap (\bigcap_{a \in A} \uparrow\{a\})$. Then $u \geq a$ for every $a \in A$, so u is an upper bound of A which is contained in K. Hence, K is an SCB set. □

One can state a natural question whether the filtered version of the de Groot dual applied on the generalized Scott topology always yields back to the original upper interval topology of the poset similarly as we described in Example 2.1 for frames. The author so far has no definitive answer for that simple question, although the expected answer is 'no'. The general iteration properties of this modified de Groot dual still remain open, too.

3. Hofmann-Mislove posets

The Hofmann-Mislove Theorem says that there is a 1-1 correspondence between Scott open filters of a frame and compact saturated sets of its abstract points which can be naturally represented as the prime elements of the frame. The set of the prime elements is known as the spectrum of the frame. However, if we have a more general poset than a frame, this correspondence either need not work at all or, at least, not so straightforward.

In this chapter we attempt to find the most general class of posets that satisfy a proper generalization of the Hofmann-Mislove Theorem. To be able to do this, we need to adjust some notions which are very simple and clearly understood in frames but are rather more complicated in a general setting. In the following definition we modify and extend the notion of a prime element to be relevant also for those posets which do not necessarily have the finite meets.

Definition 3.1. *Let $(X, \leq)$ be a poset, $L \subseteq X$. We say that L is prime if $\downarrow L \neq X$ and for every $a, b \in X$*

$$\downarrow\{a\} \cap \downarrow\{b\} \subseteq \downarrow L \Rightarrow (a \in \downarrow L) \vee (b \in \downarrow L).$$

It can be easily seen that if $(X, \leq)$ has finite meets, any element $p \in X$ is prime if and only if the singleton $\{p\}$ is prime as a set. Hence, we can extend the notion of a prime element also to those posets which do not necessarily have finite meets. Thus in the following text we mean that an element p of a poset $(X, \leq)$ is prime if and only if $\{p\}$ is prime in the sense of the previous definition. As the following proposition shows, the notions of a prime set and of a filter are dual.

Proposition 3.1. *Let $(X, \leq)$ be a poset. Then $L \subseteq X$ is prime if and only if $F = X \smallsetminus \downarrow L$ is a filter.*

Proof. Let $L \subseteq X$ be prime. Then $F = X \smallsetminus \downarrow L$ is a nonempty upper set. Let $a, b \in F$. Then $a, b \notin \downarrow L$, which implies that there is some $c \in \downarrow\{a\} \cap \downarrow\{b\}$ such that $c \notin \downarrow L$, i.e. $c \in F$. Conversely, let F be a filter. Then $\downarrow L = X \smallsetminus F \neq X$. Suppose that $\downarrow\{a\} \cap \downarrow\{b\} \subseteq \downarrow L$ for some $a, b \in X$. Then $a, b \notin \downarrow L$ implies $a, b \in F$, which means that there is some $c \leq a$, $c \leq b$, $c \in F$. Then $c \in \downarrow\{a\} \cap \downarrow\{b\}$, but $c \notin \downarrow L$, which is a contradiction. □

In the direct proof of the topological formulation of the Hofmann-Mislove Theorem (see, e.g. [4]), the sobriety of the topological space is needed to ensure that if an open set contains an intersection of a Scott open filter, then this open set is an element of the filter. We may say that such a filter is "wide" enough. If a topological space is not sober, its topology can have Scott-open filters which are not wide in this sense, but, on the other hand, the Scott open filters generated by compact sets are always wide. We want to model this situation in a poset equipped with the upper-interval topology.

However, all the elements of a poset need not necessarily correspond to the points of a certain topological space in the analogy that we want to model. It will be more convenient to relativize the "wideness" of a filter to subsets of posets and then study which subsets have the desired properties, whatever they are.

Definition 3.2. *Let $(X, \leq)$ be a poset, $P \subseteq X$. Denote $\psi(x) = P \smallsetminus \uparrow\{x\}$ for every $x \in X$. We say that non-empty set $F \subseteq X$ is wide relative to P, if for every $a \in X$, $\bigcap_{x \in F} \psi(x) \subseteq \psi(a) \Rightarrow a \in F$.*

Since we will often work with the prime sets rather than with filters, according to Proposition 3.1 we need a notion dual to relative wideness. In particular, a Scott closed prime set should have that property if and only if its complement is a relatively wide Scott open filter. This is a motivation for the next definition and the consecutive proposition, which only shows that the new notion has the expected and desired properties. There is only one exception – the notion of relative narrowness defined below make sense even for the case $K = X$. On the other hand, the empty set always (and, unfortunately, independently on P) satisfies the condition of relative wideness.

Definition 3.3. *Let $(X, \leq)$ be a poset, $P \subseteq X$. We say that $K \subseteq X$ is narrow relative to P if $K \subseteq \downarrow(P \cap K)$.*

Proposition 3.2. *Let $(X, \leq)$ be a poset, $K \subsetneq X$ a lower set. Then K is narrow relative to P if and only if $F = X \smallsetminus K$ is wide relative to P.*

Proof. Let $K \subsetneq X$ narrow relative to P, $K = \downarrow K$. Then $\varnothing \neq F = X \smallsetminus K$ is an upper set and $\bigcap_{x \in F} \psi(x) = \bigcap_{x \in F}(P \smallsetminus \uparrow\{x\}) = P \smallsetminus \bigcup_{x \in F} \uparrow\{x\} = P \smallsetminus F = P \cap K$. Suppose that $\bigcap_{x \in F} \psi(x) \subseteq \psi(a)$ for some $a \in X$. Then $P \cap K \subseteq P \smallsetminus \uparrow\{a\}$, which means that $P \cap K \cap \uparrow\{a\} = \varnothing$. Then $a \notin \downarrow(P \cap K)$, and since K is narrow relative to P, $a \notin K$. It follows $a \in F$, which yields that F is wide relative to P. Conversely, suppose that F is wide relative to P. Let $a \in K = X \smallsetminus F$. Then $\bigcap_{x \in F} \psi(x) = P \cap K \nsubseteq \psi(a) = P \smallsetminus \uparrow\{a\}$. Then $P \cap K \cap \uparrow\{a\} \neq \varnothing$, so there exists some $t \in P \cap K \cap \uparrow\{a\}$. Then $a \leq t$ and $t \in P \cap K$, which gives $a \in \downarrow(P \cap K)$. Hence, $K \subseteq \downarrow(P \cap K)$. □

Another, also useful characterization of relative narrowness is given by the following proposition.

Proposition 3.3. *Let $(X, \le)$ be a poset, $K \subseteq X$ a lower set. Then K is narrow relative to P if and only if there exists $L \subseteq P$ such that $K = \downarrow L$.*

Proof. Let $K \subseteq X$ be narrow relative to P. Then $K \subseteq \downarrow(P \cap K)$. We put $L = P \cap K$ and since K is a lower set, we have $K = \downarrow L$. Conversely, suppose that $K = \downarrow L$, where $L \subseteq P$. Let $x \in K$. Then there exists some $t \in L \subseteq P \cap K$ such that $x \le t$. Then $x \in \downarrow(P \cap K)$, which gives $K \subseteq \downarrow(P \cap K)$. Hence, K is narrow relative to P. □

Now, we are ready to say more precisely what we mean by the analogy with frames or sober topological spaces that we want to model for a certain class of posets. The desired situation is described by the conditions (i) and (ii) of the following proposition equivalently in terms of the SOB filters and SCB prime sets.

Proposition 3.4. *Let $(X, \le)$ be a poset, ω the upper interval topology on X, ω_P the induced topology on $P \subseteq X$. The following conditions (i) and (ii) are equivalent:*

- (i) *There exists $P \subseteq X$ such that:*
  - (1) *For every SOB filter $F \subseteq X$, if we denote $\psi(x) = P \setminus \uparrow\{x\}$ and $L = \bigcap_{a \in F} \psi(a)$, the set L is up-filtered compact, saturated in (P, ω_P) and $F = \{x \mid x \in X, L \subseteq \psi(x)\}$.*
  - (2) *For every up-filtered compact and saturated $L \subseteq P$ in (P, ω_P), the set $F = \{x \mid x \in X, L \subseteq \psi(x)\}$ is a SOB filter.*
- (ii) *There exists $P \subseteq X$ such that:*
  - (1) *For every SCB prime set $K \subseteq X$, the set $L = P \cap K$ is up-filtered compact, saturated in (P, ω_P) and $K = \downarrow L$.*
  - (2) *For every up-filtered compact and saturated $L \subseteq P$ in (P, ω_P), the set $\downarrow L$ is an SCB prime set.*

Proof. As it follows from Proposition 3.1 and Definition 2.2, $F \subseteq X$ is an SOB filter if and only if $K = X \setminus F$ is an SCB prime set. Further, we have $\bigcap_{a \in F} \psi(a) = \bigcap_{a \in F}(P \setminus \uparrow\{x\}) = P \setminus \bigcup_{a \in F} \uparrow\{a\} = P \setminus F = P \cap K$, and $X \setminus \downarrow L = \{x \mid x \in X, x \notin \downarrow L\} = \{x \mid x \in X, L \cap \uparrow\{x\} = \varnothing\} = \{x \mid x \in X, L \subseteq \psi(x)\}$. Now it is clear that (ii) is only a reformulation of (i). □

Definition 3.4. *Let $(X, \leq)$ be a poset. We say that $(X, \leq)$ is Hofmann-Mislove, if $(X, \leq)$ satisfies any of the conditions (i) or (ii) of Proposition 3.4. The set $P \subseteq X$ from (i) or (ii) we call a generalized spectrum of $(X, \leq)$; the topology ω_P we call the generalized hull-kernel topology on P.*

The natural question that we immediately have to ask just after the definition is, which posets are Hofmann-Mislove and how many generalized spectra such a poset can have. The next proposition shows, as one can expect, that the generalized spectrum is determined uniquely.

Proposition 3.5. *Let $(X, \leq)$ be a Hofmann-Mislove poset, $S \subseteq X$ any generalized spectrum. Then $S = \{p \mid p \in X, p \text{ is prime}\} = \{m \mid m \text{ is a maximal element of an SCB prime subset of } X\}$.*

Proof. Let $M = \{m \mid m$ is a maximal element of an SCB prime set$\}$, $P = \{p \mid p \in X, p \text{ is prime}\}$. Let $p \in P$ be a prime element. Then $\downarrow\{p\}$ is an SCB prime set and p is its maximal element. Thus $P \subseteq M$. Let $m \in M$ and let $K \subseteq X$ be an SCB prime set such that $m \in K$ is a maximal element. Then $K = \downarrow(K \cap S)$, so $m \in \downarrow(K \cap S)$. Then there exists $t \in K \cap S$ with $m \leq t$. But m is maximal in K, so $m = t \in S$. Hence, $M \subseteq S$. Let $s \in S$. Then $\{s\}$ is clearly up-filtered compact, as well as $\downarrow\{s\}$. The set $L = S \cap \downarrow\{s\}$ is saturated in (S, ω_S) and $\downarrow L = \downarrow\{s\}$. Hence, L is up-filtered compact. Then $\downarrow L$ is an SCB prime set, which means, in particular, that s is prime. Therefore, $S \subseteq P$. Now we have $P \subseteq M \subseteq S \subseteq P$, which completes the proof. □

The rest of the section will be devoted to simplifying the condition of being Hofmann-Mislove.

Proposition 3.6. *A poset $(X, \leq)$ is Hofmann-Mislove if and only if there exists $P \subseteq X$ such that the following statements are fulfilled:*

(1) *Every SCB prime set is narrow relative to P.*
(2) *Every $L \subseteq P$ such that $\downarrow L$ is an SCB set is prime.*

Proof. We will show that the conditions (1) and (2) are equivalent to the corresponding conditions of Proposition 3.4. It is clear that the condition (1) of (ii) in Proposition 3.4 implies (1). Conversely, suppose (1). Then, for every SCB prime set K, it holds $K \subseteq \downarrow(P \cap K)$. Since $K = \downarrow K$, we have $K = \downarrow L$, where $L = P \cap K$.

It follows from Proposition 2.2 that L is up-filtered compact and, clearly, L is saturated in (P, ω_P). Hence, (1) and the condition (1) of (ii) in Proposition 3.4 are equivalent.

Now, suppose (2) of (ii) in Proposition 3.4. Let $L \subseteq P$ such that $\downarrow L$ is an SCB set. Then, by Proposition 2.3, $\downarrow L$ is up-filtered compact and, of course, saturated in the upper-interval topology. We put $M = P \cap \downarrow L$. Then $L \subseteq M \subseteq \downarrow L$, so $\downarrow M = \downarrow L$. Then, by Proposition 2.2, M is up-filtered compact. Since $\downarrow L$ is saturated in (X, ω), M is saturated in (P, ω_P). By (2) of (ii) in Proposition 3.4, $\downarrow M = \downarrow L$ is prime. By Definition 3.1, L is prime. Hence, (2) is fulfilled. Conversely, suppose (2). Let $L \subseteq P$ be up-filtered compact and saturated in (P, ω_P). Then $\downarrow L$ is also up-filtered compact by Proposition 2.2 and saturated in (X, ω) as a lower set. By Proposition 2.3, $\downarrow L$ is an SCB set. It follows from (2) that $\downarrow L$ is prime, so the condition (2) of (ii) in Proposition 3.4 is fulfilled. This completes the proof. □

Theorem 3.1. *A poset $(X, \leq)$ is Hofmann-Mislove if and only if there exists $P \subseteq X$ such that for every SCB set $K \subseteq X$, the following statements are equivalent:*

(i) *K is prime.*
(ii) *K is narrow relative to P.*

Proof. Suppose that $(X, \leq)$ is Hofmann-Mislove and let P be its generalized spectrum. Let $K \subseteq X$ be an SCB set. If K is prime, then by Proposition 3.6 K is narrow relative to P. Conversely, if K is narrow relative to P, by Proposition 3.3 $K = \downarrow L$ for some $L \subseteq P$. By Proposition 3.6, L is prime and so K is also prime. Therefore, the conditions (i) and (ii) are equivalent.

Conversely, suppose that there exists $P \subseteq X$ such that for every SCB set the conditions (i) and (ii) are equivalent. Let $K \subseteq X$ be an SCB prime set. By the implication (i) $\rightarrow$ (ii), K is narrow relative to P, so the condition (1) of Proposition 3.6 is fulfilled. Now, let $L \subseteq P$ be such that $K = \downarrow L$ is an SCB set. Then K is narrow relative to P by Proposition 3.3. Therefore, K is prime by the implication (ii) $\rightarrow$ (i). It follows just from Definition 3.1 that L is prime. Hence, the condition (2) of Proposition 3.6 holds. By Proposition 3.6, $(X, \leq)$ is Hofmann-Mislove. □

Combining the previous theorem with Proposition 3.5, we have the following corollary.

Corollary 3.1. *Let* $(X, \leq)$ *be a poset,* $P \subseteq X$ *the set of prime elements. Then* $(X, \leq)$ *is Hofmann-Mislove if and only if for every SCB set* $K \subseteq X$*, the following statements are equivalent:*

(i) K *is prime.*
(ii) K *is narrow relative to* P *(i.e.,* $K = \downarrow L$ *for some* $L \subseteq P$*).*

The previous result can also be reformulated without the explicit use of the generalized spectrum, as we can see from the following theorem.

Theorem 3.2. *A poset* $(X, \leq)$ *is Hofmann-Mislove if and only if for every SCB set* $K \subseteq X$ *the following statements are equivalent:*

(i) K *is prime.*
(ii) *Every maximal element of* K *is prime.*

Proof. Let P be the set of prime elements of $(X, \leq)$. Suppose that $(X, \leq)$ is Hofmann-Mislove. Let K be an SCB set. If K is prime, by Corollary 3.1 $K = \downarrow L$, where $L \subseteq P$. Let $m \in K$ be a maximal element of K. Then there exists some $p \in L$ such that $m \leq p$ and from maximality we have $p = m$. Hence, every maximal element of K is prime. Conversely, suppose that every maximal element of K is prime. Let $M \subseteq K$ be the set of maximal elements of K. Then $M \subseteq P$. Since K is an SCB set, it follows from Zorn's Lemma that every element $x \in K$ is comparable with some maximal element $m \in M$ – we have $x \leq m$. Then $K = \downarrow M$, so by Proposition 3.3 K is narrow relative to P. It follows from Corollary 3.1 that K is prime. Hence, the conditions (i) and (ii) are equivalent.

On the other hand, suppose that the conditions (i) and (ii) are equivalent for every SCB set $K \subseteq X$. Let K be prime. Then, by the implication (i) $\to$ (ii) it follows that every maximal element of K is prime. If $M \subseteq K$ is the set of maximal elements of K, then $K = \downarrow M$ and $M \subseteq P$. By Proposition 3.3, K is narrow relative to P. Conversely, let K be narrow relative to P and let $m \in K$ be a maximal element. We have $K = \downarrow L$ for some $L \subseteq P$, which yields $m \in L$ because of maximality of m. Then every maximal element of K is prime. It follows from the implication (ii) $\to$ (i) that K is prime. By Corollary 3.1, $(X, \leq)$ is Hofmann-Mislove. ☐

Corollary 3.2. *Let $(X, \leq)$ be a directed poset with binary meets. Then $(X, \leq)$ is Hofmann-Mislove if and only if every maximal element of an SCB prime set is prime.*

Proof. By Theorem 3.2, it is sufficient to show that if $(X, \leq)$ has finite meets, then an SCB set, whose maximal elements are prime, is prime. Let $K \subseteq X$ be an SCB set and suppose that every maximal element of K is prime. First we need to show that $K \neq X$. Suppose conversely, that $K = X$. Then X is a directed set in the SCB set K, so X has an upper bound (and, of course, the greatest element) $k \in K$. Then k is a non-prime maximal element of K, which is a contradiction. Thus $K \neq X$. Now, let $\downarrow\{a\} \cap \downarrow\{b\} \subseteq \downarrow K = K$ for some $a, b \in X$. Then $a \wedge b \in K$, so there is a maximal element $m \in K$ such that $a \wedge b \leq m$. By the assumption, m is prime, so $a \leq m$ or $b \leq m$. Then $a \in K$ or $b \in K$, which means that K is prime. □

The following corollary is the reformulated Hofmann-Mislove Theorem.

Corollary 3.3. *Let $(X, \leq)$ be a non-empty distributive lattice. Then $(X, \leq)$ is Hofmann-Mislove.*

Proof. Every non-empty poset with binary joins is directed, so Corollary 3.2 can be applied. Let $K \subseteq X$ be an SCB prime set, $m \in K$ a maximal element of K. Suppose that $a \wedge b \leq m$ for some $a, b \in X$. Then $(a \vee m) \wedge (b \vee m) = (a \wedge b) \vee m = m$. Then $\downarrow\{a \vee m\} \cap \downarrow\{b \vee m\} \subseteq K$, but K is lower and prime, so $a \vee m \in K$ or $b \vee m \in K$. But then $a \vee m = m$ or $b \vee m = m$, i.e. $a \leq m$ or $b \leq m$, since m is maximal. Hence, m is a prime element. □

On the other hand, the words 'directed' or 'lattice' cannot be omitted in Corollary 3.2 or Corollary 3.3, respectively, even if some kind of relaxed distributivity holds. The poset in the following example is close to a distributive lattice, but the top element is missing. The distributive laws hold in this poset for those expressions which are correctly defined.

Example 3.1. Let $(X, \leq)$ be the poset with $X = \{1, 2, 3\}$, where $1 \leq 2$, $1 \leq 3$ and 2, 3 are incomparable. The SCB prime sets are $\{1, 2\}$, $\{1, 3\}$ and their maximal elements 2, 3 are prime.

On the other hand, X is a non-prime SCB set, whose maximal elements are prime. By Theorem 3.2, $(X, \leq)$ is not Hofmann-Mislove. □

As we can also expect, a modular lattice need not be Hofmann-Mislove, which can be easily seen from the following example.

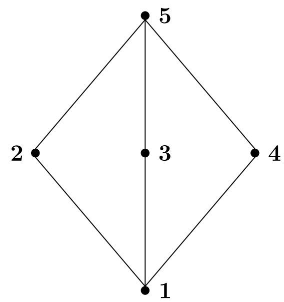

Figure 1.

Example 3.2. In the diamond lattice M_5 depicted in Figure 1, the set $\{1, 2, 3\}$ is prime and SCB. However, its maximal elements 2 and 3 are not prime. Hence, the diamond lattice is not Hofmann-Mislove. □

On the other hand, there are Hofmann-Mislove lattices which are not modular (and, of course, not distributive).

Example 3.3. The lattice $(X, \leq)$ on Figure 2 is not modular because it has a pentagonal sublattice isomorphic to N_5 with the underlying set $\{-2, 0, 1, 2, 3\}$.

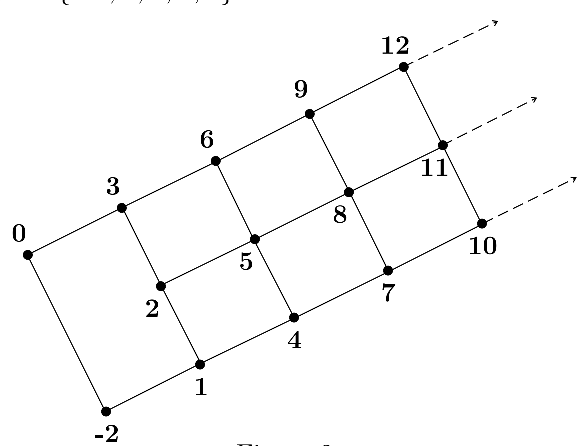

Figure 2.

It will be more illustrative if we show that $(X, \leq)$ is a Hofmann-Mislove lattice directly from the definition. The generalized spectrum of this lattice is $P = \{0, 3, 6, 9, 12, \dots\}$, which is topologized by the generalized hull-kernel topology

$$\omega_P = \{\varnothing, \{0\}, \{0, 3\}, \{0, 3, 6\}, \{0, 3, 6, 9\}, \{0, 3, 6, 9, 12\}, \dots\}.$$

Then the up-filtered compact sets, which coincide with the compact sets since $(X, \leq)$ is a lattice, are precisely the finite sets. Hence, the compact saturated sets are exactly the elements of ω_P. The lattice $(X, \leq)$ is $\wedge$-complete, so the filters in $(X, \leq)$ have the form $F = \uparrow\{f\}$, where $f \in X$. But not all the filters are SOB sets. The filters of the form $\uparrow\{3k\}$ or $\uparrow\{3k + 2\}$ where $k = 0, 1, 2, \dots$ are not SOB sets, since the linearly ordered chain $\{1, 4, 7, \dots\}$ has no upper bound in $(X, \leq)$, but it does not meet these filters (cf. Definition 2.2). Therefore, the SOB filters are precisely the sets $\uparrow\{3k+1\}$, where $k \in \{-1, 0, 1, \dots\}$. Let $F = \uparrow\{3k+1\}$ be an SOB filter. Then $L(F) = \bigcap_{a \in F} \psi(a) = \bigcap_{a \geq 3k+1} (P \smallsetminus \uparrow\{a\}) = P \smallsetminus \bigcup_{a \geq 3k+1} \uparrow\{a\} = P \smallsetminus \uparrow\{3k + 1\} = P \smallsetminus F = \{0, 3, 6, \dots, 3k\}$ if $k \in \{0, 1, 2, \dots\}$, or $L(F) = \varnothing$ if $k = -2$ (cf. notation in Proposition 3.4). In any case, $L(F)$ is compact saturated and $\{x \mid x \in X, L(F) \subseteq \psi(x)\} = \{x \mid x \in X, \uparrow\{x\} \cap P \subseteq \uparrow\{3k + 1\}\} = \uparrow\{3k + 1\} = F$. Conversely, if $L = \{0, 3, 6, \dots, 3k\}$ is a compact saturated set, the set $F(L) = \{x \mid x \in X, L \subseteq \psi(x)\} = \{x \mid x \in X, \uparrow\{x\} \cap P \subseteq P \smallsetminus L\} = \{x \mid x \in X, \uparrow\{x\} \cap P \subseteq \uparrow\{3k + 1\}\} = \uparrow\{3k + 1\}$ is an SOB filter. Hence, by Definition 3.4, the lattice $(X, \leq)$ is Hofmann-Mislove. Of course, alternatively one can prove that $(X, \leq)$ is Hofmann-Mislove by checking that the SCB prime sets are just the sets $\downarrow\{p\}$, where $p \in P$, and then applying Corollary 3.2. □

It is well-known that the spectrum of a frame equipped with the hull-kernel topology is a sober topological space. One may ask what happens with the sobriety of the generalized spectrum of a general poset. We close the paper with an example, which shows that even slightly more general Hofmann-Mislove posets than frames may very naturally lead to non-sober topologies on their generalized spectra.

Example 3.4. Let Y be an infinite set, $X = \{K \mid K \subseteq Y \text{ is finite}\}$. For every $a, b \in X$ we put $a \leq b$ if and only if $a \supseteq b$. Then $(X, \leq)$ has all finite meets including $\bigwedge \varnothing = \bigcup \varnothing = \varnothing \in X$, which is the top element of $(X, \leq)$. On the other hand, $\bigvee \varnothing = \bigcap \varnothing = Y$ is not a finite set, so $(X, \leq)$ has not the empty join. In particular, $(X, \leq)$ is a distributive lattice with all non-empty joins, it is a DCPO since directed sets are non-empty, but it is not a frame.

Let $p = \{y\}$, where $y \in Y$ and suppose that $a \wedge b \leq p$ for some $a, b \in X$. Then $a \cup b \supseteq p$, which means that $y \in a$ or $y \in b$. Hence, $a \supseteq p$ or $b \supseteq p$, which gives $a \leq p$ or $b \leq p$. Then p is a prime element of X. Conversely, let $p \in X$ be an element with $|p| \geq 2$. Then there exist $x, y \in p$ such that $x \neq y$. We put $a = p \smallsetminus \{x\}$, $b = p \smallsetminus \{y\}$. We have $a \cup b = p$, which implies $a \wedge b \leq p$, but also $a \nleq p$ and $b \nleq p$. It means that p is not prime. Since $\varnothing$ is not prime by the definition as the top element, the prime elements of $(X, \leq)$ are precisely the singletons. By Corollary 3.3, $(X, \leq)$ is a Hofmann-Mislove poset and its generalized spectrum is $P = \{\{y\} \mid y \in Y\}$. For every $a \in X$, $P \cap \uparrow\{a\} = \{f \mid f \subseteq a, |f| = 1\} = \{\{y\} \mid y \in a\}$. Now we can see that the generalized hull-kernel topology on P is the cofinite topology, which obviously is not sober. $\square$

References

1. Burdick, B. S., *A note on iterated duals of certain topological spaces*, Theor. Comput. Sci. **275**, 1-2, (2002), 69-77.
2. Cameron, D. E., *A survey of maximal topological spaces*, Topology Proc. **2** (1977), 11-60.
3. Gierz G., Hofmann K. H., Keimel K., Lawson J. D., Mislove M., Scott D. S. *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, Heidelberg, New York, 1980, pp. 372.
4. Keimel K., Paseka J., *A direct proof of the Hofmann-Mislove theorem*, Proc. Amer. Math. Soc. **120** (1994), 301-303.
5. Kovár, M. M., *On iterated dualizations of topological spaces*, Topology Appl. 146-7, (2005), 83-89.
6. Lawson J. D., *The upper interval topology, property $\mathcal{M}$ and Compactness*, Electronic Notes in Theoretical Computer Science, http://www.elsevier.nl/locate/entcs/volume13.html, **13** (1998) 1-15.
7. Lawson J.D., Mislove M., *Problems in domain theory and topology*, Open Problems in Topology, (van Mill J., Reed G. M., eds.), North-Holland, Amsterdam, 1990, pp. 349-372.

8. Vickers S., *Topology Via Logic*, Cambridge University Press, Cambridge, 1989, pp. 200

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