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CONTINUOUS ITINERARY FUNCTIONS ON DENDROIDS

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Abstract. It is well known that most of the information about the dynamics of a unimodal interval map can be obtained from its "kneading sequence" (the itinerary of its turning point with respect to the map), and similar results are known for trees and dendrites having exactly one "turning point" (a point where the function is not locally one-to-one). We show here that these ideas can be extended to a large class of unimodal dendroid maps (with an appropriate extension of the term "unimodal") satisfying the unique itinerary property, and provide a routine method for constructing many examples of such maps. In this case, the basic invariants are the kneading sequence and a zero-dimensional compact Hausdorff space which tells how the various components of $D \setminus \{t\}$ limit on each other (where t is the "turning point").

1. Introduction

Let $f : [0, 1] \to [0, 1]$ be a unimodal map of the interval (i.e., a continuous map of the interval having exactly one relative extremum $t \in (0, 1)$, called the "turning point"). If we let $L = [0, t)$, $C = \{t\}$, and $R = (t, 1]$, then we can define the *itinerary* of a point (defined more formally below) as a sequence from $\{L, C, R\}$ which identifies the sets visited by the orbit of t. It is well known that most of the information about the dynamics of such a unimodal interval map can be obtained from its *kneading sequence*, i.e., the

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itinerary of its turning point (see, e.g., [7], [6]), and similar results are available for piecewise monotone interval maps (see, e.g., [8], [1]). Other generalizations have extended in the direction of trees and dendrites (see, e.g., [3], [4], [5], [2]).

The present paper is a natural followup to [2], where it was shown that the assumption that every pair of distinct points had distinct itineraries allowed for a simple classification of such maps, up to conjugacy if certain additional hypotheses were added (such as restricting to the smallest invariant dendrite containing the turning point). This classification included the fact that for each finite set of symbols and each "acceptable" sequence τ of symbols from that set (see below for the definition), there was a natural dendrite map $\sigma_\tau : D_\tau \to D_\tau$ realizing that kneading sequence in which all other examples using the same symbols could be naturally embedded. This construction introduced a new tool, the continuous itinerary function, in which a natural (but non-Hausdorff) topology was placed on the set of all possible itineraries for a given set of symbols, and greatly simplified many of the constructions and proofs. Although the lack of the Hausdorff property can be unsettling at first, the unique itinerary property turns out to guarantee that the itinerary map has a Hausdorff range and is in fact a homeomorphism onto its range.

In this paper, we show how these results can be extended to dendroids (a generalization of dendrites which abandons the assumption of local connectivity). In addition to the invariant used in the dendrite case, i.e., the kneading sequence, we also have a zero-dimensional compact Hausdorff space which codes how various pieces of the dendroid limit on each other, and these non-Hausdorff itinerary topologies will play a key role in the constructions.

The remainder of this section gives some of the standard definitions used in this paper. In section 2, we show how these methods lead to a classification of a large class of unimodal dendroid maps. Section 3 covers the case of dendrite maps in which the kneading sequence has infinite range, which was not covered in [2] because it needed some of the results here. In section 4, we give several examples for the simplest kneading sequence having infinite range (i.e., a one-to-one kneading sequence), showing how the constructed examples change as the zero-dimensional Hausdorff space coding it changes.

Definition 1.1. We let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ the set of positive integers, and $\omega = \mathbb{N} \cup \{0\}$ the set of nonnegative integers. A *sequence of length* n is a function with domain $\{0, 1, 2, \ldots, n-1\}$, and an *infinite sequence* is a function with domain ω. If α is a finite sequence, and β is a finite or infinite sequence, then $\alpha\beta$ is the obvious concatenated sequence consisting of the finite sequence α followed by the sequence β. An *arc* is any space homeomorphic to the unit interval $[0, 1]$. A *continuum* is a compact connected metric space. A space X is *arcwise connected* if for every $x, y \in X$ there is an arc $A \subseteq X$ such that $x, y \in A$. A space is *uniquely arcwise connected* if it is arcwise connected and contains no circle. A uniquely arcwise connection continuum will be abbreviated "u.a.c.c." If a and b are two points in a uniquely arcwise connected space, we let $[a, b]$ be the unique arc having a and b as endpoints, and let $(a, b) = [a, b] \setminus \{a, b\}$, noting that the latter will not always be an open set. A *tree* is a uniquely arcwise connected union of finitely many arcs. A *dendrite* is a locally connected, uniquely arcwise connected continuum. A continuum C is *tree-like* if for every $\epsilon > 0$ there is a tree T (which may depend on ϵ) and a continuous function $f : C \to T$ such that $f^{-1}(x)$ has diameter less than ϵ for every $x \in T$. (It is well known that this definition is independent of the metric used for C.) A *dendroid* is a tree-like, uniquely arcwise connected continuum. It is well known that every tree is a dendrite, and that every dendrite is a dendroid. If X is a topological space, then the *cone* over the space X is defined to be the quotient space obtained from the product space $[0, 1] \times X$ by identifying all points of the form $(0, x)$, letting open neighborhoods of the identified point $o = \{(0, x) : x \in X\}$ be all sets of the form $U \times X$, where U is an open neighborhood of 0 in $[0, 1]$ (note that this is the same as the quotient topology if and only if X is compact). The cone over a Cantor set is often called the *Cantor fan.*

2. Classification of unimodal dendroid maps with the unique itinerary property

Definition 2.1. If X is a topological space, $f : X \to X$ is a continuous function, Σ is a set of symbols, and $\mathcal{S} = \{S_a : a \in \Sigma\}$ is a partition of X (i.e., the S_a's are pairwise disjoint and their union is X), then we define the *itinerary* of a point $x \in X$ with respect to the function f and the partition $\mathcal{S}$ as follows. Let $q : X \to \Sigma$ be the function defined by letting $q(x)$ be the unique $a \in \Sigma$ such that $x \in S_a$. Then $\iota_f^{\mathcal{S}}(x)$ is defined to be the sequence $\langle q(f^n(x)) : n \in \omega \rangle$, and we shall usually suppress the superscript and subscript of ι when there is no danger of ambiguity. We put a topology on the set Σ of symbols by putting the quotient topology on Σ with respect to the map q, and we call this the *symbol topology*. The *itinerary topology* is defined to be the usual product topology Σ^ω (using the symbol topology on each coordinate). Note that these topologies will depend on the partition $\mathcal{S}$ but not on the function f, and that they will often be non-Hausdorff. The *shift function* $\sigma : \Sigma^\omega \to \Sigma^\omega$ is defined by $\sigma(\langle x_0, x_1, x_2, \dots \rangle) = \langle x_1, x_2, x_3, \dots \rangle$ and is clearly continuous in the itinerary topology.

The type of partition in which we are interested here is one in which one member of the partition $\mathcal{S}$ is a singleton $S_a = \{t\}$ (corresponding to a "turning point"), and all other members of the partition have t as a limit point. In that case, the symbol topology can be written as $\Sigma = C \cup \{a\}$, where Σ is the only open set containing a, so that the topology on Σ is completely determined by the topology on C. It is clear that the exact symbols used for Σ are unimportant, so we shall adopt the convention that the symbol 0 will stand for this distinguished point t (i.e., $S_0 = \{t\}$). In addition, we say that the partition $\mathcal{S}$ satisfies the *labelling convention* with respect to a function f if $f^n(t) \in S_b$ implies that $b \in \omega$ and $f^n(t) \in S_i$ implies that $f^m(t) \in S_{i-1}$ for some $m < n$ if $i > 0$ (i.e., the members of the partition which are visited by the orbit of t are labelled in the order in which they are visited). The labelling convention is merely a convenience and will be used when it simplifies the statement of a theorem. If D is a uniquely arcwise connected topological space and $t \in D$, then the partition (or itinerary) with respect to the point t refers to the partition Σ (and its associated itineraries) obtained by taking $\{t\}$ and each of the arc-components of $D \setminus \{t\}$. The arc-components of $D \setminus \{t\}$ will be called the *legs* of D (with respect to the point t), and we abbreviate $D_t = D \setminus \{t\}$. We also let $D_t^{**} = \Sigma$ be the symbol topology, and we let D_t^* be the symbol topology restricted to D_t. In this setting where $\Sigma = C \cup \{0\}$, we define the *strong topology* on Σ to be the topology which is obtained by adding an isolated point 0 to C, and the strong topology on Σ^ω will then be defined as the corresponding product topology. These stronger topologies are sometimes useful in intermediate stages of our proofs, but the default topologies used for Σ and Σ^ω will always be the symbol and itinerary topologies, respectively, unless it is explicitly stated otherwise.

The itinerary topologies on Σ^ω will always be non-Hausdorff in the cases of interest to us, but, as in [2], the range of X with respect to $\iota_f^{\mathcal{S}}$ is often Hausdorff as a subspace and often even homeomorphic to x.

Proposition 2.2. *Let X, Σ, $\mathcal{S}$ be as above, $f : X \to X$ continuous, and $\iota = \iota_f^{\mathcal{S}}$. Then the itinerary topology is the strongest topology on Σ^ω such that the itinerary function $\iota : X \to \Sigma^\omega$ is continuous (and this statement is independent of the choice of f).*

Proposition 2.3. *If $\iota = \iota_f^{\mathcal{S}}$, and σ is the shift function of the corrresponding itinerary space, then $\iota \circ f = \sigma \circ \iota$.*

Definition 2.4. A continuous map f on a uniquely arcwise connected continuum (u.n.c.c.) D is said to *locally arcwise one-to-one* at a point $a \in D$ if and only if a has a neighborhood U such that for every arc $A \subseteq U$, $f \mid A$ is one-to-one. A *turning point* of f is a point at which f is not locally arcwise one-to-one. The function f is said to be *unimodal* if and only if it has exactly one turning point.

Note that if D is a dendrite, then f is locally one-to-one at a if and only if f is locally arcwise one-to-one at a, so this does not change the definitions given for dendrites in [2].

In the corresponding theory on dendrites in [2], two main properties of unimodal dendrite maps were the main point of interest. The stronger property, called *tentlike*, was a natural generalization of the "tent" maps on the interval and required that there be a constant $\lambda > 1$ such that for every subarc A missing the turning point, the length of $f(A)$ was exactly λ times the length of A, where a "taxicab metric" was required for the word "length" to make sense. Since a uniquely arcwise connected continuum admits a taxicab metric if and only if it is a dendrite, this definition does not generalize to more general u.a.c.c.'s in a nice way. The other, weaker, property covered in [2], called *tentish*, which required only that different points had different itineraries, has an obvious generalization to u.a.c.c.'s.

Definition 2.5. Let D be a u.a.c.c.. A continuous function $f : D \to D$ on a nondegenerate uniquely arcwise connected continuum D will be called *tentish* if and only if there is a point $t \in D$ such that no two points of D have the same itinerary with respect to f and t, and such that D_t^* is Hausdorff.

In the dendrite case covered in [2], the spaces corresponding to D_t^* were finite discrete spaces, making the Hausdorff requirement redundant there. While it is easy to find dendroids D and points $t \in D$ such that D_t^* is not Hausdorff (e.g., many points in the Cantor fan), it is not clear that a non-Hausdorff D_t^* is possible if all other parts of the above definition hold. Thus, we have the following question.

Question. Is the requirement that D_t^* be Hausdorff redundant in the above definition? (That is, is there an example in which D_t^* is not Hausdorff and there is a continuous function on D which has the unique itinerary property with respect to t?)

Theorem 2.6. *Let $f : D \to D$ be a tentish map on a u.a.c.c. D, and assume the labelling convention. Then*

(1) *The point t of the definition is the unique turning point of f.*
(2) *f is one-to-one on all arc-components of D_t.*
(3) *The itinerary of t begins $01^{n-1}2$ for some $n \geq 2$.*
(4) *There is a fixed point $z \in S_1$ such that D_z has exactly n arc-components, each containing exactly one point of $\{t, f(t), \ldots, f^{n-1}(t)\}$.*

Proof: If $x \neq y$ and $f(x) = f(y)$, then x and y cannot be in the same arc-component of D_t, since they would then have the same itinerary. Thus, f must be one-to-one on all arc components of D_t, and therefore no point other than t can be a turning point. We cannot have $f(t) = t$, since points in the same arc-component of D_t would then have the same itinerary. Also, $f(t)$ cannot be a fixed point, since f is one-to-one on $[t, f(t)]$. Thus, there must be a positive integer n such that $f^n(t)$ is in a different arc-component of D_t than $f(t)$, for otherwise, $f(t)$ and $f^2(t)$ would both have itinerary $\overline{1}$. Pick n least such that this is the case.

Since f maps $\{t, f(t), \ldots, f^{n-1}(t)\}$ one-to-one into $\{f(t), f^2(t), \ldots, f^n(t)\}$, it is easy to see that there is a fixed point z in S_1

(use the "dog chases rabbit" trick, showing that you never leave the subtree $[\{f(t), f^2(t), \ldots, f^n(t)\}]$); and one-to-oneness of f in S_1, along with the unique itinerary property, shows that the points must be arranged as in (4). D_z cannot have additional components, because any member of one of those components would have to have the same itinerary as z, contadicting the definition.

We will be done if we can show that t is a turning point of f. Suppose not. Then f is a homeomorphism onto its range (since D is a dendroid), as is f^n. Thus, f^n maps $[z, t]$ one-to-one onto $[z, f^n(t)]$, from which it is then easy to see that every $x \in (z, f^n(t)]$ has the same itinerary $\overline{21^{n-1}}$, a contradiction. $\square$

Definition 2.7. Let Σ be a collection of symbols. Then a sequence $\alpha \in \sigma^\omega$ will be called *consistent* if and only if whenever $\alpha_n = \alpha_0$ we have that $\alpha_{n+k} = \alpha_k$ for all $k \in \omega$. If $\alpha, \beta \in \Sigma^\omega$ and α is consistent, then we say that β is *α-consistent* if and only if $\beta_n = \alpha_0$ implies that $\sigma^n(\beta) = \alpha$.

The definition of the term "consistent" is motivated by the following trivial observation regarding the behavior of itineraries involving singleton equivalence classes in a partition.

Proposition 2.8. *Let X be a topological space, let $f : X \to X$, let $\mathcal{S}$ be a partition of X such that $S_0 = \{t\} \in \mathcal{S}$ for some $t \in X$, and let $\iota = \iota_f^{\mathcal{S}}$. Then $\iota(t)$ is consistent, and for every $x \in X$, $\iota(x)$ is $\iota(t)$-consistent.*

Theorem 2.9. *If D is a u.a.c.c. and $f : D \to D$ is tentish with turning point t and with $\iota = \iota_f^{\mathcal{S}}$, then for every distinct $x, y \in D$, there is an $n \in \omega$ such that $\iota_0(t) \neq \iota_n(x) \neq \iota_n(y) \neq \iota_0(t)$.*

Proof: As in [2], if the conclusion were false, then every point in (x, y) would have the same itinerary. $\square$

Definition 2.10. Let Σ be a set of symbols, with $0 \in \Sigma$ and suppose that Σ has a topology such that Σ is the only neighborhood of 0, and $\Sigma \backslash \{0\}$ is open in Σ and Hausdorff in the subspace topology. Let $\alpha.\beta \in \Sigma^\omega$. Then α is said to be an *acceptable* sequence if and only if $\alpha_0 = 0$, α is consistent, and whenever n is such that $\gamma = \sigma^n(\alpha) \neq \alpha$, there is a $k \in \omega$ such that $0 \neq \alpha_k \neq \gamma_k \neq 0$. If α is an acceptable sequence, we say that β is *α-admissable* if and only if β is α-consistent and whenever n is such that $\gamma = \sigma^n(\beta) \neq \alpha$, there is a $k \in \omega$ such that $0 \neq \alpha_k \neq \gamma_k \neq 0$. If C is a Hausdorff topological space such that $0 \notin C$, $\Sigma = C \cup \{0\}$ is as above, and τ is an acceptable sequence, then we define $D_{(C,\tau)}$ to be the set of all τ-admissable sequences from Σ^ω, with the subspace topology induced from the itinerary topology on Σ^ω. (Note that although the subspace topology on $D_{(C,\tau)}$ does depend on the topology of C, the set $D_{(C,\tau)}$ itself does not depend on the topology of C.)

Theorem 2.11. *Let C be a Hausdorff space with $0 \notin C$, let $\Sigma = C \cup \{0\}$, and extend the topology of C to Σ by letting Σ be the only neighborhood of 0.*

If τ is acceptable, then $D = D_{(C,\tau)}$ satisfies the following properties.

(1) *There is only one element α of D such that $\alpha_0 = 0$ (i.e., $\alpha = \tau$);*
(2) *D is closed under the shift operation σ;*
(3) *for every distinct $\alpha, \beta \in D$, there are disjoint sets A and B, both open in Σ^ω, such that $\alpha \in A$ and $\beta \in B$ (so, in particular, D is Hausdorff); and*
(4) *D is maximal in Σ^ω with respect to properties 1, 2, and 3.*

Proof: (1) The only element α of D such that $\alpha_0 = 0$ would be $\alpha = \tau$, since no other such α would be τ-consistent.

(2) Trivial from the definition of τ-admissable.

(3) Let $\alpha, \beta \in D_{(C,\tau)}$, with $\alpha \neq \beta$, and let n be least such that $\alpha_n \neq \beta_n$. If both α_n and β_n are different from 0, then they are both in C, and therefore α and β can be separated by open sets in Σ^ω. If one of α_n and β_n is 0, then we may assume by symmetry that $\beta_n = 0$, so that $\sigma^n(\beta) = \tau$, and letting $\gamma = \sigma^n(\alpha)$, τ-admissability of γ gives us a positive integer m such that $0 \neq \gamma_m \neq \tau_m \neq 0$, from which we immediately get that $0 \neq \alpha_{n+m} \neq \beta_{n+m} \neq 0$, so that α and β can be separated in Σ as before.

(4) Any element α of $\Sigma^\omega \setminus D$ is either not τ-consistent, in which case some shift of α violates (1), or α violates the main part of the definition of τ-admissability, in which case some shift of α cannot be separated from τ. □

The following two results from [2] generalize easily.

Proposition 2.12. *Let $\tau \in \Sigma^\omega$ be acceptable, and let α be τ-admissable, and let $a \in \Sigma = C \cup \{0\}$, where C is Hausdorff. Then*

(1) *if* $\alpha = \sigma(\tau)$*, then* $\langle a\rangle\alpha$ *is* τ*-admissable if and only if* $a = 0$*;*
(2) *if* $\alpha \neq \sigma(\tau)$*, then* $\langle a\rangle\alpha$ *is* τ*-admissable if and only if* $a \neq 0$*.*

Proof: Let $\beta = \langle a\rangle\alpha$.

(1) Since $\alpha = \sigma(\tau)$, $\beta_n = \tau_n$ for all n other than 0, so since $\tau_0 = 0$, β can be τ-admissable iff $\beta = \tau$, i.e., $a = 0$.

(2) If $a = 0$, then β is not even τ-consistent. In the other direction, suppose $a \neq 0$, and since α is τ-admissable, and β is clearly τ-consistent, we need only to check that there is an n such that $0 \neq \beta_n \neq \tau_n \neq 0$. Since $\alpha \neq \sigma(\tau)$, there is a $k > 0$ such that $\beta_k \neq \tau_k$, and we fix the least such k. By definition, we are done unless one of β_k or τ_k is 0. If both are 0, then τ-consistency of both τ and β would give that $\alpha = \sigma(\tau)$, a contradiction. If one of $\{\tau_k, \beta_k\}$ is 0 and the other is not, then τ-consistency gives us that one of $\{\sigma^k(\tau_k), \sigma^k(\beta_k)\}$ is τ and the other (call it γ) is a τ-acceptible sequence different from τ, so there is an m so that $0 \neq \tau_m \neq \gamma_m \neq 0$, which gives us $0 \neq \tau_{k+m} \neq \beta_{k+m} \neq 0$ as desired. □

Proposition 2.13. *Let* $\tau \in \Sigma^\omega$ *be acceptable, let* α *be* τ*-consistent, and assume that* $C = \Sigma\backslash\{0\}$ *is Hausdorff. Then there is exactly one* τ*-admissable* $\beta \in \Sigma^\omega$ *such that every open set in* Σ^ω *which contains* β *also contains* α *(or, equivalently,* $\alpha_n \neq \beta_n$ *implies* $\beta_n = 0$*).*

Proof: Uniqueness follows from the fact that $D_{(C,\tau)}$ is Hausdorff. To see existence, if α is τ-admissable, then we let $\beta = \alpha$ and we are done. Thus, suppose that α is not τ-admissable, and let n be such that τ and $\sigma^n(\alpha)$ cannot be separated. Then define β by letting $\sigma^n(\beta) = \tau$ and then, using Proposition 2.12 to define the remaining β_i's by backwards induction, $i = n-1, n-2, ..., 2, 1, 0$. □

Definition 2.14. The unique β of the previous proposition is called $\chi_\tau(\alpha)$.

Theorem 2.15. *If* τ *is an acceptable sequence, then* $D_{(C,\tau)}$ *is a compact metric space if and only if* C *is a compact metric space.*

Proof: Since C is homeomorphic to $\sigma^{-1}(\tau)$, the $(\Rightarrow)$ direction is trivial. The other direction, which we outline here, is as in [2]. If S is a sequence from $D_{(C,\tau)}$, then there must be a subsequence which converges to some α in the strong topology in Σ^ω (since this strong topology is also compact), and therefore converges to α in the itinerary topology (since the itinerary topology is weaker). α is clearly τ-consistent (but perhaps not τ-admissable), and $\chi_\tau(\alpha)$ will be the τ-admissable limit of the same subsequence (keeping in mind that the limit of a sequence need not be unique in a non-Hausdorff space). Thus, since there is a countable basis and every sequence has a convergent subsequence, $D_{(C,\tau)}$ is compact. The Urysohn Metrization Theorem completes the proof. □

Theorem 2.16. *If C is a Hausdorff space, and τ is acceptable, then $D_{(C,\tau)}$ is arcwise connected, with unique arcwise connectedness holding if and only if C contains no arcs.*

Proof: Arcwise connectedness follows as in [2]: Given $\alpha, \beta \in D_{(C,\tau)}$, with $\alpha \neq \beta$, we define a function $g : [0,1] \to D_{(C,\tau)}$ by letting $g(0) = \alpha$ and $g(1) = \beta$ and then defining f on the dyadic rationals $j/2^k$ ($0 < j < 2^k$, j odd) by induction on the denominator 2^k. If a and b are consecutive dyadic rationals having denominator less than or equal to 2^k, and $g(a) \neq g(b)$ have been defined, then let n be least such that $g(a)$ and $g(b)$ can be separated on the nth coordinate, and assume as an induction hypothesis that we will have $n \geq k$. There cannot be an $m < n$ such that both $(g(a))_m = (g(b))_m = 0$, since we would then have $(g(a))_n = (g(b))_n$ by τ-consistency of $g(a)$ and $g(b)$. Thus, let γ be the unique sequence of length n containing no 0's which cannot be separated from either $g(a)$ or $g(b)$ on coordinates less than n (trivial since if we have $(g(a))_m \neq (g(b))_m$ for $m < n$, then exactly one of $(g(a))_m$ and $(g(b))_m$ is nonzero), and let $g(\frac{a+b}{2}) = \chi_\tau(\gamma\tau)$. Since $((g\frac{a+b}{2}))_n = 0$, $g(\frac{a+b}{2})$ is distinct from both $g(a)$ and $g(b)$, and it is easy to see that $g(\frac{a+b}{2})$ cannot be separated from $g(a)$ (or from $g(b)$) on the coordinates $0, 1, 2, ..., n$. Thus, the induction hypothesis will remain true at the next step, and the points of disagreement must occur at larger and larger values of n.

Thus, let $x \in [0,1]$ which is not a dyadic rational, and for each $k \geq 1$, let c_k and d_k be the unique consecutive dyadic rationals having denominator less than or equal to 2^{k+1} such that $x \in [c_k, d_k]$. Then by the construction, there is a unique $\gamma_k \neq 0$ which cannot be separated from either $(g(c_k))_k$ or $(g(d_k))_k$. Let γ be the (necessarily τ-consistent) sequence thus constructed, and let $g(x) = \chi_\tau(\gamma)$. Using the argument of the last theorem, it is easy to see that $x_n \to x$ in $[0,1]$ implies that $g(x_n) \to g(x)$ in $D_{(C,\tau)}$, so the function g is continuous. Thus, the range of g must at least contain an arc from

$f(0)$ to $f(1)$, and $D_{(C,\tau)}$ is arcwise connected. (With a bit more work, it is not difficult to show that g is in fact one-to-one.)

Suppose that C contains an arc A. Then $A' = \{x\tau : x \in A\}$ is an arc which does not contain τ. On the other hand, doing the process of the first part of the proof on the endpoints of A' produces an arc with the same endpoints containing τ. Thus, the presence of an arc in C implies that $D_{(C,\tau)}$ contains circles.

In the other direction, suppose that $D_{(C,\tau)}$ contains a circle S. Let n be least such that two points of S can be separated on the nth coordinate. Then no more than one point of S can have nth coordinate 0 (for two distinct points having nth coordinate 0 would have to be separated on some smaller coordinate, violating the choice of n). Thus, S contains an arc A in which no 0 appears in the nth coordinate, and the endpoints have different nth coordinates. Thus $\pi_n : A \to C$ is a continuous function having range more than one point, where π_n is a projection onto the nth coordinate. Since C is Hausdorff, $\pi_n(A)$ is a compact Hausdorff space which is a continuous image of an arc, and therefore contains an arc. □

Lemma 2.17. *If $f : D \to D$ is a tentish map of a dendroid D with turning point t, then $f^{-1}(x)$ is a zero-dimensional compact Hausdorff space for every $x \in D$.*

Proof: By contradiction. Suppose that $x \in D$ and $f^{-1}(x)$ is not zero-dimensional. Then $f^{-1}(x)$ contains a nondegenerate continuum C, say $a, b \in C$ with $a \neq b$. Then $[a, b] = [a, t] \cup [t, b]$ and C are two subcontinua of D, whose intersection is exactly the two points $\{a, b\}$ (since C cannot meet a leg in more than one point). This contradicts that D is a dendroid. □

Theorem 2.18. *If τ is an acceptable sequence, then $D_{(C,\tau)}$ is a dendrite if and only if C is a finite discrete space.*

Proof: ($\Leftarrow$) Suppose that C is a finite discrete space. Let $B = \Pi_{i\in\omega} U_i$ be a basic open set of $(C \cup \{0\})^\omega$, and let B' be the closure of B in the itinerary topology of $(C \cup \{0\})^\omega$. If $\alpha, \beta \in B' \cap D_{(C,\tau)}$, it is easy to see, by taking a closer look at the proof of 2.16, that the function g constructed there has range in the closure of B. Thus, $B' \cap D_{(C,\tau)}$ is arcwise connected and therefore connected. Since this is true for the closure of any basic open set, it follows that $D_{(C,\tau)}$ is locally connected, and therefore a dendrite (since compact, metric, and uniquely arcwise connected were shown in previous proofs).

($\Rightarrow$) For the other direction, since $\sigma^{-1}(\tau)$ is homeomorphic to C, we know by the previous lemma that C is a zero-dimensional compact Hausdorff space. If C had a non-isolated point a, then no neighborhood of $a\tau$ in $D_{(C,\tau)} \setminus \{\tau\}$ would be connected, so C is discrete, and therefore finite (since it is compact). □

Proposition 2.19. *If X is a separable metric space, and $x_1, x_2, \ldots, x_n \in X$, then there is a tree T and a continuous $f : X \to T$ such that $f^{-1}(f(x_i)) = \{x_i\}$, $1 \leq i \leq n$.*

Proof: Let T be a tree with n endpoints, say $T = \bigcup_{i=1}^{n}[z, t_i]$, where t_i are distinct endpoints of T, and z is not an endpoint. Let U_i be disjoint closed neighborhoods of x_i, $1 \leq i \leq n$. Then it is easy to find continuous $f_i : U_i \to [z, t_i]$ with $f_i^{-1}(t_i) = \{x_i\}$ and such that $f_i^{-1}(z)$ contains the boundary of U_i. Define $f|U_i = f_i$ and $f(x) = z$ for $x \notin \bigcup_{i=1}^{n} U_i$. □

Theorem 2.20. *If τ is an acceptable sequence, then $D_{(C,\tau)}$ is a dendroid if and only if C is a zero-dimensional compact Hausdorff space.*

Proof: Lemma 2.17 immediately gives us the ($\Rightarrow$) direction. For the ($\Leftarrow$) direction, suppose that C is a zero-dimensional compact Hausdorff space. Fix a metric on $D = D_{(C,\tau)}$, and let $\epsilon > 0$. Cover $D_{(C,\tau)}$ with finitely many open sets U_i of diameter less than ϵ, each of the form $U_i = \Pi_{j\in\omega} U_{i,j}$, where all but finitely many $U_{i,j}$'s are all of $\Sigma = C \cup \{0\}$, and the rest are clopen subsets of C. (For convenience, let us call the latter the "nontrivial" $U_{i,j}$'s.) Let N be a positive integer such that $U_{i,n} = \Sigma$ for all i and all $n \geq N$. Let $\mathcal{A}$ be the collection of subsets of C obtained by starting with all of the nontrivial $U_{i,j}$'s, and closing under unions, intersections, and complements. $\mathcal{A}$ is clearly finite, so let $W_1, W_2, \ldots, W_q$ be the minimal nonempty elements of $\mathcal{A}$. Then the W_m's form a partition of C by clopen sets such that each $U_{i,j}$ is the union of finitely many W_m's. Let $Q = \{1, 2, 3, \ldots, q\}$, with the discrete topology. For each $a \in C$, let $k(a)$ be the unique i such that $a \in W_i$. Using the previous proposition, let $f_0 : D \to T_0$ be a continuous function from D into a tree T_0 such that $f^{-1}(f(\sigma^i(\tau))) = \{\sigma^i(\tau)\}$ for $0 \leq i \leq N$. Supposing that $n < N$ and $f_n : D \to T_n$ has been defined for some tree T_n, let f_{n+1} be defined as follows. Let T_{n+1} be the quotient of the product space $Q \times T_n$ which is formed by identifying the q points of $Q \times \{f_n(\sigma^{n+1}(\tau))\}$ to a point p_{n+1}. Define $f_{n+1} : D \to T_{n+1}$ by letting $f_{n+1}\tau = p_{n+1}$ and $f_{n+1}(\alpha) = (k(\alpha_0), f_n(\sigma(\alpha)))$. To see that f_N is as desired, let $z \in T_N$. If $\sigma^i(\alpha) \neq \tau$ for all $\alpha \in f_N^{-1}(z)$ and all $i < N$, then $f_N^{-1}(z) \subseteq \Pi_{i=0}^{N-1} W_{k(\sigma^i(\alpha))} \times \Pi_{i=N}^{\infty}\Sigma$, for every $\alpha \in f_N^{-1}(z)$. If $\sigma^n(\alpha) = \tau$ for some $\alpha \in f_N^{-1}(z)$, then, letting n be least such, $f_N^{-1}(z) = \Pi_{i=0}^{n-1} W_{k(\sigma^i(\alpha))} \times \Pi_{i=n}^{\infty}\{\tau_{i-n}\}$. In either case, $f_N^{-1}(z)$ is contained in one of the U_i's, so D is tree-like. □

Definition 2.21. Given a zero-dimensional compact Hausdorff space C not containing 0 and an acceptable sequence τ whose range is contained in $C \cup \{0\}$, we define $D''_{(C,\tau)}$ to be the smallest subdendroid of $D_{(C,\tau)}$ containing τ and invariant under σ, and we let $\sigma''_{(C,\tau)} = \sigma | D''_{(C,\tau)}$. A tentish map $f : D \to D$ is said to be *minimally tentish* if and only if no restriction of f to a proper subdendroid containing the turning point is invariant. Clearly, $\sigma''_{(C,\tau)}$ is minimally tentish. A tentish map $f : D \to D$ is said to be *self-similar* if and only if the closure of each leg maps homeomorphically onto all of D by f. It is easy to see that each $D_{(C,\tau)}$ is self-similar.

The following two results show that the spaces $D_{(C,\tau)}$ give us a complete classification of the conjugacy types of tentish maps on uniquely arcwise connected continua.

Theorem 2.22. *If $f : D \to D$ is a tentish map on a u.a.c.c. with turning point t, $C = D^*_t$ with corresponding itinerary function ι, and $\tau = \iota(t)$, then $\iota : D \to D_{(C,\tau)}$ is a homeomorphism onto its range, and $\iota(f(x)) = \sigma(\iota(x))$ for every $x \in D$.*

Proof: $\iota(f(x)) = \sigma(\iota(x))$ is clear from the definition of itinerary. The definition of the itinerary topology guarantees continuity of ι, and the unique itinerary property gives that ι is one-to-one. Since the domain is compact and the range is Hausdorff, ι must therefore be a homeomorphism onto its range. □

Corollary 2.23. *Let $f : D \to D$ and $g : E \to E$ be self-similar tentish maps on uniquely arcwise connected continua D and E with turning points t and u respectively. Then f and g are conjugate if and only if there is a homeomorphism $h : D^{**}_t \to E^{**}_u$ such that*

$\upsilon_n = h(\tau_n)$ for all $n \in \omega$, where τ and υ are the kneading sequences of t and u, respectively.

3. Dendroid maps having kneading sequences with infinite range

In the results on dendrites which appeared in [2], one complication which was avoided was the case in which the orbit of the turning point intersects infinitely many legs. As seen above, $D_{(C,\tau)}$ cannot be a dendrite unless C is a finite discrete space, but cases do exist in which τ has infinite range in a Hausdorff space C and some σ-invariant subset of $D_{(C,\tau)}$ containing τ is a dendrite. As it turns out, there are only certain τ for which such a tentish dendrite map is possible, and in those cases, the minimally tentish map is independent of the Hausdorff space C in which the range of τ is embedded, so that a minimally tentish dendrite map having kneading sequence τ, if it exists at all, depends (up to conjugacy) only on the sequence τ. The main goal of this section is to show this, and to characterize for which τ such a dendrite map exists. We start with some trivial observations regarding how the spaces $D_{(C_1,\tau)}$ and $D_{(C_2,\tau)}$ are related when C_1 and C_2 are closely related.

Proposition 3.1. *If τ is acceptable, C_2 is a Hausdorff space not containing 0 such that the range of τ is contained in $C_2 \cup \{0\}$, and C_1 is a subspace of C_2 such that $C_1 \cup \{0\}$ contains the range of τ, then $D_{(C_1,\tau)}$ is a subspace of $D_{(C_2,\tau)}$.*

Proposition 3.2. *If τ is acceptable, C_1 is a Hausdorff space not containing 0 such that the range of τ is contained in $C_2 \cup \{0\}$, and C_2 is a stronger (finer) topology on the same set, then $D_{(C_2,\tau)}$ is a stronger topology than $D_{(C_1,\tau)}$ (also with the same underlying set).*

Note that if D is a dendrite and t is any point of D, then D_t^* is a countable discrete space, and therefore not compact if D_t^* is infinite, so that one direction of Theorem 2.15 would not necessarily work for invariant subdendrites of $D_{(C,\tau)}$. Thus, it is easy to see how $D''_{(C_1,\tau)}$ and $D''_{(C_2,\tau)}$ might be the same, even when C_1 and C_2 are different.

Theorem 3.3. *Let τ be an acceptable sequence with infinite range, and let C be a Hausdorff space with $0 \notin C$ such that $C \cup \{0\}$ contains the range of τ. For convenience, assume the labelling convention.*

Then $D''_{(C,\tau)}$ is a dendrite if and only if there exists a strictly increasing function $r : \omega \to \omega$ such that whenever $m \geq r(n)$ and $\tau_i = m$, $\tau_{i+j} = \tau_j$ for all j such that $1 \leq j \leq n$. In the case where $D''_{(C,\tau)}$ is a dendrite, $D''_{(C,\tau)}$ and $D''_{(C',\tau)}$ are homeomorphic (via the identity map) for all Hausdorff spaces C, C'.

Proof: ($\Rightarrow$) Suppose that $D = D''_{(C,\tau)}$ is a dendrite, and work inside D. For each $n \geq 1$, let $A_n = \{\alpha \in D : \alpha_0 = n\}$. Note that since D is a dendrite, each A_n is open in D, and every neighborhood of τ contains all but finitely many A_n's. Since $U_n = \bigcap_{i=1}^{n} \sigma^{-i}(A_{\tau_i})$ is a neighborhood of τ, we can define $r(n)$ so that $A_m \in U_n$ for all $m \geq r(n)$, and we can clearly make r strictly increasing. It is easy to check that such an r works.

($\Leftarrow$) Suppose that $r : \omega \to \omega$ is strictly increasing such that whenever $m \geq r(n)$ and $\tau_i = m$, $\tau_{i+j} = \tau_j$ for all j such that $1 \leq j \leq n$. We first show that each leg S_i is open in $D''_{(C,\tau)}$. Thus, fix $i \in \mathbb{N}$. Let $U \subseteq D''_{(C,\tau)}$ be any neighborhood of τ. Then there is an $n \in \omega$ and open sets V_j in the symbol topology of $\Sigma = C \cup \{0\}$ such that $\tau_j \in V_j$, $V = D''_{(C,\tau)} \cap \Pi_{j\in\omega} V_j \subseteq U$, and such that $V_j = \Sigma$ for all $j \geq n$. Thus, by the definition of $r(n)$, $S_j \subseteq \overline{V}$ for all $j > r(n)$, so that $\overline{U}$ contains all but finitely many legs. Thus, since C is Hausdorff, every finite subset of C is discrete, so $S_i \setminus \overline{U}$ is open in $D''_{(C,\tau)}$. Since U was an arbitrary neighborhood of τ, S_i must be open in $D''_{(C,\tau)}$. Thus, for every τ-consistent finite sequence α of length n from Σ, the set $D''_{(C,\tau)} \cap A_\alpha = \bigcap i < n \sigma^{-i}(S_{\alpha_i})$ is open, so these sets form a basis of $D''_{(C,\tau)}$, each of whose closures in $D''_{(C,\tau)}$ is connected, so $D''_{(C,\tau)}$ is locally connected.

To complete the proof of the ($\Leftarrow$) direction, we must show that the space $D''_{(C,\tau)}$ is compact in this case, even if C is not. We combine this with the proof that the homeomorphism type of the space is independent of the Hausdorff space C. First, let Σ be the range of τ, let $C = \Sigma \setminus \{0\}$, put the discrete topology on C, and let $C' = C \cup \{\infty\}$ be the one-point compactification of C, where ∞ is an element not in Σ, letting $\Sigma' = \Sigma \cup \{\infty\}$ with the obvious symbol topology on Σ'. Since C' is compact, $D''_{(C',\tau)}$ is compact, and therefore a dendrite, so since every neighborhood of τ contains all but finitely many legs of $D''_{(C',\tau)}$, $S_\infty \cap D''_{(C',\tau)} = \emptyset$, and thus

$D''_{(C,\tau)} = D''_{(C',\tau)}$ is also compact. Thus, if C'' is any other Hausdorff space such that $C'' \cup \{0\}$ contains the range of τ, the identity map from $D''_{(C,\tau)}$ into $D_{(C'',\tau)}$ will be a homeomorphism onto its range, which is easily seen to be $D''_{(C'',\tau)}$. □

4. Examples when τ is the identity function

The statement of Corollary 2.23 has an obvious followup question: How do $D_{(C_1,\tau)}$ and $D_{(C_2,\tau)}$ differ from each other when C_1 and C_2 are not equivalent? This is not immediately obvious, because the existence theorems for $D_{(C,\tau)}$ do not really reveal much about the structure of these spaces other than the fact that they are dendroids if C is a zero-dimensional compact Hausdorff space. In general, it is not clear to what extent there might be a uniform method which would reveal more about the structure of an individual $D_{(C,\tau)}$, but it is not difficult to find ad hoc ways of dealing with some specific examples. The obvious example to look at first would be the identity kneading sequence τ defined by $\tau_n = n$ for all $n \in \omega$, because that is the simplest such function having infinite range, and it also fails to satisfy the criterion of Theorem 3.3 above, so that $D''_{(C,\tau)}$ is not a dendrite.

In this section, we shall look at how a few examples for $D''_{(C,\tau)}$ differ from each other when τ is the identity function and C ranges over a few different examples. Note that knowing what $D''_{(C,\tau)}$ looks like will also give us a good impression of the structure of $D_{(C,\tau)}$, because if $D''_{(C,\tau)} \subseteq A \subseteq D_{(C,\tau)}$, then $\bigcup_{n\in\omega} \sigma^{-n}(A)$ is dense in $D_{(C,\tau)}$, and for any such A, $\sigma^{-1}(A)$ is homeomorphic to the result of taking $C \times A$ and identifying all points of the form (c, τ). We will assume that each of the spaces C contains $\mathbb{N}$ but not 0, so that $D_{(C,\tau)}$ will be well defined, and we let τ be the identity map for the remainder of this section.

Example 4.1. Let $C_1 = \mathbb{N}$, with the discrete topology. Although this case will not even give us a locally compact $D''_{(C_1,\tau)}$ (but it will be locally connected), it is the simplest example to construct. We let D_1 be the cone over C, and define $f_1 : D_1 \to D_1$ by $f_1(o) = (1, 1)$, $f_1(n, x) = (1, 1 - 2x)$ if $0 < x < \frac{1}{2}$ (for all n), $f_1(n, \frac{1}{2}) = o$, and $f_1(n, x) = (n + 1, 2x - 1)$ if $\frac{1}{2} < x < 1$. Then $(D_1)^*_o$ is discrete, so f_1 is conjugate to $\sigma_{(C_1,\tau)}$.

Note that D_1 will map naturally (using the above conjugacy, along with the identity maps on the $D_{(C,\tau)}$'s) into each of the remaining examples which we construct, and that this natural map will be continuous (since C_1 is discrete). This will also map D_1 densely into $D''_{(C,\tau)}$ for any such C, but it will not map D_1 homeomorphically onto its range unless $\mathbb{N}$ is discrete as a subspace of C. In our following examples where C is compact, we want to see how the range of D_1 gets compactified, either by adding new points on which the legs of D_1 can limit, by weakening the topology on D_1 so that the result is compact, or by a combination of these.

Example 4.2. Let C_2 be the one point compactification of C_1, by adding a new point ∞. Let D_2 be the cone over C_2, $f_2|C_1 = f_1$, $f(\infty, x) = (1, 1-2x)$ if $0 < x < \frac{1}{2}$, $f(\infty, \frac{1}{2}) = o$, and $f(\infty, x) = (\infty, 2x-1)$ if $\frac{1}{2} < x < 1$. This space, with a single "limit leg" to serve as the limit of the remaining legs, is the easiest compact example for this τ.

Example 4.2 a. Modify the topology of Example 4.2 by making the set $\{(\infty, x) : \frac{1}{2} < x \leq 1\}$ an open subset (i.e., we "peel away" the top half of the limit bar so that the top halves of the legs S_n no longer limit on anything). This does not change $(D_o)^*_t$, which is still compact, but the space, although still uniquely arcwise connected, is no longer compact. This gives us an example that compactness is necessary in Theorem 2.21 above.

Example 4.3. Let C be any infinite zero-dimensional compact Hausdorff space (i.e., something homeomorphic to a closed infinite subset of a Cantor set), and let $h : C \to C$ be continuous with $z \in C$ such that the points $h^n(z)$ are all distinct. Assume that $0 \notin C$ and that the points $h^n(z)$ have been renamed so that $h^n(z) = n+1$, $n \in \omega$. Let D_3 be the cone over C_3 and define $f_3 : D_3 \to D_3$ by $f_3(o) = (1,1)$, $f_3(y, x) = (1, 1-2x)$ if $0 < x < \frac{1}{2}$, $f_3(y, \frac{1}{2}) = o$, and $f_3(y, x) = (h(y), 2x-1)$ if $\frac{1}{2} < x < 1$.

The number of possibilities here is vast and clearly allows any finite number of "limit legs" as in Example 4.2, or even infinitely many (countable or uncountable). This class of examples also includes many in which the topology restricted to $\mathbb{N}$ is not discrete, so that the natural one-to-one map of D_1 into D_3 would not be a homeomorphism onto its range. However, the simple nature of the above example depends on the existence of the above function h, and it is easy to see that such a function does not exist in general if countably many members of the Cantor set are arbitrarily identified with the members of $\mathbb{N}$. In those cases, the above existence proofs only guarantee the existence of $D_{(C,\tau)}$, and it is more difficult to see the exact topology. In the last two examples, we try using $\mathbb{N}$ itself as the set C, but with weaker compact topologies so that $D_{(C,\tau)}$ will be a dendroid.

Example 4.4. Let C be the set $\mathbb{N}$ of natural numbers, but instead of the discrete topology as in Example 4.1, we let C be a one-point compactification of the set $\mathbb{N} \setminus \{2\}$, with 2 as the point of compactification, and all other points isolated. The natural embedding of D_1 into $D_{(C,\tau)}$ will not have a compact range. To see this, we note that since $f(n,1) = (n+1,1)$, the points $(n,1)$ $(n \geq 3)$ would have to converge to a fixed point in S_2, but there is no such fixed point in T_1. However, we can compactify it by adding a single interval from the leg S_2 to this range. Thus, let $D_4 = D_1 \cup I$, where I is a new interval attached to D_1 at the point $(2, \frac{1}{2})$ (on D_1) and at one endpoint (on I). Call the new points of I $(2', x)$ for $\frac{1}{2} < x \leq 1$, and define $f_4(2', x) = (2, 2x-1)$ if $\frac{1}{2} < x \leq \frac{3}{4}$ and $f_4(2', x) = (2', 2x-1)$ if $\frac{3}{4} < x \leq 1$. Then $(2', 1)$ is a fixed point, which will be a limit point of the points $(n,1)$ in the new topology. For convenience, let $z = (2', 1)$. The legs S_n are all intervals for $n \neq 2$, while S_2 is a simple triod (tree with three endpoints) with endpoints o (the turning point), $(2,1) = f_4^2(o)$, and z. The topology will be the same as T_1 at all points of T_4 except for the arc $[o, z]$, which will be the limit of the arcs $[o, (n,1)]$.

It is not difficult to see that if we fix any $n \neq 1$ and let n be the point of compactification instead of 2, then we get a similar example. (See the next example for the reason for the restriction $n \neq 1$.)

Example 4.5. This is the same as Example 4.4, except that we let 1 be the point of compactification instead of 2. This is different from Example 4.4 for two reasons. One is that the fixed point in S_1 is between the turning point o and $f_1(o)$, and therefore already a member of T_1. As in 4.4, all legs other than S_1 will have the same topology as in T_1, and the points $(n,1)$ will limit on this fixed point (in this case, $z = (1, \frac{1}{3})$). In this case, it will not be necessary to add additional points, so that we will have $T_5 = T_1$ *as sets*. However, what complicates things further is that $f_5 = f_1$ reverses orientation in S_1 near the point z, so that the T_5 topology on S_1 has to be defined in a more complicated way in order to compactify the topology while not losing continuity of f_5.

To define neighborhoods of a point in S_1, we first define $g : [-1, 1] \to [-1, 1]$ by $g(x) = 1 - 2|x|$. We now define a function $h : [0, 1] \to [0, 1]$ as follows. For each $x \in [0, 1)$, we let n be least such that $g^n(-x) \in [0, 1]$, and we then let $h(x)$ be the unique $y \in [0, 1]$ such that $g^n(-x) = g^n(y)$ and such that $g^i(y) \in [0, 1]$ for all i such that $0 \leq i \leq n$. We then let $h(1) = \frac{1}{3}$. Stated in another way, if we let L be a symbol standing for the interval $[-1, 0)$, and let M be a symbol standing for the interval $[0, 1]$, then the point whose negative has itinerary $L^n RW$ (with respect to g) is sent by h to the point having itinerary $R^{n+1}W$ (where W is an infinite word of L's and R's), and the point whose negative has itinerary L^∞ is sent to the point having itinerary R^∞. If $(1, x)$ is a point of S_1, and $x > \frac{1}{2}$, then $(1, x)$ has the same neighborhoods as in T_1. If $0 < x \leq \frac{1}{2}$, we let $U \subseteq (0, 1]$ be open in $[0, 1]$, fix a positive integer N, and let $V(U, N) = \{1\} \times U \cup ([N, \infty) \cap \mathbb{N}) \times h^{-1}(U)$ be a basic open neighborhood of $(1, x)$.

D_5 is best pictured where the legs S_n for $n \neq 1$ form a sequence of narrower and narrower spirals limiting on $[o, (1, \frac{1}{2})] \subseteq S_1$, with the points $(n, \frac{1}{2})$ limiting on $(1, \frac{1}{2})$, the points $(n, \frac{3}{4})$ limiting on $(1, \frac{1}{4})$, the points $(n, \frac{7}{8})$ limiting on $(1, \frac{3}{8})$, and so forth, with the points $(n, 1)$ (the ends of the spirals) limiting on $(1, \frac{1}{3})$. In the map, each spiral "unwinds" partly and maps partly to S_1 and partly to the next spiral. The top half of S_1 (the part with the same topology as T_1) maps to S_2.

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